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EDITED BY

WILLIAM THOMSON, M.A.,

PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW, AND LATE FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE, F.R.S., F.R.S.E., FOREIGN MEMBER OF THE ROYAL SWEDISH ACADEMY OF SCIENCES, HONORARY MEMBER OF THE MANCHESTER LITERARY AND PHILOSOPHICAL SOCIETY;

AND

N. M. FERRERS, B.A.,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

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THE

CAMBRIDGE AND DUBLIN
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ON RECIPROCAL METHODS IN THE DIFFERENTIAL CALCULUS.

By George Boole, LL.D.

(*Continued from* Vol. VII. p. 166.)

We pass now to the case in which the number of quantities in the two sets corresponding to parameters and coordinates is not the same.

(1). Let the two sets of connected quantities be $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_m$, and let m be greater than n. The following theorem may then be established.

Theorem II. If two sets of quantities $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_m$, m being greater than n, are connected by any relations $u_1 = 0$, $u_2 = 0, \ldots u_r = 0$, and if the former set varying in subjection to a certain condition among themselves $X = 0$, establish among the other set $a_1, a_2 \ldots a_m$, with which they are connected, a relation $A = 0$; then the set $a_1, a_2 \ldots a_n$, varying either in subjection to the single condition $A = 0$, or to that condition together with any other $m - n$ arbitrary conditions among themselves, will establish among the set $x_1, x_2 \ldots x_n$ the relation $X = 0$.

Let us suppose the quantities $a_1, a_2 \ldots a_m$ connected with another set of quantities $b_1, b_2 \ldots b_m$ by m arbitrary functional relations, as for example,

$$b_1 = f_1(a_1, a_2 \ldots a_m),\ b_2 = f_2(a_1, a_2 \ldots a_m), \ldots b_m = f_m(a_1, a_2 \ldots a_m) \ldots (21).$$

By virtue of these relations we can transform $u_1, u_2 \ldots u_r$ into functions of $x_1, x_2 \ldots x_n$, $b_1, b_2 \ldots b_m$.

If we do this, and cause $x_1, x_2 \ldots x_n$ to vary in subjection to the condition $X = 0$, we shall establish among the constants $b_1, b_2 \ldots b_m$ a relation which may be represented by $B = 0$. And it appears from the general Theorem I., that if we regard in B tities $b_1, b_2 \ldots b_n$ as

parameters, and cause them to vary in subjection to the condition $B = 0$, the remaining $m - n$ of those quantities remaining constant, we shall establish among the quantities $x_1, x_2 \dots x_n$ the relation $X = 0$. Or, which amounts to the same thing, we may suppose the whole series of quantities $b_1, b_2 \dots b_m$ to vary in subjection to the conditions

$$B = 0,$$

$$b_{n+1} = \text{const.}, \quad b_{n+2} = \text{const.}, \dots b_m = \text{const.} \;\dots\dots(22),$$

and the resulting relation among $x_1, x_2 \dots x_n$ will establish the condition $X = 0$.

Now it is obviously indifferent, whether in the above process we make $b_1, b_2 \dots b_m$ vary independently in subjection to the above conditions, or whether we write for $b_1, b_2 \dots b_m$ their values in terms of $a_1, a_2 \dots a_m$ in the several equations, and then make $a_1, a_2 \dots a_m$ vary in subjection to the conditions consequent upon this change; just as in seeking the maximum value of a function of certain quantities, it is indifferent whether we equate to 0 the differentials of these quantities, or transform them by functional relations into another system of quantities, and equate to 0 the differentials taken with respect to the latter. If we effect, then, the proposed change, B will obviously become A, and the equations (22) will become

$$f_{n+1}(a_1, a_2 \dots a_m) = \text{const.}, \quad f_{n+2}(a_1, a_2 \dots a_m) = \text{const.},$$

Rule. Determine the equation of the envelope of the given surface regarding the coordinates x, y, z as parameters, subject to the condition (25), and the parameters $a_1, a_2 \ldots a_m$ as if they were coordinates. The resulting equation $A = 0$ will express the only *necessary* condition, but it may be associated according to choice with any number not exceeding $m - 3$ of arbitrary relations among the quantities $a_1, a_2 \ldots a_m$.

There is a remarkable circumstance attending the employment of the equation $A = 0$, which may serve as a verification of the method by which in any particular instance that equation has been determined. It is this: when we seek the envelope of ϕ subject to the single condition $A = 0$, we employ in the process a system of equations of m equations

$$\frac{d\phi}{da_1} + \lambda \frac{dA}{da_1} = 0, \quad \frac{d\phi}{da_2} + \lambda \frac{dA}{da_2} = 0, \ldots \frac{d\phi}{da_m} + \lambda \frac{dA}{da_m} = 0,$$

where for convenience ϕ stands for $\phi(x, y, z, a_1, a_2 \ldots a_m)$. On elimination of λ these give $m - 1$ equations connecting $x, y, z, a_1, a_2 \ldots a_m$. Taking into account the primitive equation (24), we have thus implicitly a system of m equations connecting these quantities, *i.e.* the two sets x, y, z, and $a_1, a_2 \ldots a_m$, together.

Now if the above equations were independent of each other and of the equation $A = 0$, they would furnish us, on elimination of x, y, z with $m - 3$, new equations connecting $a_1, a_2 \ldots a_m$. But in reality there is only one *necessary* relation connecting $a_1, a_2 \ldots a_m$, viz. the equation $A = 0$. Hence the m equations previously referred to are not independent of each other and of the equation $A = 0$, and in fact it will be found that the elimination of x, y, z will only reproduce the equation $A = 0$, verifying the observation that this is the only *necessary* condition among the constants $a_1, a_2 \ldots a_m$, and serving as a test of the correctness of the process by which that condition is determined. The above conclusion admits of formal proof. The following example will serve to illustrate both the rule and the accompanying remark.

Ex. Required the conditions under which the straight line whose equation is $ax + by = c$ shall have for its envelope the circle whose equation is $x^2 + y^2 = \frac{1-m}{m}$, a, b, and c being the variable parameters.

We have
$$ax + by = c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),$$
$$x^2 + y^2 = \frac{1-m}{m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$
$$bx = ay \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$
the last equation being got by eliminating dx and dy from the differentials of the two preceding ones.

Eliminating x and y from the above, we get
$$(m-1)(a^2+b^2) + mc^2 = 0 \ldots\ldots\ldots\ldots (29).$$
Now if we seek the envelope of (26), subjecting the parameters a, b, c, to the above condition (29), then, whether we regard any two of the parameters as varying (according to the first case), or the three as varying (according to the second case), we obtain (27), the equation of the envelope. Let us confine ourselves to the latter case.

Our equations then are
$$ax + by - c = 0 \ldots\ldots\ldots\ldots\ldots\ldots (30),$$
$$(m-1)(a^2+b^2) + mc^2 = 0 \ldots\ldots\ldots\ldots (31),$$
whence, in the ordinary way, we get
$$\frac{(m-1)\,a}{x} = \frac{(m-1)\,b}{y} = -\,mc \ldots\ldots\ldots\ldots (32);$$
and eliminating a, b, and c, the parameters from the above four equations, we find
$$x^2 + y^2 = \frac{1-m}{} \ldots\ldots\ldots\ldots\ldots\ldots (33).$$

Thus, in the case just mentioned, we have the equations

$$ax + by - c = 0,$$
$$(m - 1)(a^2 + b^2) + mc^2 = 0 \ldots\ldots\ldots\ldots (34),$$
$$a^2 + b^2 = 1;$$

and eliminating da, db, dc from the differentials, we get

$$ay = bx,$$

from which four equations, on eliminating a, b, c, there results, as before,

$$x^2 + y^2 = \frac{1 - m}{m}.$$

Suppose that the added condition was

$$pa - qb = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (35).$$

Omitting the steps of a very tedious process of elimination and reduction, I shall simply indicate the result. The final equation of the envelope is

$$\{m(x^2 + y^2) + m - 1\}$$
$$\{(my^2 + m - 1)p^2 + 2mxypq + (mx^2 + m - 1)q^2\} = 0 \ldots (36),$$

the first factor of which, equated to 0, gives

$$x^2 + y^2 = \frac{1 - m}{m},$$

the equation of a circle as before; the second gives

$$(py + qx)^2 = \frac{1 - m}{m}(p^2 + q^2);$$

or $$\frac{py + qx}{\sqrt{(p^2 + q^2)}} = \pm\sqrt{\left(\frac{1 - m}{m}\right)},$$

and represents two straight lines touching the circle at the two extremities of that diameter which makes with the axes x and y angles whose respective cosines are

$$\frac{q}{\sqrt{(p^2 + q^2)}} \text{ and } \frac{p}{\sqrt{(p^2 + q^2)}}.$$

What is implied in this interpretation? I think it represents the following circumstance.

If the envelope of a straight line, the parameters in the equation of which are subjected to given conditions, is a complete re-entering curve, the constants representing the cosines of the angles which the straight line in its different

positions makes with the coordinate axes must be susceptible of the whole series of their values from -1 to 1.

But if the relations among the constants are such that some particular values are excluded, a breach of continuity in the curve occurs. Let us suppose that there are two points for which this happens. Then the whole curve, supposed to be a re-entering one, is divided into two portions mutually separated at their extremities by the points in question.

Now the envelope of the tangent to the curve supposed to remain in contact with one of the above portions will be that portion or arc itself, together with those portions of the tangents at the extremities of the arc which may be regarded as continuations of the arc. Hence when there are two arcs of a curve separated at their extremities by intermediate points, the envelope will consist of the two arcs, together with the four branches of the tangent at their four extremities, that is, together with the two complete tangents at the separating points.

Now in the case we have been considering, all the equations of the system are homogeneous with reference to a, b, and c, except the added equation

$$pa - qb = s.$$

Supposing that s does not vanish, this equation may consist with any ratio between a and b, except the ratio of q to p. The straight line represented by the equation $ax + by = c$ may pass, in contact, consistently with the conditions which are imposed upon it, over every part of the circle represented by the equation

$$x^2 + y^2 = \frac{1-m}{m},$$

except the two points in which a and b would have the above ratio, *i.e.* the two points which form the extremities of the diameter which makes with the axis of x an angle whose cosine is $\frac{q}{\sqrt{(p^2+q^2)}}$. Hence the complete envelope consists of the two separated portions of the circle, together with the two complete tangents at the separating points. This is in fact what the general equation (65) implies.

If $s = 0$, all the equations are homogeneous with respect to a, b, and c, and we get, on elimination, one additional equation, viz. $\frac{(py + qx)}{\;} = \pm\sqrt{\left(\frac{1-m}{m}\right)}$.

Now this agrees with the equation afforded by the second factor of (36). The envelope is here reducible to the pair of tangents described in the previous section.

2. Let m be less than n, we have then the following theorem.

THEOREM III. If two sets of quantities $x_1, x_2 \dots x_n$ and $a_1, a_2 \dots a_m$, m being less than n, are connected by any equations $u_1 = 0$, $u_2 = 0 \dots u_r = 0$. And if the former set, varying in those equations in subjection to the condition $X = 0$, establish among the other set a relation $A = 0$, together with any $n - m$ additional relations $X_1 = 0$, $X_2 = 0 \dots X_{n-m} = 0$, among the quantities $x_1, x_2 \dots x_m$; then the set $a_1, a_2 \dots a_m$, varying in the original equations $u_1 = 0, u_2 = 0 \dots u_r = 0$, in subjection to the condition $A = 0$, will establish among the quantities $x_1, x_2 \dots x_n$ a relation $W = 0$, of which, and of the previous $n - m$ relations among those quantities, the relation $X = 0$ will be a consequence, the relation $W = 0$ being moreover the essential condition of the existence of the relation $X = 0$.

If, as before, we represent the function $\lambda_1 u_1 + \lambda_2 u_2 \dots + \lambda_r u_r$ by U, we have the following set of equations $r + n + 1$ in number, viz.

$$u_1 = 0, \quad u_2 = 0, \dots u_r = 0, \quad X = 0,$$

$$\frac{d(U+X)}{dx_1} = 0, \quad \frac{d(U+X)}{dx_2} = 0, \dots \frac{d(U+X)}{dx_n} = 0 \dots (37);$$

from which, if in the first instance we eliminate $a_1, a_2 \dots a_m$, $\lambda_1, \lambda_2 \dots \lambda_r$, we obtain $n + 1 - m$ equations among $x_1, x_2 \dots x_n$, that is, $n - m$ equations additional to the equation $X = 0$. We shall represent these equations by $X_1 = 0$, $X_2 = 0 \dots X_{n-m} = 0$. Eliminating $\lambda_1, \lambda_2 \dots \lambda_r$, $x_1, x_2 \dots x_n$, we also get the condition $A = 0$.

Now if in the original system $u_1 = 0$, $u_2 = 0, \dots u_r = 0$, we cause $a_1, a_2 \dots a_m$ to vary in subjection to the condition $A = 0$, the resulting relation $W = 0$ will, by Theorem II., be such that if $x_1, x_2 \dots x_n$ vary in the same system, subject to the single condition $W = 0$, or to that condition associated with any other $n - m$ arbitrary conditions among $x_1, x_2 \dots x_n$, the relation $A = 0$ will result. Moreover this relation $W = 0$ is an essential condition of the existence of $A = 0$.

Hence the relation $W = 0$ is implicitly involved in the system (37); since of that system A is a consequence. But the relations among $x_1, x_2 \dots x_n$, involved in that system,

are $X = 0$, $X_1 = 0 \dots X_{n-m} = 0$. Of these relations, therefore, W is a consequence. Conversely, then, the relation $X = 0$ is a necessary consequence of the equations

$$W = 0, \quad X_1 = 0, \quad X_2 = 0, \dots X_{n-m} = 0 \dots\dots(38),$$

that is, of the $n - m$ equations furnished in the process for determining A, and the one essential condition $W = 0$, subsequently furnished by A.

Lastly, as the relation $W = 0$ is essentially involved in the system (37), it is an essential condition of the existence of $X = 0$.

When the relations $X_1 = 0$, $X_2 = 0, \dots X_{n-m} = 0$, are identical with the relation $X = 0$, then that relation unfettered by any other condition will be established by the variation of $a_1, a_2 \dots a_m$ in the original system in subjection to $A = 0$. W and X are then identical.*

The geometrical application of the above results is sufficiently obvious. The problem to be solved, and the rule furnished by the general theorem, may be thus stated.

The equation or equations of a given locus involving both variable parameters and coordinates, the number of the former being less than the number of the latter; required the conditions among the parameters under which the locus shall, by its successive mutual intersections, either generate a given fixed surface as its envelope, or execute the greatest possible amount of motion in contact with the given fixed surface. Required also, in the latter case, the equation of the envelope actually generated, and the equations of the track marked out by it upon the given fixed surface.

Rule. Determine the equation of the envelope of the moveable locus when the coordinates are regarded as parameters, and *vice versâ*, and the equation of the given fixed surface is regarded as expressing the condition among the parameters thus assumed. Find also by elimination the additional relations among the parameters employed in the above process.

If those relations are identical with the equation of the fixed surface, then the equation of the envelope above determined expresses the condition under which the move-

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* I am led to suspect that when there is but a single original equation $u = 0$ connecting the two sets of quantities $x_1, x_2 \dots x_n$ and $a_1, a_2 \dots a_m$, the equations $X = 0$, $W = 0$ will together virtually comprise the whole system of relations $X_1 = 0$, $X_2 = 0 \dots X_{n-m} = 0$. At present, however, I have not leisure to pursue the inquiry. Should any one else be disposed to take it up, the examination of [illegible] would be productive of interest.

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able surface will have the proposed fixed surface for its envelope.

But if the relations are not identical with the equation of the fixed surface, then the equation of the envelope above determined expresses the condition under which the given moveable surface will execute the greatest possible amount of motion in contact with the given fixed surface. And the equation of the envelope actually described may be determined in the ordinary way. The locus of contact will be determined by the equation of the fixed surface, together with the relations above referred to. If, however, there exist but one such relation, the equation of the envelope actually described may be used in its place.

It is probable that the above theory may appear complex from the great number of circumstances involved in its complete exposition. But it belongs to a harmonious and connected system, and one or two examples will suffice to elucidate every difficulty.

Ex. 1. To ascertain if possible the conditions under which the plane, whose equation is

$$ax + by + z = 1 \quad\text{..................}\quad (39),$$

shall generate the surface whose equation is

$$4xy = 27m(1 - z)^2 \text{..................} (40),$$

a and b being the variable parameters.

Here we are directed first to investigate the envelope of (39), subject to the condition (40), when x, y, and z are regarded as parameters, and a and b as coordinates.

Differentiating both equations with reference to x, y, and z, we have

$$adx + bdy + dz = 0,$$

$$4ydx + 4xdy - 54m(z - 1)\, dz = 0.$$

Whence, in the usual way,

$$\frac{2y}{a} = \frac{2x}{b} = -27m(z - 1) \text{..............} (41),$$

from which

$$x = \frac{-27bm(z - 1)}{2}, \quad y = \frac{-27am(z - 1)}{2}.$$

Substituting these values in (39) and dividing the result by the common factor $y - 1$, we have

$$27mab = 1 \quad\text{.....................}\quad (42).$$

Now to ascertain whether under this condition the plane (39)

will generate (40) as its envelope, let us, according to the rule, eliminate a and b from (39) and (41); we get

$$\frac{-2xy}{27m(z-1)} + \frac{-2xy}{27m(z-1)} + z = 1,$$

which gives $$4xy = 27m(z-1)^2.$$

This result is identical with (40), whence we infer that (40) will be the true envelope of (39) under the conditions determined.

To verify this conclusion, let us take the equations

$$ax + by + z = 1 \quad\text{..................}\quad (43),$$

$$27mab = 1 \quad\text{.....................}\quad (44),$$

and seek the envelope of the former when a and b vary in subjection to the latter.

Differentiating with respect to a and b, we have

$$xda + ydb = 0,$$

$$27mbda + 27madb = 0;$$

and, eliminating da and db,

$$ax - by = 0 \quad\text{.....................}\quad (45).$$

Now if from this equation and (43) and (44) we eliminate a and b we get

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The first of these equations gives

$$\frac{x}{x-a} = \frac{y}{y-b}, \text{ or } bx = ay \text{ .... } (49).$$

We also readily deduce from (48)

$$z = \sqrt{\{(x-a)^2 + (y-b)^2\}},$$

whence, by (46), $\qquad z = \sqrt{(1-z)^2};$

therefore $\qquad z = \pm\sqrt{\tfrac{1}{2}}.$

(47) then gives $\qquad x^2 + y^2 = (r \pm \sqrt{\tfrac{1}{2}})^2;$

whence, by (49), we find

$$x = \frac{a}{\sqrt{(a^2+b^2)}}(r \pm \sqrt{\tfrac{1}{2}})^2, \quad y = \frac{b}{\sqrt{(a^2+b^2)}}(r \pm \sqrt{\tfrac{1}{2}})^2.$$

And if the values of x, y, and z, thus found, be substituted in (46), we find, after reduction,

$$a^2 + b^2 = (r \pm \sqrt{2})^2 \text{ .... } (50)$$

for the condition between the parameters a and b.

Now we have seen in the course of this investigation that the elimination of a and b between (46) and the two equations of (48), gave us

$$z = \pm\sqrt{\tfrac{1}{2}} \text{........................} (51),$$

an equation which is not identical with (47). Hence it is not possible that the sphere (46) should generate the cone (47) as its envelope, a and b being the only variable parameters. Equation (50), however, expresses the condition under which the sphere can move in contact with the cone, the equations of the locus of contact being (47) and (51).

To find the equation of the actual envelope, we have

$$(x-a)^2 + (y-b)^2 = 1 - z^2 \text{ .... } (52),$$

$$a^2 + b^2 = (r \pm \sqrt{2})^2 \text{..................} (53).$$

Differentiating with respect to a and b,

$$(x-a)\,da + (y-b)\,db = 0,$$

$$ada + bdb = 0;$$

whence, eliminating da and db, we get

$$bx = ay.$$

Hence, and from (53), we have

$$a = \frac{x(r \pm \sqrt{2})}{\sqrt{(x^2+y^2)}}, \quad b = y\frac{(r \pm \sqrt{2})}{\sqrt{(x^2+y^2)}};$$

and substituting these values in (52), and reducing, we get

$$\sqrt{(x^2+y^2)} - \sqrt{(1-z^2)} = r \pm \sqrt{2}$$

as the equation of the true envelope, which is in fact a pair of tubular rings.

We have seen that the equations of the locus of contact are

$$\sqrt{(x^2+y^2)} + z = r,$$
$$z = \pm\sqrt{\tfrac{1}{2}}.$$

Now it is easy to shew that these equations represent two circles traced on the above rings, the one at a uniform distance of $\sqrt{\frac{1}{2}}$ above the plane of xy on the inner ring, the other at a like distance below the plane of xy on the outer ring. It must be observed that $\sqrt{(1-z^2)}$ has the same sign as z.

All these conclusions may be verified by geometrical considerations. It is obvious that a sphere of invariable radius (46), the centre of which is restricted to motion in a plane (since a and b are the only variable parameters), cannot have a cone for its envelope. But it may move so as to preserve contact with the cone; and it is evident that the locus of contact will be a pair of circles, the one above and the other below the plane of xy, and both of them parallel to that plane. It is evident also that the envelope of the sphere, while executing these motions, will consist of two hollow rings, the one girding the cone, the other touching its inner surface.

We may now compare together the different cases which present themselves in the theory which has passed under discussion.

We may state it to be the general object of this investigation, so far as it may be expressed in the language of geometry, to cause a given moveable locus to execute the greatest possible amount of motion in contact with another fixed locus, by the variation of certain constant elements in its equation called parameters.

This is the most general object of the inquiry as respects geometry. It includes the determination of the condition under which a given envelope will be generated, for this will be effected whenever the given moveable locus is permitted to pass in contact over every part of the fixed locus which thus becomes its envelope. It includes also those cases in which the given moveable locus can only have contact with the fixed lo[illegible]ng some unknown line or at some unknown point [illegible] to be determined.

And it appears that in all these cases the required condition among the parameters will be found by seeking the equation of the envelope of the moveable surface, regarding in the equation of the latter the parameters as coordinates and the coordinates as parameters.

When the number of the original parameters is equal to the number of the coordinates, the conditions above determined suffice to cause the moveable locus to generate the fixed locus as its envelope. When the former number exceeds the latter number by m, we may introduce at liberty m arbitrary equations among the parameters. When it falls short of it by m, we obtain in the process m additional, but not necessarily new, equations among the coordinates which define the trace of the moveable surface upon the fixed one.

Such is the theory of the inverse problem of envelopes. From the direct one it differs in some important particulars. In the direct problem the relation of the number of parameters to that of the coordinates is comparatively unimportant; the number of relations to which the parameters are subject is equally unimportant. One method (Lagrange's one of indeterminate multipliers) serves for all cases. In the inverse method we have necessarily but two things given, the equation of the moveable and the equation of the fixed surface. The relation of the number of the parameters to that of the coordinates is here however all-important. It presents us three cases for consideration, the characters of which are quite distinct. The arbitrary functions which in one of those cases enter into the solution, seem to indicate an approximation between the results of the differential and those of the integral calculus.

Application of the Method to some Problems connected with the Wave Surface.

It is known that the wave surface is the envelope of the plane whose equation is

$$lx + my + nz = v \quad \ldots\ldots\ldots\ldots (54),$$

the parameters l, m, n, v being made to vary in subjection to the conditions

$$l^2 + m^2 + n^2 = 1 \quad \ldots\ldots\ldots\ldots (55),$$

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \quad \ldots\ldots\ldots (56).$$

Its equation is

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=1 \ldots\ldots\ldots\ldots (57),$$

in which r^2 stands for $x^2+y^2+z^2$.

There are in the above case three coordinates x, y, z, and four parameters l, m, n, v. It appears, then, from the theory developed in the previous pages, that if we seek the conditions under which the wave surface (57) shall be generated by the ultimate intersections of the plane (54), we shall obtain one *essential* condition among the parameters to which we shall be permitted to add one other condition chosen *arbitrarily*. Let us consider these two equations as a system involving one arbitrary element. Then, as this system represents the most general solution of the problem proposed, it is obvious that the two equations (55) and (56), expressing the conditions by which the wave surface has *actually* been determined, must be a particular case of the above system obtained by giving a particular form to the arbitrary equation which it involves. The solution of the inverse problem will therefore be something more than a reproduction of the original equations of condition. It will shew us what element in those conditions is the essential one, and what element is arbitrary and might be rejected or replaced.

We are then to seek the conditions under which the plane

coefficients of dx, dy, dz, dr, we have

$$\lambda l + \mu x = \frac{x}{r^2 - a^2} \dots\dots\dots\dots\dots\dots (61),$$

$$\lambda m + \mu y = \frac{y}{r^2 - b^2} \dots\dots\dots\dots\dots\dots (62),$$

$$\lambda n + \mu z = \frac{z}{r^2 - c^2} \dots\dots\dots\dots\dots\dots (63),$$

$$\mu = \frac{x^2}{(r^2 - a^2)^2} + \frac{y^2}{(r^2 - b^2)^2} + \frac{z^2}{(r^2 - c^2)^2} \dots\dots\dots (64).$$

$$(61) \times \frac{x}{r^2 - a^2} + (62) \times \frac{y}{r^2 - b^2} + (63) \times \frac{z}{r^2 - c^2}$$

gives, by (59) and (64),

$$\lambda\left(\frac{lx}{r^2 - a^2} + \frac{my}{r^2 - b^2} + \frac{nz}{r^2 - c^2}\right) + \mu = \mu,$$

therefore
$$\frac{lx}{r^2 - a^2} + \frac{my}{r^2 - b^2} + \frac{nz}{r^2 - c^2} = 0 \dots\dots\dots\dots (65).$$

Again, $(61) \times l + (62) \times m + (63) \times n$ gives, on putting ρ^2 for $l^2 + m^2 + n^2$,

$$\lambda\rho^2 + \mu v = 0 \dots\dots\dots\dots\dots\dots\dots (66).$$

Lastly, $(61) \times x + (62) \times y + (63) \times z$ gives

$$\lambda v + \mu r^2 = 1 \dots\dots\dots\dots\dots\dots\dots (67).$$

From the two last equations, we find

$$\lambda = \frac{-v}{r^2\rho^2 - v^2}, \quad \mu = \frac{\rho^2}{r^2\rho^2 - v^2}.$$

Substitute these values in (61), and we get

$$\frac{\rho^2 x - lv}{r^2\rho^2 - v^2} = \frac{x}{r^2 - a^2};$$

whence, if we determine the value of x, we shall find

$$\frac{x}{r^2 - a^2} = \frac{lv}{v^2 - a^2\rho^2} \dots\dots\dots\dots\dots\dots (68);$$

and, in like manner, from (62) and (63) we shall have

$$\frac{y}{r^2 - b^2} = \frac{mv}{v^2 - b^2\rho^2},$$

$$\frac{z}{r^2 - c^2} = \frac{nv}{v^2 - c^2\rho^2}.$$

Multiply (68) and the two following equations by l, m, n, respectively, and attending to (65), we shall have, on dividing the result by v,

$$\frac{l^2}{v^2 - a^2\rho^2} + \frac{m^2}{v^2 - b^2\rho^2} + \frac{n^2}{v^2 - c^2\rho^2} = 0 \text{ ....(69),}$$

wherein ρ^2 stands for $l^2 + m^2 + n^2$.

Such is the necessary equation of condition among the parameters l, m, n, v, in order that the plane 91 may have the wave surface 94 for its envelope. With this equation of condition we can, as has been observed, associate any other arbitrary equation. If we choose for this purpose the equation $l^2 + m^2 + n^2 = 1$, the equation (69) is then reducible to

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0,$$

and the original conditions (55) and (56) are produced.

It may be interesting, however, to verify the equation (69) and to prove its single sufficiency, by directly investigating the envelope of the plane given, subject to the condition which that equation expresses.

Our equations are

$$lx + my + nz = v \text{ ....(70),}$$

$$\frac{l^2}{v^2 - a^2\rho^2} + \frac{m^2}{v^2 - b^2\rho^2} + \frac{n^2}{v^2 - c^2\rho^2} = 0 \text{ ....(71),}$$

in which $\rho^2 = l^2 + m^2 + n^2$. These equations we may differentiate with reference to l, m, n, v, and apply, as before, the method of indeterminate multipliers. The result, as I have found by actually carrying out this process, is the equation of the wave surface (57). But that result is obtained with a little more convenience as follows.

Let $\frac{l}{\rho} = l'$, $\frac{m}{\rho} = m'$, $\frac{n}{\rho} = n'$, $\frac{v}{\rho} = v'$. Then the equations (70), (71) may be replaced by the following system:

$$l'x + m'y + n'z = v' \text{ ....(72),}$$

$$\frac{l'^2}{v'^2 - a^2} + \frac{m'^2}{v'^2 - b^2} + \frac{n'^2}{v'^2 - c^2} = 0 \text{ ....(73),}$$

$$l'^2 + m'^2 + n'^2 = 1 \text{ .... (74).}$$

Now these are the very equations from which the equation of the wave surface is known to be deduced, excepting that l', m', n', v', stand for l, m, n, v, a difference which does not affect the result.

As to the essential condition (69), we are permitted to annex one other condition arbitrary in its character; we may assume for this condition that one of the parameters l, m, n, v is constant. It would hence appear that in the equation (69) it is not necessary to suppose more than three of the parameters to vary, and whichsoever three of the set we choose, the wave surface will be the resulting envelope of the plane $lx + my + nz = v$. This conclusion may be established by other considerations. It might in fact be deduced as a consequence of the homogeneity of the two equations (70) and (71) with respect to the four parameters l, m, n, v. But this circumstance is accidental. The variation of any three of the parameters in (71), or of the whole four, subject or not to an additional relation, would secure the object proposed quite independently of any condition of homogeneity in the equation given.

In (70) and (71) for $\frac{l}{v}, \frac{m}{v}, \frac{n}{v}$, put λ, μ, ν, respectively, and for $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ write α, β, γ.

Then it will appear that the envelope of the plane

$$\lambda x + \mu y + \nu z = 1 \;\ldots\ldots\ldots\ldots\; (75),$$

subject to the condition

$$\frac{\alpha^2\lambda^2}{\rho^2 - \alpha^2} + \frac{\beta^2\mu^2}{\rho^2 - \beta^2} + \frac{\gamma^2\nu^2}{\rho^2 - \gamma^2} = 0 \;\ldots\ldots\ldots\; (76),$$

in which $\rho^2 = \lambda^2 + \mu^2 + \nu^2$ will be the wave surface, whose equation is

$$\frac{x^2}{r^2 - \frac{1}{\alpha^2}} + \frac{y^2}{r^2 - \frac{1}{\beta^2}} + \frac{z^2}{r^2 - \frac{1}{\gamma^2}} = 1 \;\ldots\ldots\ldots\; (77).$$

Now (71) may be reduced to the form

$$\frac{\lambda^2}{\rho^2 - \alpha^2} + \frac{\mu^2}{\rho^2 - \beta^2} + \frac{\nu^2}{\rho^2 - \gamma^2} = 1 \;\ldots\ldots\ldots\; (78),$$

which would be the equation of the wave surface if λ, μ, ν were regarded as parameters. On comparing (77) and (78) it becomes apparent that the polar reciprocal of a wave surface is another wave surface whose axes are the reciprocals of the axes of the given one. The equation (75) exhibits the relation connecting the coordinates λ, μ, ν of the one surface with the coordinates x, y, z of the other,

and indicates that the surface of transformation is a sphere whose radius is unity, and whose centre is placed at the common origin of coordinates. The same conclusion has, I believe, been deduced by MacCullagh from the known reciprocal properties of the ellipsoid.

If two only of the parameters l, m, n, v were permitted to vary, it would no longer be possible that the plane (70) should generate the wave surface as the locus of its successive intersections, but we might still ascertain the condition under which the plane might move in contact with the surface and the equations of that line of double curvature, which would be the locus of its successive points of contact.

As another example, let us investigate the conditions under which the wave surface can be generated by the mutual intersections of ellipsoids, concentric with itself, and having axes coincident with the axes of coordinates.

We will suppose the equation of the ellipsoids to be

$$\frac{x^2}{a^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1 \quad \ldots\ldots\ldots\ldots \quad (79),$$

a, β, γ, the semiaxes, being the variable parameters.

We must in this equation, and in the equation of the wave surface, now regard x, y, z as the variable parameters. But as x, y, z only enter these equations under the forms

$$\lambda l + \mu = \frac{1}{r-a}, \quad \lambda m + \mu = \frac{1}{r-b}, \quad \lambda n + \mu = \frac{1}{r-c} \ldots (83),$$

$$\mu = \frac{x}{(r-a)^2} + \frac{y}{(r-b)^2} + \frac{z}{(r-c)^2} \ldots\ldots\ldots (84).$$

Multiply the three equations of (83) by $\frac{x}{r-a}$, $\frac{y}{r-b}$, and $\frac{z}{r-c}$ respectively, and reducing by (82) and (84), we get

$$\lambda\left(\frac{lx}{r-a} + \frac{my}{r-b} + \frac{nz}{r-c}\right) + \mu = \mu;$$

therefore
$$\frac{lx}{r-a} + \frac{my}{r-b} + \frac{nz}{r-c} = 0 \ldots\ldots\ldots\ldots\ldots (85).$$

From equation (82) subtract (81) divided by r, the result is

$$\frac{ax}{r-a} + \frac{by}{r-b} + \frac{cz}{r-c} = 0 \ldots\ldots\ldots\ldots\ldots (86).$$

From the two last equations and (82), regarded as linear with respect to $\frac{x}{r-a}$, $\frac{y}{r-b}$, $\frac{z}{r-c}$, we get

$$\frac{x}{r-a} = \frac{p}{p+q+s}, \quad \frac{y}{r-b} = \frac{q}{p+q+s}, \quad \frac{z}{r-b} = \frac{s}{p+q+s} \ldots (87),$$

in which, for brevity,

$$p = cm - bn, \quad q = an - cl, \quad s = bl - am.$$

From (87) we have

$$lx + my + nz = \frac{lp(r-a) + mq(r-b) + ns(r-c)}{p+q+s},$$

$$= \frac{r(lp + mq + ns) - lap - mbq - ncs}{p+q+s}.$$

Substituting for p, q, and s their values, and observing that $lx + my + nz = 1$, we get

$$1 = \frac{-la(cm-bn) - mb(an-cl) - nc(bl-am)}{cm - bn + an - cl + bl - am},$$

or, as it may be written,

$$(la+1)(cm-bn) + (mb+1)(an-cl) + (nc+1)(bl-am) = 0,$$

wherein, if for l, m, n we substitute their values and replace

C 2

a by a^2, b by b^2, c by c^2, we get

$$\left(\frac{a^2}{\alpha^2}+1\right)\left(\frac{c^2}{\beta^2}-\frac{b^2}{\gamma^2}\right)+\left(\frac{b^2}{\beta^2}+1\right)\left(\frac{a^2}{\gamma^2}-\frac{c^2}{\alpha^2}\right)+\left(\frac{c^2}{\gamma^2}+1\right)\left(\frac{b^2}{\alpha^2}-\frac{a^2}{\beta^2}\right)=0$$

as the expression of the required condition among the semi-axes α, β, γ.

I found it somewhat difficult to verify this result, but at length succeeded in doing so by a process of which I shall indicate the principal steps.

If in the equation of the ellipsoid and the equation expressing the condition among the axes we make the same substitutions as before, we get

$$lx + my + nz = 1 \text{ . . . . } (89),$$

$$(la+1)(cm-bn)+(mb+1)(an-cl)+(nc+1)(bl-am)=0 \ldots (90),$$

in which l, m, n are to be regarded as variable parameters. We hence deduce

$$\lambda x + a(cm-bn) + b(nc+1) - c(mb+1) = 0 \ldots (91),$$

and two other similar equations.

Again, from (90) and the identical equation

$$a(cm-bn) + b(an-cl) + c(bl-am) = 0,$$

we get

$$\frac{cm-bn}{c(mb+1)-b(nc+1)} = \frac{an-cl}{a(nc+1)-c(la+1)} = \frac{bl-am}{b(la+1)-a(mb+1)}.$$

[redacted]

whence, by (90),

$$\lambda = cm - bn + an - cl + bl - am \;\ldots\ldots\ldots\; (95),$$

and substituting in (94), we get

$$r = k.$$

Substitute this value of k in (92), and we get

$$\frac{\lambda x}{cm - bn} = r - a,$$

therefore $$\frac{x}{r - a} = \frac{cm - bn}{\lambda}.$$

Similarly $$\frac{y}{r - b} = \frac{an - cl}{\lambda},$$

$$\frac{z}{r - c} = \frac{bl - am}{\lambda};$$

and adding these results together, and substituting for λ its value given in (95), we have

$$\frac{x}{r - a} + \frac{y}{r - b} + \frac{z}{r - c} = 1.$$

Replacing then x by x^2, y by y^2, &c., we have

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1,$$

the equation desired.

Application to Inverse Problems of Maxima and Minima.

That problems of maxima and minima and those of envelopes are intimately connected, must have occurred to every one who has particularly attended to them. But the precise nature of their connexion does not appear to have been pointed out.

Let us confine ourselves to the supposition that there are but two variables x and y, and let us consider by what process we should seek the maximum value of a proposed function $\phi(x, y)$ or u, the variables being subject to the condition $\psi(x, y) = 0$.

That process consists in eliminating the variables x, y, and their differentials dx, dy, from the equations

$$u = \phi(x, y) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (96),$$

$$\psi(x, y) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (97),$$

$$\frac{d\phi(x, y)}{dx}\,dx + \frac{d\phi(x, y)}{dy}\,dy = 0,$$

$$\frac{d\psi(x, y)}{dx}\,dx + \frac{d\psi(x, y)}{dy}\,dy = 0.$$

The result will be a relation connecting u with the constants which enter into $\phi(x, y)$ and $\psi(x, y)$. We will however consider it solely with reference to u and the constants which enter into $\phi(x, y)$.

Now the process above explained is precisely that to which we should be led if, regarding x and y as parameters, and the abovementioned constants as coordinates, we sought the envelope of (96) subject to the condition (97). The converse problem of maxima and minima therefore, or the discovery of the condition (97) when the form of the function $\phi(x, y)$ to be made a maximum, and also its maximum value or the relation which connects u with the constants in $\phi(x, y)$ are known, will be solved in the same way as the inverse problem of envelopes.

The reasoning here exemplified is applicable, *mutatis mutandis*, to cases in which the number of variables exceeds two. We are thus conducted to the following rule.

To ascertain the condition which must connect the variables $x_1, x_2 \ldots x_n$, in order that a proposed function $\phi(x_1, x_2 \ldots x_n,$ $a_1, a_2 \ldots a_n)$ may have its maximum or minimum value deter-

[redacted]

One end of a straight beam rests against a vertical wall, the other rests upon an unknown curve. Required the form of that curve in order that, in consistence with the laws of mechanics, a given relation may exist between the length of the beam and the height of its centre of gravity.

Let the axis of y coincide with the vertical line in which the top of the beam is found, and let the altitude of the top be y'; also let x and y be the coordinates of its other extremity which moves on the unknown curve. Also let l be the length of the beam, and $\frac{1}{2}u$ the height of its centre of gravity. Then we have

$$l^2 = x^2 + (y' - y)^2,$$

and, by the principles of statics,

$$u = y + y' = \text{a minimum}.$$

If from these equations we eliminate y', we get

$$x^2 + (2y - u)^2 = l^2.$$

This is an equation connecting the quantity u, which is to be made a minimum with the variable coordinates xy and the length l. If we knew the relation between x and y, we should in the ordinary way be able to determine the relation between the minimum value of u and l. This would be the direct problem. But in the present instance the relation between u and l is supposed to be given, and that between x and y to be sought.

Let the relation given between u and l be represented by $\chi(u, l) = 0$, we have then the two equations

$$x^2 + (2y - u)^2 = l^2,$$

$$\chi(u, l) = 0.$$

And if from these equations and their differentials relative to u and l, we eliminate the latter quantities, we shall obtain the required relation between x and y.

Thus if the relation between u and l be

$$u = al,$$

we get the following equations,

$$x^2 + (2y - u)^2 - l^2 = 0,$$

$$u - al = 0,$$

$$(2y - u)\,du + l\,dl = 0,$$

$$du - a\,dl = 0,$$

the two last of which give

$$l + a(2y - u) = 0.$$

From this equation and the two first of the previous system, we get, by elimination of u and l,

$$y = \pm \frac{\sqrt{(a^2 - 1)}}{2} x.$$

This indicates that the beam rests upon a straight line.

Suppose in the next place that the relation between u and l is of the form

$$8pu - p^2 - 16l^2 = 0,$$

a form to which the solution of a certain direct problem leads. We get the system

$$x^2 + (2y - u)^2 - l^2 = 0,$$
$$8pu - p^2 - 16l^2 = 0,$$
$$p - 4(u - 2y) = 0,$$

the last equation obtained by eliminating the differentials. From these we readily get

$$u = \frac{p + 8y}{4}, \quad l^2 = \frac{8pu - p^2}{16} = \frac{p^2 + 16py}{16};$$

and, substituting in the first equation of the system, we find

$$x^2 + \frac{p^2}{16} = \frac{p^2 + 16py}{16},$$

or

$$x^2 = py,$$

[redacted]

ON CERTAIN THEOREMS IN THE CALCULUS OF OPERATIONS.

By W. Spottiswoode, M.A., Balliol College, Oxford.

In the *Philosophical Transactions* for the year 1844, Professor Boole enunciated certain theorems relating to the operative symbol $D = x\dfrac{d}{dx}$, and applied them to the integration of a large class of differential equations, and to the evaluation of certain definite integrals: and in the *Cambridge and Dublin Mathematical Journal*, Nov. 1851, Mr. Carmichael has extended this method by demonstrating the analogous theorems relating to the operative symbol

$$\nabla = x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} + \ldots$$

I propose to make some still further extensions: to the cases (1) in which the order of the variables by which the symbols of differentiation are multiplied is not the same as that of the variables with respect to which the differentiations are to be performed; (2) in which the variables by which the symbols of differentiation are multiplied are any linear functions of the given variables.

Section I.

Suppose that the series

$$i_1, i_2, \ldots$$
$$j_1, j_2, \ldots$$
$$\ldots\ldots$$

represents any permutations of the series

$$1, 2, \ldots$$

and that $\nabla_i, \nabla_j, \ldots$ are operative symbols analogous to ∇, thus:

$$\nabla_i = x_{i_1} \frac{d}{dx_1} + x_{i_2} \frac{d}{dx_2} + \ldots$$

$$\nabla_j = x_{j_1} \frac{d}{dx_1} + x_{j_2} \frac{d}{dx_2} + \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Again, adopting a similar notation, the permutation

$$j_1, j_2, \ldots$$

of the series $\qquad i_1, i_2, \ldots$

will be represented by the series

$$j_{i_1}, j_{i_2}, \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and the corresponding operative symbol by $\nabla_{j,i}$; so that

$$\nabla_{j,i} = x_{ji_1}\frac{d}{dx_1} + x_{ji_2}\frac{d}{dx_2} + \dots\dots\dots\dots \quad (4),$$

and generally

$$\nabla_{l,..j,i} = x_{l..ji_1}\frac{d}{dx_1} + x_{l..ji_2}\frac{d}{dx_2} + \dots\dots\dots \quad (5).$$

And further, writing

$$\left.\begin{aligned} \nabla_{..j+i} = \dots\, x_{j_1}x_{i_1}\frac{d^n}{dx_1^{\,n}} + \dots\, x_{j_2}x_{i_2}\frac{d^n}{dx_2^{\,n}} + \dots \\ + (\dots\, x_{j_1}x_{i_2} + \dots\, x_{j_2}x_{i_1} + \dots)\frac{d^n}{dx_1\,dx_2\dots} + \dots \end{aligned}\right\} \dots\dots(6),$$

it will be found that $\nabla_{j+i} = \nabla_j\nabla_i - \nabla_{j,i}$ ………………… (7).

Similarly,

$$\nabla_{k+j+i} = \nabla_k\nabla_j\nabla_i - \nabla_{k,j}\nabla_i - \nabla_j\nabla_{k,i} - \nabla_k\nabla_{j,i} + \nabla_{k,j,i} + \nabla_{j,k,i} \dots (8),$$

and so on for any number of operations. But on account of the non-interchangeability of the order of the permutations, it does not seem generally possible to give a concise expression for the operation $\nabla_{l+..j+i}$.

If, however, the permutations are of such a character as to admit of an interchange of order, the operation in question may be expressed in a very brief and elegant way. Suppose,
[redacted]

[redacted]

It is not difficult to form the permutational equations corresponding to (9) and (11) for the four systems $P(i)$, $P(j)$, $P(k)$, $P(l)$; and when they are satisfied, there will result

$$\left.\begin{array}{c} \nabla_{l+k+j+i} = \nabla_l\nabla_k\nabla_j\nabla_i \\ -\nabla_j\nabla_k\nabla_{i,l} - \nabla_k\nabla_i\nabla_{j,l} - \nabla_i\nabla_j\nabla_{k,l} \\ -\nabla_i\nabla_l\nabla_{j,k} - \Delta_j\nabla_l\nabla_{k,i} - \nabla_k\nabla_l\nabla_{i,j} \\ +2\nabla_i\nabla_{j,k,l} + 2\nabla_j\nabla_{k,i,l} + 2\nabla_k\nabla_{i,j,l} + 2\nabla_l\nabla_{i,j,k} \\ +\nabla_{i,l}\nabla_{j,k} + \nabla_{j,l}\nabla_{k,i} + \nabla_{k,l}\nabla_{i,j} \\ -6\nabla_{l,k,j,i} \end{array}\right\}\ldots(13).$$

The analogy between the structure of these functions and that of determinants suggests the following symbolical notation:

$$\left.\begin{array}{c} (i,i) = \nabla_i,\ (j,j) = \nabla_j,\ (k,k) = \nabla_k,\ \ldots \\ (j,k)(k,j) = \nabla_{j,k},\ (k,i)(i,k) = \nabla_{k,i},\ (i,j)(j,i) = \nabla_{i,j},\ldots \\ (j,k)(l,j)(l,k) = \nabla_{j,k,l},\ (k,i)(l,k)(l,i) = \nabla_{k,i,l}, \\ (i,j)(l,i)(l,j) = \nabla_{i,j,l},\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}\ldots(14),$$

in which it will be observed, that whenever a letter occurs twice in the left-hand members of these equations, it is to be inserted once as a suffix in the right-hand members; a remark which will sufficiently explain the general formation of the new symbols. The equations (7), (12),... may then be written as follows:

$$\nabla_{j+i} = \begin{vmatrix} (i,i) & (i,j) \\ (j,i) & (j,j) \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

$$\nabla_{k+j+i} = \begin{vmatrix} (i,i) & (i,j) & (i,k) \\ (j,i) & (j,j) & (j,k) \\ (k,i) & (k,j) & (k,k) \end{vmatrix} \ldots\ldots\ldots\ldots(16),$$

and generally

$$\nabla_{l+\cdots j+i} = \begin{vmatrix} (i,i) & (i,j) & \ldots & (i,l) \\ (j,i) & (j,j) & \ldots & (j,l) \\ \ldots & \ldots & \ldots & \ldots \\ (l,i) & (l,j) & \ldots & (l,l) \end{vmatrix} \ldots\ldots\ldots\ldots(17).$$

Or, adopting Mr. Sylvester's notation,

$$\nabla_{j+i} = \begin{Bmatrix} i, j \\ i, j \end{Bmatrix},\quad \nabla_{k+j+i} = \begin{Bmatrix} i, j, k \\ i, j, k \end{Bmatrix},\ldots$$

$$\nabla_{l+\cdots j+i} = \begin{Bmatrix} i, j, \ldots l \\ i, j, \ldots l \end{Bmatrix}.$$

These formulæ enable us to find a symbolical solution of a certain class of partial differential equations; in fact, if

$$x_1^{\alpha} x_2^{\beta} \dots \frac{d^n}{dx_1^n} + x_1^{\alpha_1} x_2^{\beta_1} \dots \frac{d^n}{dx_2^n} + \dots\dots$$

$$+ \Sigma x_1^{\lambda} x_2^{\mu} \dots \frac{d^n}{dx_1^{\rho} dx_2^{\sigma} \dots} + \dots = \nabla_{l, \dots j, i} \dots\dots (18),$$

in which the number of quantities $i, j, \dots l$, is n, and

$$\alpha + \beta + \dots = \alpha_1 + \beta_1 + \dots = \lambda + \mu + \dots = \rho + \sigma + \dots = n \dots (19).$$

Then, supposing the order of the permutations to be interchangeable, *i.e.* supposing the case to be a symmetrical one, the operative function on the left-hand side of (18)

$$= \begin{Bmatrix} i, j, \dots l \\ i, j, \dots l \end{Bmatrix} \dots\dots\dots\dots\dots\dots\dots\dots (20):$$

and if there be formed other similar expressions

$$\begin{Bmatrix} i', j', \dots l' \\ i', j', \dots l' \end{Bmatrix} \begin{Bmatrix} i'', j'', \dots l'' \\ i'', j'', \dots l'' \end{Bmatrix}, \dots$$

the differential equation

$$(Ax_1^{\alpha} x_2^{\beta} \dots + Bx_1^{\alpha'} x_2^{\beta'} \dots + \dots) \frac{d^n u}{dx_1^n}$$

Section II.

Let

$$\left.\begin{aligned} \Xi_1 &= \xi_{11}\frac{d}{dx_1} + \xi_{12}\frac{d}{dx_2} + \dots \\ \Xi_2 &= \xi_{21}\frac{d}{dx_1} + \xi_{22}\frac{d}{dx_2} + \dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\}\dots\dots\dots\dots(25),$$

where

$$\left.\begin{aligned} \xi_{11} &= \alpha_{11}x_1 + \beta_{11}x_2 + \dots, & \xi_{12} &= \alpha_{12}x_1 + \beta_{12}x_2 + \dots \\ \xi_{21} &= \alpha_{21}x_1 + \beta_{21}x_2 + \dots, & \xi_{22} &= \alpha_{22}x_1 + \beta_{22}x_2 + \dots \\ &\dots\dots\dots\dots\dots\dots & &\dots\dots\dots\dots\dots\dots \end{aligned}\right\}\dots(26).$$

Then if

$$\Xi_{2+1} = \xi_{21}\xi_{11}\frac{d^2}{dx_1^2} + \xi_{22}\xi_{12}\frac{d^2}{dx_2^2} + \dots + (\xi_{22}\xi_{11} + \xi_{21}\xi_{12})\frac{d^2}{dx_1dx_2} + \dots(27),$$

$$\left.\begin{aligned} \Xi_{2+1} &= \Xi_2\Xi_1 - (\alpha_{11}\xi_{21}+\beta_{11}\xi_{22}+\dots)\frac{d}{dx_1} - (\alpha_{12}\xi_{21}+\beta_{12}\xi_{22}+\dots)\frac{d}{dx_2} + \dots \\ &= \Xi_2\Xi_1 - \{(\alpha_{11}\alpha_{21}+\beta_{11}\alpha_{22}+\dots)x_1 + (\alpha_{11}\beta_{21}+\beta_{11}\beta_{22}+\dots)x_2 + \dots\}\frac{d}{dx_1} \\ &\quad - \{(\alpha_{12}\alpha_{21}+\beta_{12}\alpha_{22}+\dots)x_1 + (\alpha_{12}\beta_{21}+\beta_{12}\beta_{22}+\dots)x_2 + \dots\}\frac{d}{dx_2} \\ &\quad - \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\}$$

$$\dots\dots(28),$$

$$\left.\begin{aligned} \alpha_{11}\alpha_{21}+\beta_{11}\alpha_{22}+\dots &= \genfrac{}{}{0pt}{}{21}{11}, & \alpha_{11}\beta_{21}+\beta_{11}\beta_{22}+\dots &= \genfrac{}{}{0pt}{}{21}{12},\dots \\ \alpha_{12}\alpha_{21}+\beta_{12}\alpha_{22}+\dots &= \genfrac{}{}{0pt}{}{21}{21}, & \alpha_{12}\beta_{21}+\beta_{12}\beta_{22}+\dots &= \genfrac{}{}{0pt}{}{21}{22},\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\}\dots(29),$$

all which may be comprised in the following formula:

$$\left.\begin{aligned} &\left|\begin{matrix} \alpha_{21} & \alpha_{22}\dots \\ \beta_{21} & \beta_{22}\dots \\ \dots & \dots \end{matrix}\right| \left|\begin{matrix} \alpha_{11} & \beta_{11}\dots \\ \alpha_{12} & \beta_{12}\dots \\ \dots & \dots \end{matrix}\right| \\ &= \left|\begin{matrix} \genfrac{}{}{0pt}{}{21}{11} & \genfrac{}{}{0pt}{}{21}{12} & \dots \\ \genfrac{}{}{0pt}{}{21}{21} & \genfrac{}{}{0pt}{}{21}{22} & \dots \\ \dots & \dots & \end{matrix}\right| \end{aligned}\right\}\dots\dots\dots\dots(30),$$

which equation is to be understood disjunctively, *i.e.* each constituent of the product of the two determinants on the

left-hand side of the equation, developed as indicated in (29), is to be equated to the corresponding constituent of the determinant on the right-hand side. Then

$$\left.\begin{aligned}\Xi_{2+1}&=\Xi_2\Xi_1-\left\{\left(\begin{matrix}21\\11\end{matrix}\,x_1+\begin{matrix}21\\12\end{matrix}\,x_2+\ldots\right)\frac{d}{dx_1}+\left(\begin{matrix}21\\21\end{matrix}\,x_1+\begin{matrix}21\\22\end{matrix}\,x_2+\ldots\right)\frac{d}{dx_2}+\ldots\right\}\\&=\Xi_2\Xi_1-\Xi_{2,1}\end{aligned}\right\}\ldots\ldots(31),$$

suppose. In the same way we might write

$$\left.\begin{aligned}&\begin{vmatrix}\begin{Vmatrix}\alpha_{31} & \alpha_{32}\ldots\\ \beta_{31} & \beta_{32}\ldots\\ \ldots\ldots\ldots \end{Vmatrix}\begin{Vmatrix}\alpha_{21} & \alpha_{22}\ldots\\ \beta_{21} & \beta_{22}\ldots\\ \ldots\ldots\ldots \end{Vmatrix}\begin{Vmatrix}\alpha_{11} & \beta_{11}\ldots\\ \alpha_{12} & \beta_{12}\ldots\\ \ldots\ldots\ldots \end{Vmatrix}\end{vmatrix}\\ &=\begin{vmatrix}\begin{matrix}321\\11\end{matrix} & \begin{matrix}321\\12\end{matrix} & \ldots\\ \begin{matrix}321\\21\end{matrix} & \begin{matrix}321\\22\end{matrix} & \ldots\\ \ldots & \ldots & \ldots\end{vmatrix}\end{aligned}\right\}\ldots\ldots\ldots(32),$$

and there would result

$$\Xi_{3+2+1}=\Xi_3\Xi_2\Xi_1-\Xi_{3,2}\Xi_1-\Xi_2\Xi_{3,1}-\Xi_3\Xi_{2,1}+\Xi_{3,2,1}+\Xi_{2,3,1}.\ .(33).$$

And similarly, the process might be carried on to operations of higher orders. To this class of *index symbols* there corresponds *symmetrical cases* analogous to those mentioned in § 1. For supposing that, instead of (28), we consider the following equation,

$$\left.\begin{aligned}\Xi_{1+2}&=\Xi_1\Xi_2-(\alpha_{21}\xi_{11}+\beta_{21}\xi_{12}+\ldots)\frac{d}{dx_1}-(\alpha_{22}\xi_{11}+\beta_{22}\xi_{12}+\ldots)\frac{d}{dx_2}-\ldots\\&=\Xi_1\Xi_2-\{(\alpha_{21}\alpha_{11}+\beta_{21}\alpha_{12}+\ldots)x_1+(\alpha_{21}\beta_{11}+\beta_{21}\beta_{12}+\ldots)x_2+\ldots\}\frac{d}{dx_1}\\&\qquad-\{(\alpha_{22}\alpha_{11}+\beta_{22}\alpha_{12}+\ldots)x_1+(\alpha_{22}\beta_{11}+\beta_{22}\beta_{12}+\ldots)x_2+\ldots\}\frac{d}{dx_2}\\&\qquad-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&=\Xi_1\Xi_2-\left\{\left(\begin{matrix}12\\11\end{matrix}\,x_1+\begin{matrix}12\\12\end{matrix}\,x_2+\ldots\right)\frac{d}{dx_1}+\left(\begin{matrix}12\\21\end{matrix}\,x_1+\begin{matrix}12\\22\end{matrix}\,x_2+\ldots\right)\frac{d}{dx_2}+\ldots\right\}\\&=\Xi_1\Xi_2-\Xi_{1,2}\end{aligned}\right\}\ldots\ldots(34),$$

suppose. Then, if the disjunctive equation

$$\left.\begin{matrix}\begin{matrix}12\\11\end{matrix} & \begin{matrix}12\\12\end{matrix} & \ldots & = & \begin{matrix}21\\11\end{matrix} & \begin{matrix}21\\12\end{matrix} & \ldots\\ \begin{matrix}12\\21\end{matrix} & \begin{matrix}12\\22\end{matrix} & \ldots & & \begin{matrix}21\\21\end{matrix} & \begin{matrix}21\\22\end{matrix} & \ldots\\ \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots\end{matrix}\right\}\ldots\ldots\ldots\ldots\ldots(35)$$

holds good, there results

$$\Xi_{1,2} = \Xi_{2,1}.$$

And consequently, adopting a notation analogous to that given in (14), *i.e.* writing

$$(1, 1) = \Xi_1, \quad (2, 2) = \Xi_2, \quad (1, 2)(2, 1) = \Xi_{1,2} \ldots (37),$$

we have
$$\Xi_{2+1} = \begin{vmatrix} (1, 1) & (1, 2) \\ (2, 1) & (2, 2) \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Similarly, if the disjunctive equation

$$\left.\begin{array}{l} \underset{11}{pq}\ \underset{12}{pq} \ldots = \underset{11}{qp}\ \underset{12}{qp} \ldots \\ \underset{21}{pq}\ \underset{22}{pq} \ldots = \underset{21}{qp}\ \underset{22}{qp} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (39)$$

hold good for all values of p and q from 1 to 3 inclusive (p and q however always being different); there will result, from the three cases

$$p, q = 2, 3; \quad p, q = 3, 1; \quad p, q = 1, 2 \ldots\ldots (40),$$

the following relations,

$$\Xi_{2,3} = \Xi_{3,2}, \quad \Xi_{3,1} = \Xi_{1,3}, \quad \Xi_{1,2} = \Xi_{2,1} \ldots\ldots\ldots (41).$$

And further, if the system of quantities

$$\left.\begin{array}{l} \underset{11}{pqr}\ \ \underset{12}{pqr} \ldots \\ \underset{21}{pqr}\ \ \underset{22}{pqr} \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

are independent of the order of the quantities p, q, r, there will result

$$\Xi_{3,2,1} = \Xi_{2,3,1} = \Xi_{1,2,3} \ldots\ldots\ldots\ldots\ldots (43),$$

and consequently (33) becomes

$$\Xi_{3+2+1} = \Xi_3\Xi_2\Xi_1 - \Xi_1\Xi_{2,3} - \Xi_2\Xi_{3,1} - \Xi_3\Xi_{1,2} + 2\Xi_{3,2,1} \ldots (44);$$

and, extending the notation (37), this may be written

$$\Xi_{3+2+1} = \begin{vmatrix} (1, 1) & (1, 2) & (1, 3) \\ (2, 1) & (2, 2) & (2, 3) \\ (3, 1) & (3, 2) & (3, 3) \end{vmatrix} \ldots\ldots\ldots\ldots (45).$$

And generally, if the system

$$\left.\begin{matrix} \underset{11}{i_1 i_2 \ldots i_j} & \underset{12}{i_1 i_2 \ldots i_j} & \ldots \\ \underset{21}{i_1 i_2 \ldots i_j} & \underset{22}{i_1 i_2 \ldots i_j} & \ldots \\ \ldots\ldots & \ldots\ldots & \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (46),$$

for all values of j from 2 to n inclusive, and for all values of $i_1, i_2, \ldots i_j$ from 1 to m (m being the number of the variables) inclusive, remains unchanged by any permutation of the order of $i_1 i_2 \ldots i_j$; the case will be symmetrical, and

$$\Xi_{n+\ldots 2+1} = \begin{vmatrix} (1,1)\,(1,2)\ldots(1,n) \\ (2,1)\,(2,2)\ldots(2,n) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (n,1)\,(n,2)\ldots(n,n) \end{vmatrix} \ldots\ldots\ldots (47),$$

which may for convenience be written, as in §. 1, as follows,

$$\begin{Bmatrix} 1,\ 2, \ldots n \\ 1,\ 2, \ldots n \end{Bmatrix}.$$

This expression will enable us to find a symbolical solution of another class of partial differential equations, namely the following :

$$U\frac{d^n u}{\quad} + V\frac{d^n u}{\quad} + \ldots + W\frac{d^n u}{\qquad} + \ldots = \Theta \ldots (48),$$

[redacted]

Then the equation (41) may be thus transformed:

$$A\begin{Bmatrix}1, 2,\ldots n\\1, 2,\ldots n\end{Bmatrix}u + B\begin{Bmatrix}1', 2',\ldots n'\\1', 2',\ldots n'\end{Bmatrix} + \ldots = \Theta \ldots\ldots (52);$$

and, as before, its solution may be thus expressed:

$$u = \Phi(\Xi)\,\Theta + \Phi(\Xi)\,0 \ldots\ldots\ldots\ldots\ldots (53),$$

where $\Phi(\Xi) = \left[A\begin{Bmatrix}1, 2,\ldots n\\1, 2,\ldots n\end{Bmatrix} + B\begin{Bmatrix}1', 2',\ldots n'\\1', 2',\ldots n'\end{Bmatrix} + \ldots\right]^{-1} \ldots (54).$

ON A PHYSICAL PROPERTY OF THE GENERATORS OF THE WAVE SURFACE.

By William Walton.

In the May Number of the *Mathematical Journal* for 1852, I have shewn that the wave surface may be generated by the movement of a curve of double curvature, defined by the equations

$$\left.\begin{aligned}\frac{y^2}{\mu} - \frac{z^2}{\nu} &= b^2 - c^2\\ \frac{z^2}{\nu} - \frac{x^2}{\lambda} &= c^2 - a^2\\ \frac{x^2}{\lambda} - \frac{y^2}{\mu} &= a^2 - b^2\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and subjected to pass through the three directors

$$\left.\begin{aligned}x &= 0\\ y^2 + z^2 &= a^2\end{aligned}\right\},$$

$$\left.\begin{aligned}y &= 0\\ z^2 + x^2 &= b^2\end{aligned}\right\},$$

$$\left.\begin{aligned}z &= 0\\ x^2 + y^2 &= c^2\end{aligned}\right\},$$

the condition of passing through these directors being in fact, as there shewn, equivalent to the two algebraical conditions

$$1 + \lambda + \mu + \nu = 0,$$

$$\lambda a^2 + \mu b^2 + \nu c^2 = 0.$$

the coefficients and eliminating λ, we find the two conditions of similarity to be

$$\left.\begin{aligned}&\frac{A_2A_3+A_1A_3+A_1A_2-B_1^2-B_2^2-B_3^2}{(A_1+A_2+A_3)^2}=\frac{a_2a_3+a_1a_3+a_1a_2-b_1^2-b_2^2-b_3^2}{(a_1+a_2+a_3)^2}\,*\\&\frac{A_1A_2A_3+2B_1B_2B_3-A_1B_1^2-A_2B_2^2-A_3B_3^2}{(A_1+A_2+A_3)^3}\\&\qquad\qquad=\frac{a_1a_2a_3+2b_1b_2b_3-a_1b_1^2-a_2b_2^2-a_3b_3^2}{(a_1+a_2+a_3)^3}\end{aligned}\right\}\ldots\ldots(9).$$

When the surfaces (1) and (2) are referred to oblique axes, the conditions of similarity (which are of course much more complicated) may be found in the very same manner, the 'discriminating cubic' in this case being

$$(P-A_1)(P-A_2)(P-A_3)-(fP-B_1)^2(P-A_1)-(gP-B_2)^2(P-A_2)$$
$$-(hP-B_3)^2(P-A_3)+2(fP-B_1)(gP-B_2)(hP-B_3)=0\ldots(10),\dagger$$

where f, g, and h are the cosines of the angles which the axes make with each other.

The analogous condition of similarity for two conics

$$Ax^2+By^2+2Cxy=V\ldots\ldots\ldots\ldots\ldots(11),$$

and

$$ax^2+by^2+2cxy=v\ldots\ldots\ldots\ldots\ldots(12),$$

$AB-C^2 \quad ab-c^2$

where f is the cosine of the inclination of the axes; and the condition of similarity will in this case be

$$\frac{AB-C^2}{(A+B-2fC)^2}=\frac{ab-c^2}{(a+b-2fc)^2}\text{..........}(16).$$

In the preceding investigation the two surfaces (or conics) have been supposed referred to the same axes. If, however, they be referred to different axes, the conditions of similarity will evidently remain unchanged providing both systems of axes be rectangular; but if the axes be oblique the conditions, which are easily found from (10) or (15), will be somewhat different; thus (16) will become

$$\frac{(AB-C^2)(1-f^2)}{(A+B-2fC)^2}=\frac{(ab-c^2)(1-f'^2)}{(a+b-2f'c)^2},$$

or

$$\frac{(AB-C^2)\sin^2 w}{(A+B-2C\cos w)^2}=\frac{(ab-c^2)\sin^2 w'}{(a+b-2c\cos w')^2}\text{.....}(17),$$

w and w' being the inclinations of the two systems of axes.

York Town, near Bagshot,
Jan. 21, 1852.

EASY METHOD OF FINDING THE MOMENTS OF INERTIA OF AN ELLIPSOID ABOUT ITS PRINCIPAL AXES.

By the late G. W. HEARN.*

THE equation to the ellipsoid being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\text{.....................}(1),$$

make $x=ax'$, $y=by'$, and $z=cz'$; then, at the surface,

$$x'^2+y'^2+z'^2=1\text{.....................}(2).$$

Let A, B, C be the moments of inertia about the axes of x, y, z, respectively, and ρ the density; therefore

$$A=\rho\iiint(y^2+z^2)\,dx\,dy\,dz=\rho abc\iiint(b^2y'^2+c^2z'^2)\,dx'dy'dz'$$
$$=abc\{b^2.\Sigma m'y'^2+c^2.\Sigma m'z'^2\}\text{...}(3),$$

where $(x'y'z')$ denotes any point within the sphere (2), and m' the element (of the sphere) at that point.

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* Communicated by Mr. Weddle.

Now, in the case of a sphere, we evidently have

$$\Sigma m'x'^2 = \Sigma m'y'^2 = \Sigma m'z'^2 = \tfrac{1}{3}\Sigma m'(x'^2 + y'^2 + z'^2) = \tfrac{1}{3}\Sigma m'r'^2,$$

and $$m' = \rho.4\pi r'^2.dr'\,;$$

also, (2), the limits of r' are 0 and 1, therefore

$$\Sigma m'r'^2 = 4\pi\rho\int_0^1 r'^4 dr' = \tfrac{4}{5}\pi\rho\,;$$

and $$\Sigma m'x'^2 = \Sigma m'y'^2 = \Sigma m'z'^2 = \tfrac{4}{15}\pi\rho.$$

Hence, (3), $A = abc(b^2 + c^2)\tfrac{4}{15}\pi\rho = \tfrac{4}{15}\pi\rho abc(b^2 + c^2)$.

But M, the mass of the ellipsoid, $= \tfrac{4}{3}\pi\rho abc$, therefore

$$A = \tfrac{1}{5}M(b^2 + c^2)\,;$$

and B and C may of course be found in a similar manner.

THEOREMS ON COMBINATIONS.

By the Rev. Thomas P. Kirkman, M.A.

It has been observed by Mr. Cayley, somewhere in the *Philosophical Magazine*, that two systems, and only two, of triads can be made with 7 symbols *abcdefg*, so that the

[redacted]

If *abcd*, *abce*, *abcg* are three of these 21, then will *cdeg*, *bdeg*, *adeg* be three others.

With 9 symbols can be made seven different groups, each of twelve triads, every group exhibiting all the duads once. This is not new.

D. *With* 13 *symbols can be made three different groups, each of* 26 *triads, each group containing all the duads once.*

The following,

Aa_1a_2	*abc*	*aab*	*bbc*	*cca*
Aa_3a_4	111	132	133	143
Ab_1b_2	134	241	242	231
Ab_3b_4	223	144	144	134
Ac_1c_2	241	233	231	242
Ac_3c_4	343			
	312			
	424			
	432			

where the second column is $a_1b_1c_1$, $a_1b_3c_4$, &c., is such a group. If 1 and 3 be exchanged, and also *b* and *c*, a second is formed; and a third, by exchanging in the first 1 and 4, and cyclically permuting *cba*. Whether any more such groups can be made without repeating any triad of the three groups, I know no simple method of deciding.

If we write out the duads of 8 things, *hiklmnop*, thus,

hi	*hk*	*hl*	*hm*	*hn*	*ho*	*hp*
ko	*ip*	*ik*	*il*	*im*	*in*	*io*
ln	*lo*	*mo*	*kp*	*kl*	*km*	*kn*
mp	*mn*	*np*	*no*	*op*	*lp*	*lm*

we may prefix to the columns in order the seven letters *abcdefg*, thus forming the triads *ahi*, *ako*,...*bhk*, &c. We can then cyclically permute *abcdefg* in these triads five steps, and thus make five systems, each of 28 triads, in each of which systems all the seven *abcdefg* will once be combined with all the eight *hiklmnop*. If we form with the seven the 21 triads of Theorem A, and add to these three of the five systems of 28 triads, we shall have 15 symbols thrown into triads till every duad has been thrice employed, without repeating a triad. The two remaining systems of 28 form, with the two systems of 7 triads mentioned at the beginning of this paper, two distinct arrangements of 15 things in triads, each exhibiting all the duads once, and having no triad in common. The sum of the constructed triads contains every duad five times.

If we now write out the pairs of the 8 symbols *abcdefgp*, thus,

ab	*ac*	*ad*	*ae*	*af*	*ag*	*ap*
cg	*bp*	*bc*	*bd*	*be*	*bf*	*bg*
df	*dg*	*eg*	*cp*	*cd*	*ce*	*cf*
ep	*ef*	*fp*	*fg*	*gp*	*dp*	*de*

and prefix to the columns in order the two sets of 7 letters

lh, kn, mh, ko, im, ol, in,

thus forming the 56 triads *lab, hab, lcg, hcg, ...kac, nac, ... oag, lag,* &c.; it is evident that we shall twice combine every one of the 8 *abedefgp* with each of the 7 *hiklmno,* and that we shall have twice employed all the pairs possible with the eight. Also our 56 triads are all distinct from those above formed with the 15 symbols; for we have before made none containing a pair only of the seven *abcdefg,* and of those which we have just made containing *p,* none have occurred before; *lep* and *ldp,* for example, which we have just formed, are new, because *lp* was combined before only with the five letters *fgabc,* and not with *d* or *e.* If we now form two distinct sets of 7 triads with the 7 elements *hiklmno,* each set exhibiting all the duads, we shall have triads completely new, as containing each *three* of these elements: these 14, added to the twice 28 just made, are a system of triads made with the 15 symbols

From Theorem D follows easily that

G. *With* 27 *things triads can be made till every duad has been either twice or thrice employed.*

Mr. Salmon, discussing, at p. 196 of his *Higher Plane Curves*, the double tangents of curves of the fourth order, has laid bare, by a single stroke of his bright analytic wand, the pleasing property, that 28 things can be thrown into quadruplets till every duad has been five times employed. I find that this is a case of the following theorem:

H. $4(3n+1)$ *things can be arranged in quadruplets till every duad has been* $2n+1$ *times employed.*

To prove this we take $12n+3$ symbols, viz. the three bcd, the $4n$ unaccented letters $efghij\ldots$, the $4n$ accented $e'f'g'h'i'j'\ldots$, and the $4n$ subaccented $e_1f_1g_1h_1i_1j_1\ldots$, and arrange them thus:

bcd							
bef	$ce'f'$	de_1f_1	$ee'e_1$	$ff'f_1$	$ge'g_1$	$hf'h_1$	$ie'i_1$
bgh	$cg'h'$	dg_1h_1	$eg'g_1$	$fh'h_1$	$gg'i_1$	$hh'j_1$	$ig'k_1$
bij	$ci'j'$	di_1j_1	$ei'i_1$	$fj'j_1$	$gi'k_1$	$hj'l_1$	$\vdots$
bkl	$ck'l'$	dk_1l_1	$ek'k_1$	$fl'l_1$	$gk'm_1$	$hl'n_1$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Here the second vertical rows of the even columns after the second are always the same $2n$ accented letters, and their third vertical rows are the $2n$ cyclical permutations of the same series subaccented. The second vertical rows of the odd columns after the third are all one accented series, while their third rows are that series subaccented and cyclically permuted, the series being the remaining half of the $4n$ letters.

The number of the columns is $4n+3$, equal to that of the unaccented letters $abcde\ldots$, any one of which heads every triad in one column.

The number of triads in every column, excluding the triplet bcd, is $2n$.

It is evident that each of the $12n+3$ letters occurs $2n+1$ times, and that no duad is twice employed. If then we prefix a to every triad, we shall have quadruplets $abcd$, $abef$, $ace'f'$, &c., in which a is combined $2n+1$ times with every other symbol. Let us suppose that a is thus prefixed throughout.

We next execute the law, that if $\alpha\beta\delta\theta$ and $\alpha\beta\mu\phi$ be two quadruplets, $\delta\theta\mu\phi$ shall be a third. It will thus come to pass that every duad will be employed $2n+1$ times, and no more. For every unaccented duad will occur $2n+1$ times, as cd or ce. The former occurs in $abcd$, and the $2n$ quadruplets $cdef$, $cdgh$, $cdij$, &c., made by our law from the first column. The second and fourth columns give us $ace'f'$ and $aee'e_1$, whence comes $cef'e_1$; as also $ceh'g_1$ from $acg'h'$ and $aeg'g_1$; and in the same way $2n$ quadruplets containing ce can be added to $cdef$.

Also, every duad containing a and an accented or sub-accented letter, occurs $2n+1$ times, as ae', which is found in $ace'f'$, $aee'e_1$, $age'g_1$, &c., from the $2n+1$ even columns: ae_1 occurs in ade_1f_1, $aee'e_1$, and $2n-1$ times more from the 6th, 8th, 10th, &c. columns: ah_1 occurs in adg_1h_1, $afh'h_1$, &c. made from the third and $2n$ more odd columns. The duad ee_1 occurs once in $aee'e_1$; and if we write under this quadruplet the other $2n$, which contain the duad ae', our law will give us $2n$ more containing ee_1.

The duad $e'f'$ occurs once in $ace'f'$, and $2n$ times more in the quadruplets $e'f'g'h'$, $e'f'i'j'$, &c., made from the second column.

The duad $e'f_1$ does not appear in the quadruplets which contain the letter a; but $aee'e_1$ and ade_1f_1, appear, wherefore $ede'f_1$ is a quadruplet, under which, if we write the remain-

In fact, if three r^{th} places in the triplet are all different letters, the r^{th} place of the fourth symbol is the remaining fourth letter; if they are all one letter, the fourth r^{th} place is that one; if they shew two letters only, the fourth shews the odd one.

Let any of these $4^m.(4^m-1)(4^m-2):(1.2.3.4)$ quadruplets be chosen at random. It will either exhibit, or not, some four e^{th} places (for one or more values of e) comprising all the letters *abcd*. Such values of e in the above quadruplet are the first places and the sixth. Let the four places answering to the least such value of e be suffered to stand; in this case $e = 1$, the units' places, *dbac*. Permute now cyclically *abcd* in the four $(e+1)^{th}$ places: this gives, including the one before us, four 4-plets, having no symbol in common, and all alike in their e^{th} places. In these four permute *abcd* cyclically in the $(e+2)^{th}$ places, which produces 4^2 quadruplets, having no common symbol: and if the same cyclical permutations be effected in all these in their four $(e+3)^{th}$ places, there will arise 4^3 such 4-plets. Continuing these permutations of *abcd* in all the m places except the e^{th}, we obtain 4^{m-1} 4-plets, in which the 4^m symbols are exhausted. It is easily seen that every quadruplet in this group of 4^{m-1} will, if thus treated, produce this same group, and nothing more. Thus it is proved that every 4-plet, which exhibits four unlike e^{th} places, determines a group in which all the symbols are exhausted, and can only determine its own group.

Take next a quadruplet in which no four e^{th} places exhibit the four letters. There must be in it four e^{th} places exhibiting a pair, and also four e_1^{th} places exhibiting a pair; for $Aa.Ab.Ab.Aa$ cannot be one of our 4-plets, because the symbol Aa is repeated in it. Let such a quadruplet, for example, be

$$(\ldots abbb)\,(\ldots acdb)\,(\ldots bcdb)\,(\ldots bbbb).$$

We take the lowest values of e and e_1, which in this instance are $e = 2$, $e_1 = 3$, counting from the right. We conceive *abcd* to have rank or magnitude rising in that order: then we exchange the pair in the e^{th} places for the excluded pair, the less for the less and the greater for the greater. This gives two 4-plets, including that before us. Next, in the e_1^{th} places we exchange the pair exhibited for the excluded pair, the less for the less and greater for greater. This makes our two into four 4-plets, having no symbol in common, any one of which, treated in the manner laid down,

will produce all the four. If we now continually cyclically permute *abcd* in all the places except the e^{th} and the e_1^{th}, we shall make our four quadruplets into 4^{m-1}, all different in their elements, and therefore comprising the 4^m symbols.

We have thus proved that every triplet possible with the 4^m elements or m-plets, determines one quadruplet which determines one group of 4^{m-1}. That is, any three young ladies who choose to walk together on any day, determine the arrangement of all the 4^m for that day.

To prove the truth of the theorem when n is $2m+1$, we add to the former 4^m m-plets made with *abcd*, the 4^m m-plets made with abcd. We then join to our rule above given, in which *a* and a, *b* and b, &c., are like letters, the following: *no quadruplet shall have only three italic or roman symbols.* This determines, with the preceding rule, the fourth element to be added to any triplet. Thus, if out of 512 young ladies the three (abbd), (*abbb*), (*adaa*), choose to walk together to-day, their companion must be (adac), and this quadruplet determines the arrangement, for the day, of the 512.

For such a 4-plet must either have, or not, both italic and roman symbols. If it has not, let them be all italic. We can, by the preceding argument, form by it a group of 4^{m-1} 4-plets comprising all the italic symbols, and under these we can write the same group in roman letters. This completes the arrangement of the 2.4^m ladies for the day.

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as before in the other places, thus obtaining 4^{n-1} quadruplets having no symbol in common. Then, putting in the same e^{th} places *dd* for *bb*, and *cc* for *aa*, less for less, as before, we obtain, by a like course of cyclical permutations in the remaining places, another group of 4^{n-1} 4-plets, which, added to the preceding, completes the arrangement for the day. And it is easily seen that any quadruplet of any of these groups of 2.4^{n-1}, if treated in the manner prescribed, will give rise to its own complete group and to nothing more. Thus the Theorem J is established.

When $n = 4$ in J, the triplets combined with any letter a, are those required for the solution of the problem of the 15 young ladies. But I do not see in the case of $n = 6$ any step towards a like handling of 63.

Croft Rectory, near Warrington.

ON A CLASS OF RULED SURFACES.

By the Rev. George Salmon.

The remarks of Mr. Cayley in the last No. on "Ruled Surfaces in General" (vol. VII. p. 171) have led me to examine more particularly the nature of the surface generated by a right line resting on three directrices, which we shall suppose to be curves of the degrees m_1, m_2, m_3.

First, then, it is plain that the directrices are in general multiple lines on the surface of the degrees respectively m_2m_3, m_3m_1, m_1m_2. For through any point on the first curve pass m_2m_3 lines of the system; the intersections, namely, of the two cones having this point for a common vertex, and resting on the curves m_2, m_3.

The degree of the surface is equal to the number of lines of the system which meet an arbitrary right line; it is therefore equal to the number of intersections of the curve m_3 with the ruled surface, having for directrices the arbitrary right line and the curves m_1, m_2; or it is equal to m_3 times the degree of this latter ruled surface. And by a repetition of the same argument it appears that the degree of the surface is $2m_1m_2m_3$.

The intersection of two lines of the system gives rise to a double point on the surface. Let us therefore examine how many other lines of the system can intersect a given

one. And it is evident that this number is equal to that of the intersections of the curve m_3 with the ruled surface whose directrices are the curves m_1, m_2, and a right line resting on both. The degree of this latter ruled surface is found by examining the nature of a plane section through the right line. And it is found that the right line is a multiple line of the degree $m_1m_2 - 1$, and that there are besides in any such plane section $(m_1 - 1)(m_2 - 1)$ lines of the new system. The degree of the latter ruled surface is therefore $2m_1m_2 - m_1 - m_2$, which meets the curve m_3 in points not on the right line, whose number is

$$(2m_1m_2-m_1-m_2)\,m_3-(m_1m_2-1)=2m_1m_2m_3-m_2m_3-m_3m_1-m_1m_2+1.$$

If we consider the section of the ruled surface passing through any line of the system, we know that it must be this right line and a curve of the degree $2m_1m_2m_3 - 1$, intersecting the right line in that number of points. And these points are distributed as follows:

$$m_1m_2 - 1, \quad m_2m_3 - 1, \quad m_3m_1 - 1,$$

where the right line meets each of the directrices; the $2m_1m_2m_3 - m_1m_2 - m_2m_3 - m_3m_1 + 1$ points just mentioned, where the right line meets other lines of the system; and the single point of contact of the plane of section with the ruled surface.

It is to be observed that there are a number of right lines

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I am not able to tell the order of the double curve in this case; but I may mention that when Mr. Cayley observes (p. 173, near the end) that it does not appear that there is anything to determine x, he overlooks that he has proved that the reciprocal of the skew surface is one of the same order as itself, and consequently that the theory of reciprocal surfaces must afford equations connecting the characteristics of the double curve with those of the ruled surface. When the curve x is simply a double line, the equation in question is

$$m(m-2)(m-4) = 3x(m-4).$$

The proof of this will appear in a paper on Reciprocal Surfaces, which I wrote about three years ago and which I hope soon to publish. But ordinarily I believe that the curve x will include multiple lines of a degree higher than the double (as, for example, in the case discussed in this paper): and I have not examined the effect of such multiple lines in diminishing the cuspidal and double edges of the tangent cones to surfaces.

Trinity College, Dublin,
Sept. 7, 1852.

NOTE.—I take this opportunity of saying, in reply to Mr. Walton's article in the last Number, that I see no reason why we should not close our controversy on the terms of arbitration proposed by Professor De Morgan, namely that Mr. Gregory's conventions shall be banished from the regions of Algebraic Geometry to those of Geometrical Algebra, where for my part I have no desire to follow them. In fact Mr. Walton does not deny the only point for which I am anxious to contend, viz. that the curvilinear loci obtained by Mr. Gregory's rules have *no geometrical connexion* with the plane curves represented by the same equations. And if this be so, they cannot be expected to throw any light on any difficulties, real or supposed, in the theory of plane curves. I have only to add that I believe Mr. Walton was hasty in asserting (vol. VII. p. 239) that "if $f(x, y) = 0$ be transcendental, a conjugate point not double but single may easily present itself," and that the case of a conjugate point appearing to have a real tangent is explained by observing that such a point results from the union of two or more ordinary conjugate points.

AN ACCOUNT OF SOME TRANSFORMATIONS OF CURVES.

By ANDREW S. HART.

AMONG the methods of Transformation of Curves mentioned by Mr. Salmon in his treatise on the *Higher Plane Curves*, is the method of inversion which was introduced to the notice of geometers by Dr. Ingram and Mr. Stubbs

in the *Transactions of the Dublin Philosophical Society* for 1843, and which appears worthy of further development.

The inverse curve is formed by substituting $\frac{1}{\rho}$ for ρ in the polar equation of the given curve, or, which is the same thing, substituting $\frac{x}{x^2+y^2}$, $\frac{y}{x^2+y^2}$ for x and y in the equation referred to rectangular coordinates.

Thus, if the equation of a given curve be

$$u_0 + u_1 + u_2 + \&c. + u_n = 0$$

(where u_0 is the absolute term, and u_1, u_2, &c. the aggregates of all the terms of the first, second, &c. degrees), the equation of the inverse curve will be

$$u_0(x^2+y^2)^n + u_1(x^2+y^2)^{n-1} + u_2(x^2+y^2)^{n-2} + \&c. + u_n = 0,$$

an equation of the $2n^{\text{th}}$ degree having three multiple points of the n^{th} degree, viz. the origin and the two circular points at infinity. But if the origin be on the given curve, $u_0 = 0$, and the degree of the inverse curve will be $2n-1$, the origin will still be of the n^{th} degree, and the circular points at infinity of the $(n-1)^{\text{th}}$ degree. If the origin be a multiple point on the given curve, the degree of the inverse curve and also of its circular points will be diminished by the degree of this multiple point. Again, if the given curve be circular (*i.e.* if it pass through the two circular points), the

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First it is to be observed, that while as a general rule every point on one line has its corresponding point on the inverse line, the origin and the two circular points at infinity are to be excepted, as we have seen that their presence on any curve gives rise not to corresponding points, but to a reduction of degree of the inverse curve.

Secondly, with these exceptions every multiple point corresponds to one of the same nature on the inverse curve, and every contact or intersection of two lines corresponds to a contact or intersection at the same angle of the inverse lines. Hence, also, since a focus is an infinitely small circle having double contact with the curve, its inverse is a focus of the inverse curve, except in the case where the origin is a focus to which correspond in the inverse curve two imaginary cusps at the circular points at infinity. Thus, if the origin be a focus of a conic section, the inverse curve will be a limaçon, which becomes a cardioide when the given curve is a parabola, and the inverse of a circular cubic or bicircular biquadratic, when the origin is a focus, becomes a Cartesian oval.

Thirdly, if a and b be the points inverse to A and B, $ab = \frac{AB}{OA.OB}$, O being the origin, and the distance of any point A from a right line is to the distance of the inverse point from the circle inverse to the line measured in the direction of the origin as OA to the diameter of the circle.

Examples of these properties will be found in Mr. Salmon's Treatise and in the volume of Transactions already referred to. I will only add that, as it is proved (*Higher Plane Curves*, p. 177), that confocal circular cubics cut at right angles, it follows by inversion that all confocal bicircular biquadratics (including Cartesian ovals) cut one another orthogonally at their eight points of intersection; in fact, if any of these points be taken as origin, the tangents will be parallel to the asymptotes of the inverse confocal circular cubics, and it is therefore sufficient to prove that they are perpendicular. Also, since a circular cubic has four tangents parallel to its asymptote, which touch at four of the intersections with the confocal cubic, it follows by inversion that there are four circles which touch a bicircular biquadratic at any point O, each of which touches it a second time at the points A, B, C, D, which are four of the intersections of this biquadratic with the confocal passing through O; the three remaining intersections are the intersections of three pairs of circles which pass respectively through OAB, OCD;

OAC, OBD; OAD, OBC; and each pair of these circles cut one another at right angles.

It would be easy to multiply examples. The above may serve to shew the fertility of this method in geometrical results.

Trinity College, Dublin,
July 28, 1852.

ON THE METHOD OF VANISHING GROUPS.

By James Cockle.

[*Concluded from* Vol. vii. p. 118.]

XXVI. If a system of simultaneous and homogeneous equations admit of finite algebraic determination by any process without its being necessary that more than a certain number of undetermined quantities should enter into each of the given equations, I call that number the ***explicit*** limit of the process as applied to the system. If, *à priori* and independently of the given system, each of the indeterminates involved in it can be represented as a linear and homogeneous function of not *less* than a certain number of other indeterminates, without our results being thereby rendered illusory, I call the latter number the corresponding *implicit* limit of the system.

solutions, it would not be difficult to determine it. Adopting Mr. Sylvester's nomenclature and notation (*Journal,* new series, vol. VI. p. 17—18), and also putting every equation of a given system under the form

$$Q_0\lambda^n + Q_1\lambda^{n-1} + Q_2\lambda^{n-2} + \ldots + Q_{n-1}\lambda + Q_n = 0,$$

let us assume that all the quantities Q_0 vanish. Then if for all values of m, from 1 to r inclusive, the quantities Q_m, which are K_m in number, could be made to vanish, we should obtain a linear solution of the system. This evanescence depends upon the solution of the system indicated by the change of k into K, and this again upon the linear solution of the system derived from the last by the change of K_1 and K_r into $'K_1$ and K_r' respectively. By successive reductions we should ultimately be conducted to $L+2$.

The result thus arrived at does not directly furnish us with the implicit limit, inasmuch as we have not shewn how the quantities Q_0 are to be made to vanish. And, for all that appears, the very operations by which equations are obtained in which such relations are satisfied may conduct to illusory results. But, comparing the above with Sir W. R. Hamilton's formulæ, we see that the implicit limit of Mr. Jerrard's process is the absolute limit of all linear solutions, however obtained.

XXVII. The expression $k_1k_2k_3..k_r$ or κ indicates, throughout this paper, that the system given for determination consists of k_1 linear, k_2 quadratic, k_3 cubic,..., and k_r r^ic equations. When, in place of k, numbers or letters used as numbers are inserted, commas are placed between them and their meaning ascertained by a reference to their position, as in the notation of Sir W. R. Hamilton and Mr. Sylvester. The evanescence of the formula indicates the finite algebraic solution of the system, or the reduction of its solution to that of equations involving only two indeterminates.

The symbol $(2r;\ \theta)$ denotes the double operation consisting (1) of the performance of γ_{2r}, and (2) of the grouping the resulting powers two and two together and making each of the r groups thence arising vanish. Whenever we have a vanishing group, we must use it to eliminate an indeterminate from EVERY expression into which it enters. If we call $\xi^{(r)}$ and $\xi^{(r+1)}$ the leading indeterminates of the group p_rp_{r+1}, it will in general be convenient to consider one of the leading indeterminates as eliminated.

I use the symbols j, s, and vg as characteristic of the respective methods of Mr. Jerrard, of Mr. Sylvester, and

of vanishing groups. When the characteristic of a process is prefixed to a bracketed κ, the compound symbol represents the explicit limit of the process as applied to the determination of the system. Thus,

$$j(0, 1, 1) = 5, \quad s(0, 1, 1) = 4, \quad vg(0, 1, 1) = 4.$$

I have given no symbol for the memorable process of Sir W. R. Hamilton; but in the above instance, and in general, the implicit limit of j, as determined by Sir William Hamilton, is given by s. The symbol e will denote ordinary elimination.

XXVIII. If we assume that

$$k_1 = \alpha, \quad k_2 = \beta, \ldots, k_{r-1} = b, \quad k_r = a,$$

the general explicit limit of the pure method of vanishing groups may be represented by either of the identical expressions

$$vg(k_1k_2\ldots k_r), \quad vg(\kappa), \quad \text{or } vg(\alpha, \beta, \gamma, \delta, \varepsilon, \ldots, b, a),$$

of which the last is equal to Υ, as given in IX. and corrected in X. Let w represent either of the identical expressions

$$u\begin{bmatrix} 4, 5, \ldots, r-1, r \\ \delta, \varepsilon, \ldots, \quad b, \quad a \end{bmatrix} \text{ or } vg(k_4k_5\ldots k_{r-1}k_r);$$

and suppose that

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XXIX. Consider first the system of quadratics

$$u_1 = 0,\ u_2 = 0, \ldots,\ u_{s-1} = 0,\ \ u_s = 0,$$

in which

$$vg(0, x) = 2^s \text{ and } s(0, x) = \tfrac{1}{2}(x^2 + x + 2),$$

and consequently for values of x greater than 2 the process s has an advantage over vg. There is a modification (g_2) of vg in which the explicit limit is given by

$$g_2(0, x) = 2^{x-2}.3 \ldots\ldots\ldots\ldots (18),$$

and which may be represented as follows,

$$(2^{s-2};\, \theta)\, u_1,\ (2^{s-3};\, \theta)\, u_2,\ \ldots,\ (2^2;\, \theta)\, u_{s-3},\ (2\,;\, \theta)\, u_{s-2},$$

$$u_{s-1} = f_{s-1}(3) = 0,\quad u_s = f_s(3) = 0.$$

The functions θ vanish by groups, and f by ordinary elimination, in which case $e(0, 2) = 3$. Whenever we arrive at $vg(0, 1) = 2$, we have the ordinary solution of a quadratic, of which the method of vanishing groups is the indeterminate development; e entails upon us the solution of a biquadratic. But for the values $x = 2, 3, \ldots, r$, we thereby obtain an advantage over the linear solution.*

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* The scheme of XXXII. enables us to arrive at (18). For if we have x quadratics and

$$v'(0, x) = 2^{x-1} + 2^{x-2} = y,$$

and perform $(2^{x-1};\, \theta)$, $(2^{x-2};\, \theta)$, .., $(2^2;\, \theta)$ in due order on the first $x-2$ of them, they will be reduced each to the form $f(y - 2^{x-1})$ and, with the above value of y, admit of solution by vg. The last two quadratics will be solved by the remaining two of the grouped quantities (of which

$$2^{x-1} - (2^{x-2} + 2^{x-3} + \ldots + 2^2 + 2) \text{ or } 2$$

still remain) by the aid of e.

We may now replace 2^{x-2} by $2^{x-3} + 2^{x-4}$ in the expression for y. And, pursuing this process continuously, we shall arrive at

$$v'(0, x) = y = 2^{x-1} + 2^{x-3} + 2^{x-5} + \ldots,$$

the last term being 2 when x is even, and 4 when x is odd. Of course the properties of the functions ψ, &c. would enable us to modify this formula, but I shall not pursue the inquiry.

I would however observe, that in the relation (obtained by any means we please)

$$\psi(0, p) = q,$$

whenever we find that $2^{x-p}q$ is less than $2^{x-2}.3$, we may change (18) into

$$g_2(0, x) = 2^{x-p}q = 2^{x-p}\,\psi(0, p),$$

with an advantage great in proportion to their difference. Thus, one of the results of XXXI. furnishes us with the formula

$$g_2(0, x) = 2^{x-4}.9,$$

the quantity $2^{x-4}.9$ being less than $2^{x-2}.3$ or $2^{x-4}.12$.

I have thought it for the most part unnecessary to encumber the investigations in the text with linear equations. The implicit limit (l)

XXX. A combination of Lagrange's method of multipliers with an extension of the principle of linear solution

of the system $(0, x, ..)$ is always connected with the implicit limit (l') of $(a, x, ..)$ by the relation

$$l + a = l'.$$

The explicit limits in Mr. Jerrard's process are not in general connected by a similar relation. When linear equations are introduced, Mr. Jerrard increases the explicit limit by a number greater than that of the linear equations.

In concluding the subject of quadratics I may observe, that if in XXV. we make

$$v + w + \lambda_1^{-1}m = x, \quad v - w + \lambda_2^{-1}n = y,$$
$$p - \lambda_1^{-2}m^2 - 2\lambda_1^{-1}\lambda_2^{-1}n^2 = r, \quad 2\lambda_1^{-1}n = t,$$
$$q - \lambda_2^{-2}n^2 - 2\lambda_1^{-1}\lambda_2^{-1}m^2 = s, \quad \text{and } 2\lambda_2^{-1}m = u,$$

the system there given is equivalent to

$$x^2 + ty + r = 0,$$
$$y^2 + ux + s = 0;$$

and if U and U' be functions of g undetermined quantities, then, in general, x and y are linear functions of all those quantities, and t and u linear and r and s quadratic functions of the same $g-2$ of them. Various observations (for instance, the transformation

$$x' = x + \tfrac{1}{2}u, \quad y' = y + \tfrac{1}{2}t)$$

arise upon the system last arrived at; but I must now pass on to ulterior objects.

Let U, V, and W denote three homogeneous quadratic functions of five undetermined quantities. Then by XVI. we have, omitting the indices of γ,

$$\gamma_2 U = h_1^2 + h_2^2 + f^2(3) = pq + f^2(3),$$

p and q respectively replacing the p_1 and p_2 of XI. Let r', s', and t be the

gives results not unworthy of notice. Construct out of the given system $x-z$ equations, each of the form

$$\lambda_m u_m + \lambda_n u_n + \ldots + \lambda_r u_r = 0 = \Sigma.\lambda u,$$

and let the solution of these equations, together with that of the z equations

$$u_a = 0,\quad u_b = 0, \ldots, u_l = 0,$$

involve the solution of the given system of x quadratics. Let the terms included under Σ be $z'+1$ in number, and make $\Sigma_1\lambda u = A_1\xi' + A'_1, \ldots, \Sigma_{z-1}\lambda u = A_{z-1}\xi' + A'_{z-1}$.

By means of the $x-z$ linear equations

$$A_1 = 0,\quad A_2 = 0, \ldots, A_{z-1} = 0,$$

eliminate $x-z$ indeterminates from $A_1', A_2', \ldots, A'_{z-1}$. We shall thus have $x-z$ reduced equations each of the form

$$\Sigma'.\lambda u = B\xi'' + B',$$

and if we eliminate $x-z$ other indeterminates by means of $x-z$ equations $B=0$, and continue this operation z' times, we shall ultimately arrive at $x-z$ equations of the form

$$\Sigma^{(z')}.\lambda u = \alpha'\xi'^2 + \alpha''\xi''^2 + \ldots + \alpha^{(z')}\xi^{(z')2} + f(y) = 0.$$

By means of the z' ratios of the $z'+1$ quantities λ, let each of the z' quantities $\alpha', \alpha'', \ldots, \alpha^{(z')}$ be made to vanish; and, this being done for each of the $x-z$ equations of the form last given, let our results be denoted by

$$\Sigma(\lambda u)_1 = f_1(y),\quad \Sigma(\lambda u)_2 = f_2(y), \ldots, \Sigma(\lambda u)_{z-1} = f_{z-1}(y).$$

Let the number of quantities required for the solution of the given system by the present process be $\psi(0, x)$, a yet unknown function of x. Then, from the course of our eliminations,

$$y = \psi(0, x) - z'(x - z + 1),$$

and, from the conditions of the question,

$$y = \psi'(0, x-z),\quad z' = \psi'(0, z) - 1,$$

where ψ' is the known operation best fitted for our purpose. Hence

$$\psi(0, x) = \psi'(0, x-z) + (x - z + 1)\{\psi'(0, z) - 1\} \ldots (19).$$

XXXI. Let $z = 1$; then, putting ψ_1 for ψ,

$$\psi_1(0, x) = \psi'(0, x-1) + x;$$

and if we make ψ' and ψ_1 identical, and treat this last as an equation in finite differences, availing ourselves of e in determining the constant, we find

$$\psi_1(0, x) = \tfrac{1}{2}(x^2 + x) \ldots\ldots\ldots\ldots (20),$$

which is true for all values of x greater than 1, and has always an advantage over s. In the present instance we may make

$$\Sigma_1\lambda u = \lambda_1 u_1 - \lambda_2 u_2,\ \Sigma_2\lambda u = \lambda_1 u_1 - \lambda_3 u_3, \ldots, \Sigma_{x-1}\lambda u = \lambda_1 u_1 - \lambda_{x-1} u_{x-1},$$

and then solve $u_1 = 0$ by means of ξ'.

This process is equivalent to another which should proceed by causing ξ'^2 to disappear by ordinary elimination, and ξ' by making its coefficient vanish.

In (19) let $z = 2$, then

$$\psi_2(0, x) = \psi'(0, x-2) + (x-1).2 \ldots\ldots (21);$$

and, if we make $x = 4$ and ψ' the same as e, we find

$$\psi_2(0, 4) = 3 + 3.2 = 9;\ \ \psi_1(0, 4) = 10;$$

and, in this instance, ψ_2 has an advantage over ψ_1, and a still greater one over s. The two Σ functions will be

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0,\ \text{and}\ \mu_1 u_1 + \mu_2 u_2 + \mu_3 u_4 = 0,$$

and the remaining equations to be solved will be

$$u_1 = 0\ \text{and}\ u_2 = 0.$$

So we should find

$$\psi_2(0, 5) = 6 + 4.2 = 14;\ \psi_1(0, 5) = 15;\ s(0, 5) = 16.$$

With this fragment on multilinear solution (as I should propose to call it) I shall leave the subject of simultaneous quadratics, and proceed to systems in which quadratics are

in which scheme x_m' represents

$$2^{s-1} + 2^{s-2} + \ldots + 2^{s-m+1} + 2^{s-m},$$

and consequently $x'_{s-1} = 2^s - 2$.

XXXIII. The explicit limit of this process is

$$v_2(0,\, x,\, 0^{s-3},\, 1) = 2^s + \psi(x) = z,$$

where ψ is any of the functions which we have used in the theory of quadratics. For, the determination of the system is, by the scheme of XXXII., reduced to the solution of x quadratics, each of the form

$$f(z - 2^s) = f\{\psi(x)\} = 0,$$

and which consequently admit of solution. These being solved, we have only to eliminate an indeterminate between the linear

$$'h_1 \pm 'h_2 \sqrt{(-1)} = 0,$$

and the given s^{ic} equation, and the system is determined. When $x = 2$ we have

$$v_2(0,\, 2,\, 0^{s-3},\, 1) = 2^2 + e(2) = 7 = s(0,\, 2,\, 0^{s-3},\, 1).$$

For $x = 1$ we have

$$vg(0,\, 1,\, 0^{s-3},\, 1) = 2^2 = 4 = s(0,\, 1,\, 0^{s-3},\, 1).$$

In both these cases our explicit limit is the implicit limit of Mr. Jerrard's process.

XXXIV. The combination of the theory of linear transformations with that of vanishing groups gives one or two interesting results. Let a solution of the system

$$0,\, 2,\, 0^{s-3},\, 1 = 0,$$

be required, and let z be the explicit limit. Then, using f to denote the quadratics, we have, as in XXXII.,

$$(2\,;\ \theta) f_1(z) = f_1(z-2)\,;\ f(z-1):$$

$$(2\,;\ \theta) f_2(z-1) = f_2(z-3):$$

and, if $z = 7$, we have, without the aid of equations higher than biquadratics, the following linear transformations,

$$f_2(z-3) = f_2(4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2,$$

$$f_1(z-2) = f_1(5) = a_1^2 x_1^2 + a_2^2 x_2^2 + a_3^2 x_3^2 + a_4^2 x_4^2 + a_5^2 x_5^2;$$

whence f_1 and f_2, under the last form, are satisfied by substituting $-\Sigma_4(x^2)$ for x_5^2 in $f_1(5)$, and solving the result by vanishing groups, that is to say, by breaking it up into two

equations of the form

$$b_1^2x_1^2 + b_2^2x_2^2 = 0, \quad b_3^2x_3^2 + b_4^2x_4^2 = 0.$$

After all the eliminations there will yet remain one disposable quantity in the final s^ic, by means of which it may be solved.

XXXV. A second result deduced by the aid of linear transformation is the following. Let

$$0,\ 3,\ 0^{r-1},\ 1 = 0,$$

be the system for determination, and let z be the explicit limit. Proceed as follows:

$$(2^3;\,\theta)\,f_1(z) = f_1'(z-2^3)\,;\; f(z-2^3):$$

$$\gamma_4 f_2(z-2^3) = h_1^2 + h_2^2 + h_3^2 + h_4^2 + f_3'(z-2^3)\,;$$

make $\qquad f_1'(z-2^3) = 0$ and $f_2'(z-2^3) = 0,$

which can be done by ordinary elimination and the solution of a biquadratic, provided that

$$z - 2^3 = 3, \quad \text{or} \quad z = 11.$$

The solution of the given system is now reduced to that of the given s^ic, of

$$h_1^2 + h_2^2 + h_3^2 + h_4^2 = 0 = \Sigma_4(h^2),$$

and of $\qquad f_3(z - 2^3 - 2) = 0 = f_3(z-6)\,;$

XXXVI. There are other perhaps not less important applications of linear transformation to the theory of equations. In fact it is not difficult to see, that three simultaneous homogeneous quadratics involving five unknowns admit of finite algebraic solution without demanding the solution of an equation higher than an equation of the fifth degree. For, by known processes, two of the given equations may be simultaneously transformed into pure homogeneous quadratics involving five new unknowns linearly connected with the given ones. Let $x_1, x_2, \ldots, x_5$ be the new quantities, then the transformed quadratics may be written thus,

$$\Sigma.x^2 = 0,\quad \Sigma.ax^2 = 0,\quad \Sigma.bx^2 + \Sigma.cx_1x_2 + x_5\Sigma.dx = 0,$$

where the last Σ only includes four terms, or

$$\Sigma.dx = d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = D.$$

Now if by means of $D = 0$ we eliminate x_4 from the transformed system, we shall have three resulting quadratics in x_1, x_2, x_3, and x_5, which, when respectively divided by x_5^2, may be made to form a system of pseudo-homogeneous quadratics completely resolvible by a process similar to that by which Frend solved the problem known as Colonel Titus's. This process entails upon us the necessity of solving a biquadratic: by means of others, that need not be further adverted to, we may avoid the occurrence of an equation higher than a cubic.

XXXVII. The whole indeterminate department of the theory of equations has a peculiar interest attached to it, inasmuch as it suggests an inquiry into the extent to which the limits supposed to be affixed to the solution of systems of simultaneous equations by the researches of Sir W. R. Hamilton and Mr. Sylvester may be legitimately considered as impassable. So far as their views have yet been developed they do not seem to include such cases as those discussed in XXX. and XXXI., which appear to have burst through the bounds prescribed in their investigations. However this be, I shall not now attempt to enter into the question. But, in concluding this series of papers, while I owe an apology for the imperfect manner in which the subject has been handled, I may be allowed to express a hope that the general indeterminate method here discussed will be found, from its simplicity of conception and uniformity of process, to admit of being easily grasped in its most general form, while in its application to more elementary cases (the transformation of the general equation of the fifth degree to the trinomial

form, for example) it possesses the great advantage of having its explicit limit identical with the implicit one of linear solution, while its processes are all purely algebraic. The next great question will be, how far the indeterminate processes are limited to obtaining linear solutions, and whether another range may not be given to them. But this question I must leave.

2, *Pump Court, Temple,*
October 15, 1852.

Erratum.—Art. XII. line 8, *for* $+\rho_s$ *read* $-\rho_s$.

ON A THEOREM CONCERNING THE COMBINATION OF DETERMINANTS.

By J. J. Sylvester, F.R.S.

Let ${}^1A$ represent the line of terms ${}^1a_1, {}^1a_2, \ldots {}^1a_m$,

${}^1B$ . . . . ${}^1b_1, {}^1b_2, \ldots {}^1b_m$.

Let ${}^1A \times {}^1B$ represent $\Sigma\,({}^1a_r \times {}^1b_r)$, where of course there are r terms within the symbol of summation.

$\left(\begin{array}{l}\text{and where in general } {}^rA \text{ denotes } {}^ra_1,\ {}^ra_2,\ldots{}^ra_n \\ \qquad\text{and } {}^rB \text{ denotes } {}^rb_1,\ {}^rb_2,\ldots{}^rb_n\end{array}\right)$ represent

$$\Sigma\left\{\begin{vmatrix} {}^1a_{h_1}, & {}^1a_{h_2},\ldots & {}^1a_{h_m} \\ {}^2a_{h_1}, & {}^2a_{h_2},\ldots & {}^2a_{h_m} \\ \cdots & \cdots & \cdots \\ {}^ma_{h_1}, & {}^ma_{h_2},\ldots & {}^ma_{h_m} \end{vmatrix} \times \begin{vmatrix} {}^1b_{h_1}, & {}^1b_{h_2},\ldots & {}^1b_{h_m} \\ {}^2b_{h_1}, & {}^2b_{h_2},\ldots & {}^2b_{h_m} \\ \cdots & \cdots & \cdots \\ {}^mb_{h_1}, & {}^mb_{h_2},\ldots & {}^mb_{h_m} \end{vmatrix}\right\}$$

Now let (r) be any integer less than (m), and let

$$\mu = \frac{m.(m-1)\ldots(m-r+1)}{1.2\ldots r},$$

and let $G_1,\ G_2,\ldots G_\mu$ denote the μ rectangular matrices of the forms

$$\left.\begin{vmatrix} A_{\theta_1} \\ A_{\theta_2} \\ \cdots\cdots \\ A_{\theta_r} \end{vmatrix}\right\} \text{ respectively,}$$

and let $H_1,\ H_2,\ldots H_\mu$ denote the μ rectangular matrices of the forms

$$\left.\begin{vmatrix} B_{\theta_1} \\ B_{\theta_2} \\ \cdots\cdots \\ B_{\theta_r} \end{vmatrix}\right\} \text{ respectively.}$$

Now form the determinant

$$\begin{array}{lll} G_1 \times H_1; & G_1 \times H_2\ldots; & G_1 \times H_\mu; \\ G_2 \times H_1; & G_2 \times H_2\ldots; & G_2 \times H_\mu; \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ G_\mu \times H_1; & G_\mu \times H_2\ldots; & G_\mu \times H_\mu; \end{array}$$

then, if we give r the successive values $1,\ 2,\ 3\ldots m$, (in which last case the determinant in question reduces to a single term), the values of the determinant above written will be severally in the proportions of

$$K,\ K^m,\ K^{\frac{1}{2}m(m-1)},\ldots K^m,\ K;$$

that is to say, the logarithms of these several determinants will be as the coefficients of the binomial expansion $(1+x)^m$.

When we make $r=m$, and equate the determinant corresponding to this value of r with that formed by making $r=1$, the theorem becomes identical with a theorem previously given by M. Cauchy, for the Product of Rectangular Matrixes.

It would be tedious to set forth the demonstration of the general theorem in detail. Suffice it here to say that it is a direct corollary from the formula marked (4) in my paper in the *Philosophical Magazine* for April 1851, entitled "On the Relations between the Minor Determinants of Linearly Equivalent Quadratic Functions," when that formula is particularized by making

$$\left.\begin{matrix} a_{m+1}, & a_{m+2}, & \ldots & a_{m+n} \\ b_{m+1}, & b_{m+2}, & \ldots & b_{m+n} \end{matrix}\right\}$$

represent a determinant all whose terms are zeros except those which lie in one of the diagonals, these latter being all units, which comes, in fact to defining that

$$\begin{vmatrix} a_{m+e} \\ b_{m+e} \end{vmatrix} = 1, \quad \text{and} \begin{vmatrix} a_{m+e} \\ b_{m+e} \end{vmatrix} = 0.$$

The important theorem here referred to is made almost unintelligible by an unfortunate misprint of ${}^{q}\theta_m$, ${}^{1}\theta_m$, ${}^{2}\theta_m$, ${}^{\mu}\theta_m$, in place of ${}^{q}\theta_r$, ${}^{1}\theta_r$, ${}^{2}\theta_r$, ${}^{\mu}\theta_r$. I may here take notice of another and still more inexplicable blunder in the same paper, formula (3), in the latter part of the equation belonging to which

$$\left\{\begin{matrix} a_{\theta_1}, & a_{\theta_2}, & \ldots & a_{\theta_m}, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \ldots & a_{\theta_{m+t}} \\ a_{\phi_1}, & a_{\phi_2}, & \ldots & a_{\phi_m}, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \ldots & a_{\phi_{m+t}} \end{matrix}\right\}$$

is written in lieu of

1. The discovery of Combinants; that is to say, of concomitants to systems of functions remaining invariable, not only when combinations of the variables are substituted for the variables, but also when combinations of the functions are substituted for the functions; and as a remarkable first-fruit of this new theory of double invariability, the representation of the Resultant of any three quadratic functions under the form of the square of a certain combinantive sextic invariant added to another combinant which is itself a biquadratic function of 10 cubic invariants. When the three quadratic functions are derived from the same cubic function, this expression merges in M. Aronhold's for the discriminant of the cubic. The theory of combinants naturally leads to the theory of invariability for non-linear substitutions, and I have already made a successful advance in this new direction.

2. The unexpected and surprising discovery of a quadratic covariant to any homogeneous function in x, y of the n^{th} degree, containing $(n-1)$ variables cogredient with x^{n-2}, $x^{n-3}y \dots y^{n-2}$ and possessing the property of indicating *the number of real and imaginary roots* in the given function. This covariant, on substituting for the $(n-1)$ variables the combinations of the powers of x, y with which they are cogredient, becomes the Hessian of the given function.*

3. The demonstration due to M. Hermite of a law of reciprocity connecting the degree or degrees of any function or system of functions with the order or orders of the invariants belonging to the system. The theorem itself was first propounded by me about a twelvemonth back, and communicated to Messrs. Cayley, Polignac, and Hermite, as serving to connect together certain phenomena which had presented themselves to me in the theory: unfortunately it appeared to contradict another law too hastily

* This covariant furnishes, if we please, functions symmetrical in respect to the two ends of an equation for determining the number of its real and imaginary roots. The ordinary Sturmian functions, it is well known, have not this symmetry. As another example of the successful application of the new methods to subjects which have been long before the mathematical world and supposed to be exhausted, I may notice that I obtain without an effort, by their aid, a much more simple, practical, and complete solution of the question of the simultaneous transformation of two quadratic functions, or the orthogonal transformation of one such function, than any previously given, even by the great masters Cauchy and Jacobi, who have treated this question.

assumed by myself and others as probably true, and I consequently laid aside the consideration of this great law of reciprocality. To M. Hermite, therefore, belongs the honour of reviving and establishing,—to myself whatever lower degree of credit may attach to suggesting and originating,—this theorem of numerical reciprocity, destined probably to become the corner-stone of the first part of our new calculus; that part, I mean, which relates to the generation and affinities of forms.*

4. I may notice that the Calculus of Forms may now with correctness be termed the Calculus of Invariants, by virtue of the important observation that every concomitant of a given form or system of forms may be regarded as an invariant of the given system and of an absolute form or system of absolute forms combined with the given form or system. As regards that particular branch of the theory of invariants which relates to resultants, or, in other words, to the doctrine of elimination, I may here state the theorem alluded to in a preceding Number of the *Journal,* to wit that if R be the resultant of a system of (n) homogeneous functions of (n) variables, written out in their complete and most general form (so that by definition $R=0$ is the condition that the equations got by making the (n) given functions zero, shall be simultaneously satisfiable by one system of ratios), then the condition that these equations

[redacted]

ON THE TRIGONOMETRY OF THE PARABOLA.

By the Rev. J. Booth, F.R.S.

From the *Philosophical Transactions*, for 1852. Part II. p. 385.

A fundamental theorem in the theory of elliptic integrals is

$$\cos\omega = \cos\phi\cos\chi - \sin\phi\sin\chi\sqrt{(1 - i^2\sin^2\omega)}\ldots(338).$$

The angles ϕ, χ, ω may be called conjugate amplitudes.

When the hyperconic section is a circle, $i = 0$, and $\cos\omega = \cos\phi\cos\chi - \sin\phi\sin\chi$, whence $\omega = \phi + \chi$, or the conjugate amplitudes are $\phi + \chi$, ϕ and χ. The development of this expression is the foundation of circular trigonometry.

When the hyperconic section is a parabola, $i = 1$, and (338) may be reduced to

$$\tan\omega = \tan\phi\sec\chi + \tan\chi\sec\phi\ldots\ldots\ldots(339).$$

If we make the imaginary transformations,

$$\left.\begin{array}{l}\tan\omega = \sqrt{(-1)}\sin\omega', \ \tan\phi = \sqrt{(-1)}\sin\phi', \\ \tan\chi = \sqrt{(-1)}\sin\chi', \ \sec\phi = \cos\phi', \ \sec\chi = \cos\chi'\end{array}\right\}\ldots(340).$$

The preceding formula will become, on substituting these values, and dividing by $\sqrt{(-1)}$,

$$\sin\omega' = \sin\phi'\cos\chi' + \sin\chi'\cos\phi',$$

the well-known trigonometrical expression for the sine of the sum of two circular arcs.

Hence, by the aid of imaginary transformations, we may interchangeably permute the formulæ of the trigonometry of the circle with those of the trigonometry of the parabola. In the trigonometry of the circle, $\omega = \phi + \chi$, and in the trigonometry of the parabola ω is such a function of the angles ϕ and χ, as will render $\tan[(\phi, \chi)] = \tan\phi\sec\chi + \tan\chi\sec\phi$. We must adopt some appropriate notation to represent this function. Let the function (ϕ, χ) be written $\phi \perp \chi$, so that $\tan(\phi \perp \chi) = \tan\phi\sec\chi + \tan\chi\sec\phi$. This must be taken as the *definition* of the function $\phi \perp \chi$.

In like manner, we may represent by $\tan(\phi \top \chi)$ the function $\tan\phi\sec\chi - \tan\chi\sec\phi$.

In applying the imaginary transformations, or while $\tan\phi$ is changed into $\sqrt{(-1)}\sin\phi$, $\sec\phi$ into $\cos\phi$, and $\cot\phi$ into $-\sqrt{(-1)}\operatorname{cosec}\phi$, $\perp$ must be changed into $+$ and $\top$ into $-$.

$\perp$ and $\top$ may be called logarithmic plus and minus. As examples of the analogy which exists between the trigonometry of the parabola and that of the circle, the following expressions in parallel columns are given; premising that the formulæ, marked by corresponding letters, may be derived singly, one from the other, by the help of the preceding imaginary transformations.

..(α.) $\sin(\phi+\chi) = \sin\phi\cos\chi + \sin\chi\cos\phi$ .... (a.)

..(β.) $\sin(\phi-\chi) = \sin\phi\cos\chi - \sin\chi\cos\phi$ .... (b.)

..(γ.) $\cos(\phi\pm\chi) = \cos\phi\cos\chi \mp \sin\phi\sin\chi$ .... (c.)

..(δ.) $\tan(\phi+\chi) = \dfrac{\tan\phi + \tan\chi}{1 - \tan\phi\tan\chi}$ .... (d.)

....(ε.) $\tan(\phi-\chi) = \dfrac{\tan\phi - \tan\chi}{1 + \tan\phi\tan\chi}$ .... (e.)

Let $\phi = \chi$.

..(η.) $\sin 2\phi = 2\sin\phi\cos\phi$ .... (ē.)

.. (θ.) $\cos 2\phi = \cos^2\phi - \sin^2\phi$ .... (th.)

...(ι.) $\tan 2\phi = \dfrac{2\tan\phi}{1-\tan^2\phi}$ .... (i.)

..(κ.) $\cos\phi = \dfrac{e^{\phi\sqrt{(-1)}} + e^{-\phi\sqrt{(-1)}}}{2}$, $\sin\phi = \dfrac{e^{\phi\sqrt{(-1)}} - e^{-\phi\sqrt{(-1)}}}{2\sqrt{(-1)}}$ ...... (k.)

..(λ.) $1 + \sin 2\phi = (\cos\phi + \sin\phi)^2$.................. (l.)

.. (μ.) $\sin^2\phi = \dfrac{1 - \cos 2\phi}{2}$ ....(m.)

Let the amplitudes be $\phi + \chi$ and $\phi - \chi$.

..(ν.) $\sin(\phi+\chi)\sin(\phi-\chi) = \sin^2\phi - \sin^2\chi$..................(n.)

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Since

$$\sec(\phi \perp \phi) = \sec^2\phi + \tan^2\phi, \text{ and } \tan(\phi \perp \phi) = 2\tan\phi\sec\phi,$$

$$\sec(\phi \perp \phi) + \tan(\phi \perp \phi) = (\sec\phi + \tan\phi)^2.$$

Again, as

$$\sec(\phi \perp \phi \perp \phi) = \sec(\phi \perp \phi)\sec\phi + \tan(\phi \perp \phi)\tan\phi,$$

and $\tan(\phi \perp \phi \perp \phi) = \tan(\phi \perp \phi)\sec\phi + \sec(\phi \perp \phi)\tan\phi,$

it follows that

$$\sec(\phi \perp \phi \perp \phi) + \tan(\phi \perp \phi \perp \phi) = (\sec\phi + \tan\phi)^3,$$

and so on to any number of angles. Hence

$$\sec(\phi \perp \phi \perp \phi \ldots \text{to } n\phi) + \tan(\phi \perp \phi \perp \phi \ldots \text{to } n\phi) = (\sec\phi + \tan\phi)^n \ldots (342).$$

Introduce into the last expression the imaginary transformation $\tan\phi = \sqrt{(-1)}\sin\phi$, and we get Demoivre's imaginary theorem for the circle,

$$\cos n\phi + \sqrt{(-1)}\sin n\phi = \{\cos\phi + \sqrt{(-1)}\sin\phi\}^n.$$

Let $\bar{\omega}$ be conjugate to ψ and ω, while ω, as before, is conjugate to ϕ and χ. Then we shall have

$$\tan\bar{\omega} = \tan(\phi \perp \chi \perp \psi), \text{ or}$$

$$\tan(\phi \perp \chi \perp \psi) = \tan\phi\sec\chi\sec\psi + \tan\chi\sec\psi\sec\phi$$
$$+ \tan\psi\sec\phi\sec\chi + \tan\phi\tan\chi\tan\psi \ldots . (\varpi.)$$

$$\sec(\phi \perp \chi \perp \psi) = \sec\phi\sec\chi\sec\psi + \sec\phi\tan\chi\tan\psi$$
$$+ \sec\chi\tan\psi\tan\phi + \sec\psi\tan\phi\tan\chi \ldots . (\rho.)$$

$$\text{and } \sin(\phi \perp \chi \perp \psi) = \frac{\sin\phi + \sin\chi + \sin\psi + \sin\phi\sin\chi\sin\psi}{1 + \sin\chi\sin\psi + \sin\psi\sin\phi + \sin\phi\sin\chi} \ldots (\sigma.)$$

whence, in the trigonometry of the circle,

$$\sin(\phi + \chi + \psi) = \sin\phi\cos\chi\cos\psi + \sin\chi\cos\psi\cos\phi$$
$$+ \sin\psi\cos\phi\cos\chi - \sin\phi\sin\chi\sin\psi \ldots\ldots\ldots (\text{p.})$$

$$\cos(\phi + \chi + \psi) = \cos\phi\cos\chi\cos\psi - \cos\phi\sin\chi\sin\psi$$
$$- \cos\chi\sin\psi\sin\phi - \cos\psi\sin\phi\sin\chi \ldots\ldots\ldots (\text{r.})$$

$$\tan(\phi + \chi + \psi) = \frac{\tan\phi + \tan\chi + \tan\psi - \tan\phi\tan\chi\tan\psi}{1 - \tan\chi\tan\psi - \tan\psi\tan\phi - \tan\phi\tan\chi} \ldots (\text{s.})$$

Let $(k.\omega)$, $(k.\phi)$, $(k.\chi)$ denote three parabolic arcs, measured from the vertex of the parabola whose parameter is k.

The normal angles of these arcs are ω, ϕ, and χ; ω, ϕ, and χ being conjugate amplitudes. Then

$$2(k.\phi) = k \tan\phi \sec\phi + k\int \frac{d\phi}{\cos\phi}, \quad 2(k.\chi) = k\tan\chi \sec\chi + k\int\frac{d\chi}{\cos\chi},$$

$$2(k.\omega) = k \tan\omega \sec\omega + k\int\frac{d\omega}{\cos\omega};$$

whence, since $\int\frac{d\omega}{\cos\omega} - \int\frac{d\phi}{\cos\phi} - \int\frac{d\chi}{\cos\chi} = 0$, because ω, ϕ, and χ are conjugate amplitudes,

$$(k.\omega) - (k.\phi) - (k.\chi) = k \tan\omega \tan\phi \tan\chi \ldots\ldots (343).$$

Let y, y', y'' be the ordinates of the arcs $(k.\phi)$, $(k.\chi)$, and $(k.\omega)$. Then $y = k\tan\phi$, $y' = k\tan\chi$, $y'' = k\tan\omega$, and the last expression becomes

$$(k.\omega) - (k.\phi) - (k.\chi) = \frac{yy'y''}{k^2} \ldots\ldots\ldots (344).$$

If we call an arc measured from the vertex of a parabola an *apsidal* arc, to distinguish it from an arc taken anywhere along the parabola, the preceding theorem will enable us to express an arc of a parabola, taken anywhere along the curve, as the sum or difference of an apsidal arc and a right line.

D

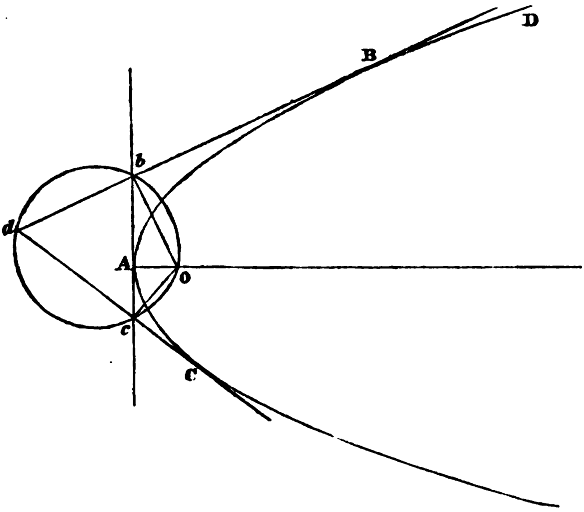
Then (343) shews that the parabolic arc $(AC+AB)$ = apsidal arc $AD-h$; and the parabolic arc $(AD-AB) = BD$ = apsidal arc $AC+h$. When the arcs AC', AB' together constitute a focal arc, or an arc whose cord passes through the focus, $\phi + \chi = \frac{1}{2}\pi$, and h is the ordinate of the conjugate arc AD. Hence we derive this theorem,

Any focal arc of a parabola is equal to the difference between the conjugate apsidal arc and its ordinate.

The relation between the amplitudes ϕ and ω, in this case, is $\sin 2\phi = \dfrac{2\cos\omega}{1-\cos\omega}$. Thus when the focal cord makes an angle of 30° with the axis, we get $\cos\omega = \frac{1}{5}$, or $y = 5k$. Here therefore the ordinate of the conjugate arc is five times the semiparameter.

We may in all cases represent by a simple geometrical construction the ordinates of the conjugate parabolic arcs, whose amplitudes are ϕ, χ, and ω.

Let ABC be a parabola whose focus is O, and whose vertex is A. Let $AO = g = \frac{1}{2}k$; moreover, let AB be the



arc whose amplitude is ϕ, and AC the arc whose amplitude is χ. At the points A, B, C draw tangents to the parabola, they will form a triangle circumscribing the parabola, whose sides represent the semi-ordinates of the conjugate arcs, AB, AC, AD.

We know that the circle circumscribing this triangle passes through the focus of the parabola. Now

$Ab = g\tan\phi$, $Ac = g\tan\chi$, $bd = g\tan\phi\sec\chi$, $cd = g\tan\chi\sec\phi$;

hence $\quad bd + cd = g(\tan\phi\sec\chi + \tan\chi\sec\phi)$,

therefore $\quad g\tan\omega = bd + cd$.

When AB, AC together constitute a focal arc, the angle adc is a right angle.

The diameter of this circle is $g\sec\phi\sec\chi$.

The demonstration of these properties follows obviously from the figure.

In the trigonometry of the circle, we find the formula

$$\vartheta = \tan\vartheta - \frac{\tan^3\vartheta}{3} + \frac{\tan^5\vartheta}{5} - \frac{\tan^7\vartheta}{7} + \&c. \ldots\ldots \text{(a.)}$$

And if we develop by common division the expression

$$\frac{1}{\cos\theta} = \frac{\cos\theta}{1-\sin^2\theta} = \cos\theta(1 + \sin^2\theta + \sin^4\theta + \sin^6\theta + \ldots \&c.),$$

and integrate,

$$\int\frac{d\theta}{\cos\theta} = \sin\theta + \frac{\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + \frac{\sin^7\theta}{7} + \&c. \ldots\ldots \text{(b.)}$$

If we now inquire, what, in the circle, is the arc which

Hence we must have, in this case, $\int \frac{d\theta}{\cos\theta} = 1$. If we now revert the series (b), putting 1 for $\int \frac{d\theta}{\cos\theta}$, we shall get from this particular value of the series,

$$1 = \sin\theta + \frac{\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + \frac{\sin^7\theta}{7} + \&c.,$$

an arithmetical value for $\sin\theta$. This we shall find to be $\sin\theta = \frac{e^1 - e^{-1}}{e^1 + e^{-1}}$, e being the number called the base of the naperian logarithms. Hence $\sec\theta + \tan\theta = e$; or, if we write ε for this particular value of θ to distinguish it from every other,

$$\sec\varepsilon + \tan\varepsilon = e \ldots\ldots\ldots\ldots\ldots (345).$$

We are thus (for the first time it is believed) put in possession of the geometrical origin of that quantity so familiarly known to mathematicians, the naperian base. From the above equations we may derive

$$\sec\varepsilon = \frac{e^1 + e^{-1}}{2}, \quad \tan\varepsilon = \frac{e^1 - e^{-1}}{2} \ldots\ldots\ldots\ldots (346),$$

$$\text{or } \tan\varepsilon = 1{\cdot}175203015, \text{ whence } \varepsilon = {\cdot}8657606,$$

$$\text{or } \varepsilon = 49^\circ\ 36'\ 15''.$$

The corresponding arc of the parabola will be found to be

$$(k.\theta) = k\left[1 + \frac{2^1}{123} + \frac{2^3}{12345} + \frac{2^5}{1234567}\ \&c.\right] \ldots (347).$$

If we assume the theory of logarithms as known, we may at once arrive at this value, for in general

$$\int \frac{d\theta}{\cos\theta} = \log(\sec\theta + \tan\theta);$$

and as this is to be 1, we must have $\sec\theta + \tan\theta = e$, as before.

If we now extend this inquiry, and ask, what is the magnitude of the amplitude of the arc of the parabola which shall render the difference between the parabolic arc and its protangent equal to n times the distance between the focus and the vertex, we shall have, as before, by the terms of the question,

$$(k.\theta) - g\sec\theta\tan\theta = ng \ldots\ldots\ldots\ldots\ldots (348).$$

But, in general,

$$(k.\theta) - g\sec\theta\tan\theta = g\int \frac{d\theta}{\cos\theta};$$

hence we must have

$$n = \int \frac{d\theta}{\cos\theta} = \log(\sec\theta + \tan\theta), \text{ or } \sec\theta + \tan\theta = e^n \ldots (349).$$

Now we may solve this equation in two ways, either by making n a given number, and then determine the value of $\sec\theta + \tan\theta$, which may be called the *base*. Or we may assign an arbitrary value to $\sec\theta + \tan\theta$, and then derive the value of n. Taking the latter course, let, for example,

$$\sec\theta + \tan\theta = 10, \quad \text{then } n = \log 10,$$

or $\frac{1}{n}$ is the modulus of the second system of logarithms. Hence, if we assume any number of systems of logarithms on the same parabola, and take their bases

$$(\sec\theta + \tan\theta), \quad (\sec\theta' + \tan\theta'), \quad (\sec\theta'' + \tan\theta''), \ldots \text{ \&c.},$$

the moduli of these successive systems will be the ratios of half the semiparameter to the successive differences between the base parabolic arcs and their protangents.

In the naperian system, g the distance from the focus to the vertex of the parabola, is taken as 1. The difference between the parabolic arc and its protangent when equal to g, gives $g(\sec\theta + \tan\theta) = eg$. In the decimal system $g(\sec\theta_{,} + \tan\theta_{,}) = 10g$, and the difference between the corresponding parabolic arc and its protangent being ng, if we

Hence, while e^1 is the *parabolic* base, $e^{\sqrt{(-1)}}$ is the *circular* base. Or as $[\sec \varepsilon + \tan \varepsilon]$ is the naperian base, $[\cos(1) + \sqrt{(-1)}\sin(1)]$ is the *circular* or imaginary base. Thus

$$[\cos(1) + \sqrt{(-1)}\,\sin(1)]^{\vartheta} = \cos\vartheta + \sqrt{(-1)}\,\sin\vartheta.$$

Hence, speaking more precisely, imaginary numbers have real logarithms, but an imaginary base. We may always pass from the real logarithms of the parabola, to the imaginary logarithms of the circle, by changing $\tan\theta$ into $\sqrt{(-1)}\sin\vartheta$, $\sec\theta$ into $\cos\vartheta$, and e^1 into $e^{\sqrt{(-1)}}$.

As in the parabola, the angle θ is non-periodic, its limit being $\frac{1}{2}\pi$, while in the circle ϑ has no limit, it follows that while a number can have only one real or *parabolic* logarithm, it may have innumerable imaginary or *circular* logarithms.

In the parabola we thus can shew the geometrical origin of the magnitudes known as the base and the modulus. We might too form systems of circular trigonometry analogous to different systems of logarithms. We might refer the arc of a circle not to the radius, but to some other arbitrary fixed line, the diameter or any other suppose. Let the circumference be referred to the diameter, then π will signify a whole circumference instead of a semicircle, and $\frac{1}{4}\pi$ will represent a right angle. Having on this system, or any similar one, found the lengths of the arcs which correspond to certain functions, such as given sines or tangents, we should multiply the results by some fixed number, which we might call a modulus (2 in this example), to reduce them to the standard system; but such systems would obviously be useless.

If ε be the angle which gives the difference between the parabolic arc and its protangent equal to $g = \frac{1}{2}k$; $(\varepsilon \dotplus \varepsilon)$ is the angle which will give this difference equal to $2g$, $(\varepsilon \dotplus \varepsilon \dotplus \varepsilon)$ is the angle which will give this difference equal to $3g$, and so on to any number of angles. Hence, in the circle, if ϑ be the angle which gives the circular arc equal to the radius, 2ϑ is the angle which will give an arc equal to twice the radius, and so on for any number of angles. This is of course self-evident in the case of the circle, but it is instructive to point out the complete analogy which holds in the trigonometries of the circle and of the parabola.

The geometrical origin of the exponential theorem may thus be shewn.

Assume two known logarithmic bases $(\sec\alpha + \tan\alpha)$, and $(\sec\beta + \tan\beta)$, and let us investigate the ratio of the differences of the corresponding parabolic arcs and their protangents.

Let $\sec\varepsilon + \tan\varepsilon$ be the naperian base, and let one difference be xg and the other yg. The ratio of these differences is therefore $\dfrac{y}{x} = z$, if we make $y = xz$. Hence

$$\sec\alpha + \tan\alpha = (\sec\varepsilon + \tan\varepsilon)^x = e^x, \text{ and } (\sec\beta + \tan\beta) = e^y.$$

Therefore $\quad (\sec\alpha + \tan\alpha)^y = e^{xy} = (\sec\beta + \tan\beta)^x.$

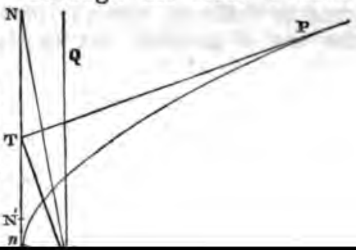
Or, as $y = xz$, $\quad (\sec\alpha + \tan\alpha)^z = \sec\beta + \tan\beta.$

Let A be the first base, and B the second. Then $B = A^z$. This is the exponential theorem.

Let A be the naperian base, then $x = 1$, and $A = e$. Hence $B = e^y$.

Given a number to find its logarithm, may be exhibited by the following geometrical construction.

Let OAP be a parabola. Through the focus O draw the perpendicular OQ to the axis AO. Through A let a tangent of indefinite length be drawn. On this tangent take the line AN to represent the given number. Join NO, and make the angle NOT always equal to the angle NOQ. Draw TP at right



When the number is 0, n coincides with A, and the angle NOQ in this case is a right angle. Therefore the point T' will be the intersection of AT' and OQ. Hence T' is at an infinite distance below the axis, and therefore the logarithm of $+0$ is $-\infty$.

Hence negative numbers have no logarithms, at least no real ones; and imaginary ones can only be educed by the transformation so often referred to, and this leads us to seek them among the properties of the circle. For as θ always lies between 0 and a right angle, or between 0 and the half of $\pm\pi$, $\sec\theta \pm \tan\theta$ is *always* positive; hence *negative* numbers can have no real or *parabolic* logarithms, but they may have imaginary or *circular* logarithms; for in the expression $\log\{\cos\vartheta + \sqrt{(-1)}\sin\vartheta\} = \vartheta\sqrt{(-1)}$, we may make $\vartheta = (2n+1)\pi$, and we shall get $\log(-1) = (2n+1)\pi\sqrt{(-1)}$.

Hence also, as the length of the parabolic arc TP, without reference to the sign, depends solely on the amplitude θ, it follows that the logarithm of $\sec\theta - \tan\theta$ is equal to the logarithm of $\sec\theta + \tan\theta$. As $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$, we may hence infer, that the logarithm of any number is equal to the logarithm of its reciprocal, with the sign changed.

When θ is very large, $\sec\theta + \tan\theta = 2\tan\theta$, nearly. Hence if we represent a large number by an ordinate of a parabola whose focal distance to the vertex is 1, the difference between the corresponding arc and its protangent will represent its logarithm.

Along the tangent to the vertex of the parabola, as in the preceding figure, draw, measured from the vertex, a series of lines in geometrical progression,

$$g(\sec\theta + \tan\theta),\ g(\sec\theta + \tan\theta)^2,$$
$$g(\sec\theta + \tan\theta)^3 \ldots\ldots g(\sec\theta + \tan\theta)^n.$$

Join N, the general representative of the extremities of these right lines, with the focus O. Erect the perpendicular OQ, and make the angle NOT *always* equal to the angle NOQ. The line OT will be $= g\sec\theta$, the line $OT_{,} = g\sec(\theta \dotplus \theta)$, the line $OT_{,,} = g\sec(\theta \dotplus \theta \dotplus \theta)$, &c., and we shall likewise have

$$AT = g\tan\theta,\ AT_{,} = g\tan(\theta \dotplus \theta),\ AT_{,,} = g\tan(\theta \dotplus \theta \dotplus \theta), \text{ \&c.}$$

This follows immediately from (342); for any integral power of $(\sec\theta + \tan\theta)$ may be exhibited as a linear function of $\sec\Theta + \tan\Theta$. If $\Theta = \theta \dotplus \theta \dotplus \theta \ldots$ &c., since

$$\sec(\theta \dotplus \theta \dotplus \theta \dotplus \theta \text{ \&c. to } n\theta) + \tan(\theta \dotplus \theta \dotplus \theta \dotplus \theta \text{ \&c. to } n\theta) = (\sec\theta + \tan\theta)^n.$$

Hence the parabola enables us to give a graphical construction for the angle $(\theta \dot{-} \theta \dot{-} \&c.)$ as the circle does for the angle $(\theta + \theta + \&c.)$.

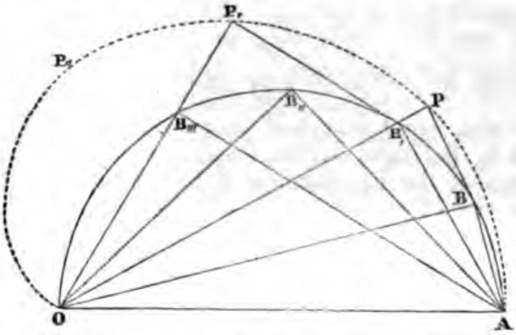
The analogous theorem in the circle may be developed as follows:—In the circle OBA take the arcs

$$AB = BB_{,} = B_{,}B_{,,} = B_{,,}B_{,,,} \ldots \&c. = 2\vartheta.$$

Let the diameter be G; then

$$OB = G\cos\vartheta,\ OB_{,} = G\cos 2\vartheta,\ OB_{,,} = G\cos 3\vartheta \ldots \&c.$$

and $AB = G\sin\vartheta,\ AB_{,} = G\sin 2\vartheta,\ AB_{,,} = G\sin 3\vartheta \ldots \&c.$



Now as the lines in the second group are always at right angles to those in the first, and as such a change is denoted

tion in (33), we shall have the arc

$$AB = s = 2G\int \cos^2\vartheta d\vartheta - G\sin\vartheta\cos\vartheta.$$

Make the imaginary transformations $\cos\vartheta = \sec\theta$, and $\sin\vartheta = \sqrt{(-1)}\tan\theta$, and we shall have

$$\frac{s}{G\sqrt{(-1)}} = 2\int\frac{d\theta}{\cos^3\theta} - \sec\theta\tan\theta,$$

the expression for an arc of a parabola, diminished by its protangent.

The protangent to the circle, which is exhibited in this formula, disappears in the actual process of integration; while in the parabola, the protangent which is involved in the differential, is evolved by the process of integration.

As in the parabola, the perpendicular, from the focus on the tangent, bisects the angle between the radius vector and the axis of the curve; so in the circle, the radius vector OB drawn from the extremity of the diameter, bisects the angle between the perpendicular OP and the diameter OA.

There are some curious analogies between the parabola and the circle, considered under this point of view.

In the parabola, the points $T, T_{,}, T_{,,}$, which divide the lines

$$g(\sec\theta + \tan\theta), \quad g[\sec(\theta \dotplus \theta) + \tan(\theta \dotplus \theta)]$$

into their component parts, are upon tangents to the parabola. The corresponding points $B, B_{,}, B_{,,}$ in the circle, are on the circumference of the circle.

In the parabola the extremities of the lines $g(\sec\theta + \tan\theta)$ are on a right line AT; in the circle the extremities of the bent lines are all in the point A.

The locus of the point T, the intersections of the tangents to the parabola with the perpendiculars from the focus, is a right line; or in other words, while one end of a protangent rests on the parabola, the other end rests on a right line. So in the circle, while one end of the protangent rests on the circle, the other end rests on a *cardioide*, whose diameter is equal to that of the circle, and whose cusp is at O. $OP_{,}A$ is the cardioide.

The length of the tangent AN to any point N is

$$g(\sec\theta + \tan\theta) = 2g\tan\theta,$$

when θ is very large. The length of the cardioide is $2G\sin\vartheta$.

It is singular that the imaginary formulæ in trigonometry have long been discovered, while the corresponding real expressions have escaped notice. Indeed, it was long ago

observed by Bernoulli, Lambert, and by others—the remark has been repeated in almost every treatise on the subject since—that the ordinates of an equilateral hyperbola might be expressed by real exponentials, whose exponents are sectors of the hyperbola, but the analogy, being illusory, never led to any useful results. And the analogy was illusory from this, that it so *happens* the length and area of a circle are expressed by the *same* function, while the area of an equilateral hyperbola is a function of an arc of a parabola. The true analogue of the circle is the parabola.

Let $\bar{\omega}$ be the conjugate amplitude of ω and ψ, while ω is the conjugate amplitude, as before, of ϕ and χ. Then, as

$$\int\frac{d\bar{\omega}}{\cos\bar{\omega}}=\int\frac{d\omega}{\cos\omega}+\int\frac{d\psi}{\cos\psi},\ \text{and}\int\frac{d\omega}{\cos\omega}=\int\frac{d\phi}{\cos\phi}+\int\frac{d\chi}{\cos\chi},$$

we shall have

$$\int\frac{d\bar{\omega}}{\cos\bar{\omega}}=\int\frac{d\phi}{\cos\phi}+\int\frac{d\chi}{\cos\chi}+\int\frac{d\psi}{\cos\psi};$$

and if $(k.\bar{\omega})$, $(k.\phi)$, $(k.\chi)$, and $(k.\psi)$ are four corresponding parabolic arcs,

$$(k.\bar{\omega})-(k.\phi)-(k.\chi)-(k.\psi)=k\tan(\phi\perp\chi)\tan(\phi\perp\psi)\tan(\chi\perp\psi)\ldots(350),$$

which gives a simple relation between four conjugate parabolic arcs.

Let, in the preceding formula, $\phi=\chi=\psi$, and we shall have

The theorem given in (342) is a particular case of this more general theorem

$$\sec(\alpha \dotplus \beta \dotplus \gamma \dotplus \delta \dotplus \&c.) + \tan(\alpha \dotplus \beta \dotplus \gamma \dotplus \delta \dotplus \&c.)$$
$$= (\sec\alpha + \tan\alpha)(\sec\beta + \tan\beta)(\sec\gamma + \tan\gamma)(\sec\delta + \tan\delta)\ \&c.$$

We might pursue this subject very much further; but enough has been done to show the analogy which exists between the trigonometry of the circle and that of the parabola. As the calculus of angular magnitude has always been referred to the circle as its type, so the calculus of logarithms may, in precisely the same way, be referred to the parabola as its type.

The obscurities, which hitherto have hung over the geometrical theory of logarithms, have it is hoped been now removed. It is possible to represent logarithms, as elliptic integrals usually have been represented, by curves devised to exhibit some special property only; and accordingly, such curves, while they place before us the properties they have been devised to represent, fail generally to carry us any further. The close analogies which connect the theory of logarithms with the properties of the circle will no longer appear inexplicable*.

ON ELECTRODYNAMIC INDUCTION.

By Riccardo Felici.

[*Extracted from a Letter to the Editor.*]

* * * Monsieur Tortolini m'a écrit que vous voudriez bien insérer, par extrait, mes travaux dans votre Journal accrédité. Je vous remercie infiniment de l'offre, de laquelle je profite dés ce moment; en vous priant d'accueillir l'extrait sui-

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* The views above developed, on the trigonometry of the parabola, throw much light on a controversy long carried on between Leibnitz and J. Bernoulli on the subject of the logarithms of negative numbers. Leibnitz insisted they were imaginary, while Bernoulli argued they were real, and the same as the logarithms of equal positive numbers. Euler espoused the side of the former, while D'Alembert coincided with the views of Bernoulli. Indeed, if we derive the theory of logarithms from the properties of the hyperbola (as geometers always have done), it will not be easy satisfactorily to answer the argument of Bernoulli—that as an hyperbolic area represents the logarithm of a positive number, denoted by the positive abscissa $+x$, so a negative number, according to conventional usage, being represented by the negative abscissa $-x$, the corresponding hyperbolic area should denote its logarithm also. All this obscurity is cleared up by the theory developed in the text, which completely establishes the correctness of the views of Leibnitz and Euler.

vant, d'un mémoire qui sera publié prochainement dans les *Annales de l'Université de Toscane.*

Mémoire sur l'Induction Electro-Dynamique. (*Extrait*).

A' l'aide d'un nouveau méthode expérimental que l'on trouverá decrit dans les *Annales des Sciences,* publiés á Rome, par M. le Professeur Tortolini, année 1851, il est facile d'établir avec toute certitude les faits suivants.

1. La force des courants induits en ouvrant, ou bien en fermant, le circuit de la pile, est simplement proportionelle à celle des courants inducteurs.

2. Le théorème relatif au conducteur sinueux, énoncé dans la théorie d'Ampère, se vérifie aussi dans le cas de l'induction.

3. Dans le cas de deux anneaux, dont l'un est l'induit et l'autre l'inducteur, égaux, parallèles et avec leurs centres sur la même droite normale à leurs plans, la force des courants induits, en interrompant le circuit de la pile, croit proportionellement aux diamètres, lorsque le rapport qui existe entre les distances des ces plans et les diamétres des mêmes anneaux, est une quantité constante.

4. La somme A de tous les courants induits sur un circuit conducteur par un circuit voltaïque, fermé et en mouvement, pendant que ce dernier circuit passe d'une

où r dénote la distance des éléments; a une quantité proportionelle à la force de la pile; k, n constantes qui doivent être déterminées par l'expérience. Mais la valeur de n est facile a connaître en vertu du 3^e fait; et par un calcul très-facile on trouve $n = 1$; c'est-à-dire la formule

$$d^2E = -a\left(\frac{d^2r}{ds.ds'} + \frac{k}{r}\frac{dr}{ds}\cdot\frac{dr}{ds'}\right)ds.ds^2 \ldots\ldots (2).$$

On peut maintenant remarquer, que le premier terme de son second membre, disparait dans les intégrations, pour un circuit fermé, et l'on ecrira

$$d^2E = -\frac{ak}{r}\cdot\frac{dr}{ds}\cdot\frac{dr}{ds'}.ds.ds^2 \ldots\ldots\ldots\ldots (3).$$

Pour le cas des courants induits par le mouvement du circuit inducteur, on voit, très-clairement que le fait 4^e resout le problême sans ajouter à l'analyse aucune nouvelle difficulté.

La formule (3) doune des resultats assez simples lorsque on l'applique au cas du magnétisme dans l'hypothése de Ampère.

Voilà, Monsieur, ce que j' avais a vous communiquer de plus important dans le dit mémoire. * * *

Pisa, March 30, 1852.

ON THE INDEX SYMBOL OF HOMOGENEOUS FUNCTIONS.

By Robert Carmichael, A.M., Fellow of Trinity College, Dublin.

[*Concluded from* Vol. vii. p. 284.]

In the first article of the following paper, the Index Symbol is employed to illustrate a useful general theorem in the Calculus of Operations. In the second and third articles, the general applicability of the same symbol to the integration of large classes of differential equations, ordinary and partial, is exhibited and illustrated. In the fourth article, this symbol is employed for the solution of systems of simultaneous partial differential equations, a department of the Integral Calculus as yet comparatively unnoticed, and possessing much that will interest the mathematician and physicist. Finally, in the fifth article, certain excep-

tional cases are discussed, which will occur in this, as in any other, method of integration.

1. ψ being a distributive symbol such that

$$\psi.uv = u\psi v + v\psi u,$$

it can be readily proved that

$$e^{\psi}.uv = e^{\psi}u.e^{\psi}v,$$

whence it follows that

$$e^{\psi}.uvw\ldots = e^{\psi}u.e^{\psi}v.e^{\psi}w\ldots$$

Hence $$e^{\psi}.u^{n} = (e^{\psi}u)^{n},$$

and therefore, if F denote any algebraic function,

$$e^{\psi}.F(u) = F(e^{\psi}u).$$

This valuable theorem is due to the Rev. Prof. Graves.

Now the distributive symbol

$$x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} + \&c. = \nabla,$$

satisfies the above law, and therefore

$$e^{\Theta\nabla}.F(U) = F(e^{\Theta\nabla}U) \ldots\ldots\ldots\ldots\ldots (\text{I.}),$$

where Θ and U are any functions whatsoever of xyz &c.

we obtain the remarkable theorem

$$e^{\nabla}.F(U) = F(u_0 + eu_1 + e^2u_2 + \&c. + e^n u_n).$$

As a second example of the general theorem (I.) we may investigate the algebraic value of the symbolic quantity

$$e^{\Theta_m \nabla}.F(\Theta_n),$$

where Θ_m, Θ_n are known homogeneous functions of the degrees m, n, respectively. Now

$$e^{\Theta_m \nabla}.\Theta_n = \left\{1 + n\Theta_m + \frac{n(n+m)}{1.2}\Theta_m^2 + \&c.\right\}\Theta_n = \frac{\Theta_n}{(1-m\Theta_m)^{\frac{n}{m}}},$$

and therefore $e^{\Theta_m \nabla}.F(\Theta_n) = F\left\{\dfrac{\Theta_n}{(1-m\Theta_m)^{\frac{n}{m}}}\right\}.$

Thus $e^{(ax+by+cz)\nabla}.F(x^2+y^2+z^2) = F\left[\dfrac{x^2+y^2+z^2}{\{1-(ax+by+cz)\}^2}\right].$

2. There is one class of linear differential equations with *constant coefficients*, whose integration, in general, presents insurmountable difficulties to the student. In it the right-hand member or absolute term consists exclusively of exponentials or circular functions, sines, cosines, &c., and may be written in the form $f(e^\theta, \sin\theta, \cos\theta)$.

For the solution of such equations, different processes have been employed, varying with the character of each example, useless in practice when the order of the equation is elevated, and unsuggestive of any susceptibilities of extended application.

It is proposed to shew that, through the instrumentality of the Index Symbol, this class of equations can be integrated by a process simple and uniform, equally susceptible of employment in equations of the higher orders as in those of the lower, and directly indicative, in either case, of a corresponding class of partial differential equations with the appropriate form of solution.

The type of the first class is

$$\frac{d^m y}{d\theta^m} + P\frac{d^{m-1}y}{d\theta^{m-1}} + Q\frac{d^{m-2}y}{d\theta^{m-2}} + \dots + Ty = f(e^\theta, \sin\theta, \cos\theta).$$

This, being reduced to the form

$$F\left(\frac{d}{d\theta}\right)y = \Sigma A e^{a\theta},$$

where a may be positive or negative, fractional or integer, real or imaginary, becomes, by the substitution $x = e^\theta$,

$$F\left(x\frac{d}{dx}\right)y = \Sigma A x^a,$$

and the solution, obtained at once in terms of x, gives in terms of θ

$$y = \Sigma A \frac{e^{a\theta}}{F(a)} + \textit{ord. comp. funct.}$$

and when a is imaginary, we may restore the circular function.

We have said that this class of ordinary differential equations has its analogue amongst partial differential equations, and that the method of solution of the former is directly suggestive of that of the latter.

Thus, the class of partial differential equations whose type is

$$\nabla^m z + P\nabla^{m-1}z + \ldots + Tz = f(e^\theta, e^\phi, \sin\theta, \sin\phi, \cos\theta, \cos\phi),$$

where $$\nabla = \frac{d}{d\theta} + \frac{d}{d\phi}$$

can be thrown into the form

$$F(\nabla)\, z = \Sigma A_{a,b}\, e^{a\theta + b\phi};$$

When there are α equal roots whose common value is n, its form is

$$\psi_n(e^\theta, e^\phi).(\theta+\phi)^{\alpha-1} + \chi_n(e^\theta, e^\phi).(\theta+\phi)^{\alpha-2} + \&c. + \psi_r(e^\theta, e^\phi) + \&c.,$$

where ψ_n, χ_n, &c. are different arbitrary homogeneous functions of the degree n.

Finally, when there are pairs of imaginary roots, the form of the arbitrary portion of the solution is

$$\psi_{n+p\sqrt{(-1)}}(e^\theta, e^\phi) + \psi_{n-p\sqrt{(-1)}}(e^\theta, e^\phi) + \&c. + \psi_q(e^\theta, e^\phi) + \&c.$$

We proceed to furnish some illustrations of the above method of solution, which would seem to establish its value as a practical good. The equations proposed for solution are selected from *Gregory's Examples*.

(I.) $$\frac{d^2y}{d\theta^2} - 2m\frac{dy}{d\theta} + m^2y = \sin a\theta.$$

The method of investigation given for the solution of this simple equation is extremely artificial, and seemingly unsusceptible of extension.

By the transformation $x = e^\theta$, it becomes

$$\left(x\frac{d}{dx} - m\right)^2 y = \frac{1}{2\sqrt{(-1)}}\left\{x^{a\sqrt{(-1)}} - x^{-a\sqrt{(-1)}}\right\};$$

therefore

$$y = \frac{1}{2\sqrt{(-1)}}\left[\frac{x^{a\sqrt{(-1)}}}{\{a\sqrt{(-1)}-m\}^2} - \frac{x^{-a\sqrt{(-1)}}}{\{a\sqrt{(-1)}+m\}^2}\right] + c_1x^m\log x + c_2x^m;$$

or, replacing the circular function,

$$y = \frac{(m^2-a^2)\sin a\theta + 2ma\cos a\theta}{(m^2+a^2)^2} + e^{m\theta}(c_1\theta + c_2).$$

By the transformations $x = e^\phi$, $y = e^\psi$, and a repetition of the same precise process which we have now employed, can be obtained the solution of the partial differential equation

$$\left(\frac{d^2z}{d\phi^2} + 2\frac{d^2z}{d\phi\, d\psi} + \frac{d^2z}{d\psi^2}\right) - 2m\left(\frac{dz}{d\phi} + \frac{dz}{d\psi}\right) + m^2z = \sin(a\theta + b\phi),$$

in the form

$$z = \begin{cases} \dfrac{\{m^2-(a+b)^2\}\sin(a\theta+b\phi) + 2m(a+b)\cos(a\theta+b\phi)}{\{m^2+(a+b)^2\}^2} \\ \qquad + \Phi_m(e^\phi, e^\psi).(\phi+\psi), \quad + \Psi_m(e^\phi, e^\psi), \end{cases}$$

where Φ_m, Ψ_m are different arbitrary homogeneous functions of the m^{th} degree.

(II.) $$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}.$$

Expanding the right-hand member, this becomes

$$\left(x\frac{d}{dx}+1\right)^2 y = 1+2x+3x^2+\&\text{c.};$$

therefore $$y=\left(1+\frac{x}{2}+\frac{x^2}{3}+\&\text{c.}\right)+\frac{c_1}{x}\log x+\frac{c_2}{x},$$

or $$y=\log\left(\frac{1}{1-x}\right)^{\frac{1}{x}}+\frac{c_1}{x}\log x+\frac{c_2}{x}.$$

Similarly, the solution of the partial differential equation

$$(x^2r+2xys+y^2t)+3(xp+yq)+z=\frac{1}{(1-\Theta_1)^2},$$

where Θ_1 is a given homogeneous function of the first degree in x, y, is, by the second fundamental theorem,

$$z=\log\left(\frac{1}{1-\Theta_1}\right)^{\frac{1}{\Theta_1}}+u_{-1}(\log x+\log y)+v_{-1},$$

where u_{-1}, v_{-1} are different arbitrary homogeneous functions in x, y, of the degree -1.

(III.) $$x\frac{dw}{dx}+y\frac{dw}{dy}+z\frac{dw}{dz}-aw=\frac{xy}{\ldots}$$

$$\text{(IV.)}\quad x^n \frac{d^n z}{dx^n} + n x^{n-1} y \frac{d^n z}{dx^{n-1}dy} + \frac{n(n-1)}{1.2} x^{n-2} y^2 \frac{d^n z}{dx^{n-2}dy^2} + \ldots = 0.$$

By the third article of the paper, to which the writer has already taken the liberty of referring, it appears most readily that this equation is susceptible of the symbolic shape

$$\nabla(\nabla - 1)(\nabla - 2)\ldots(\nabla - n + 1)\, z = 0.$$

Consequently its solution is, at once,

$$z = u_0 + u_1 + u_2 + \ldots + u_{n-1}.$$

More generally, the solution of the equation

$$x^n \frac{d^n z}{dx^n} + n x^{n-1} y \frac{d^n z}{dx^{n-1}dy} + \frac{n(n-1)}{1.2} x^{n-2} y^2 \frac{d^n z}{dx^{n-2}dy^2} + \ldots = \Theta_a + \Theta_b,$$

is

$$z = \frac{\Theta_a}{a(a-1)\ldots(a-n+1)} + \frac{\Theta_b}{b(b-1)\ldots(b-n+1)} + u_0 + u_1 + \ldots + u_{n-1}.$$

The simplicity of the method employed in this last example, when compared with the artificial and laborious processes which have hitherto been employed for its solution, seems to exhibit, in a remarkable degree, the power of the Index Symbol as an instrument of integration, and the facility with which it admits of manipulation. It is obvious that (11) in the Examples is only a particular case of the general theorem now established.

3. Hitherto we have confined our attention to the integration of classes of differential equations, ordinary and partial, in which the coefficients are constants. There are, however, two classes of equations in which the coefficients are symmetric functions of the variables, which can easily be reduced to those discussed.

In the first, the coefficients are symmetric functions of the independent variables. Its type is

$$\left.\begin{aligned} &A\left\{(m+\lambda x)^\alpha \frac{d^\alpha z}{dx^\alpha} + \alpha(m+\lambda x)^{\alpha-1}(n+\lambda y)\frac{d^\alpha z}{dx^{\alpha-1}dy}\right.\\ &\qquad\left. + \frac{\alpha(\alpha-1)}{1.2}(m+\lambda x)^{\alpha-2}(n+\lambda y)^2 \frac{d^\alpha z}{dx^{\alpha-2}dy^2} + \&c.\right\}\\ &\qquad\qquad + \\ &B\left\{(m+\lambda x)^\beta \frac{d^\beta z}{dx^\beta} + \beta(m+\lambda x)^{\beta-1}(n+\lambda y)\frac{d^\beta z}{dx^{\beta-1}dy}\right.\\ &\qquad\left. + \frac{\beta(\beta-1)}{1.2}(m+\lambda x)^{\beta-2}(n+\lambda y)^2 \frac{d^\beta z}{dx^{\beta-2}dy^2} + \&c.\right\}\\ &\qquad\qquad + \&c. \end{aligned}\right\} = \Omega.$$

Making the substitutions

$$m + \lambda x = \lambda x', \quad n + \lambda y = \lambda y',$$

and breaking up Ω into sets of homogeneous functions, its solution can be obtained at once. The corresponding class of ordinary differential equations is most easily solved in this way, and the transformation employed for its solution by Legendre (*Mémoires de l'Academie,* 1787) rendered unnecessary.

In the second class, the coefficients are symmetric functions of the *dependent* variable. Its type is

$$Az^{m-\alpha}\nabla^{\alpha}z + Bz^{m-(\alpha-1)}\nabla^{\alpha-1}z + \&c. = \Theta_n + \Theta_p + \&c.,$$

where Θ_n, Θ_p, &c. are homogeneous functions of x, y, of the degrees n, p, &c. respectively. Putting

$$z^m = z',$$

the equation is obviously reducible to the form

$$A'\nabla^{\alpha}z' + B'\nabla^{\alpha-1}z' + \&c. = \Theta_m + \Theta_n + \&c.;$$

the solution of which can be at once obtained by the method furnished in the paper before quoted.

Thus, the equation

$$z^{m-2}(x^2r + 2xys + y^2t) + Bz^{m-1}(xp + yq) + Cz^m = \Theta_n + \Theta_p$$

becomes

$$\left\{\frac{\nabla(\nabla - 1)}{m(m-1)} + B\frac{\nabla}{m} + C\right\} z^m = \Theta_n + \Theta_p.$$

4. We may, in some cases, employ the Index Symbol to great advantage in the investigation of the solutions of systems of partial differential equations. The results exhibit themselves in a remarkably symmetrical and elegant form.

Thus, if we had the system

$$\left.\begin{aligned} r &= f_1(x, y), \\ s &= f_2(x, y), \\ t &= f_3(x, y), \end{aligned}\right\}$$

multiply the first equation by x^2, the second by $2xy$, the third by y^2, and adding, we get

$$x^2 r + 2xys + y^2 t = x^2 f_1 + 2xyf_2 + y^2 f_3.$$

Break up the right-hand member, as before, into sets of homogeneous functions, and the whole assumes the symbolic shape

$$\nabla(\nabla - 1)z = \Theta_m + \Theta_n + \Theta_p + \&c.;$$

and the required solution is

$$z = \frac{\Theta_m}{m(m-1)} + \frac{\Theta_n}{n(n-1)} + \&c. + u_0 + u_1,$$

where u_0, u_1 are arbitrary homogeneous functions of the degrees 0 and 1 respectively.

The apparent method of solving such a system would be, to integrate the first equation twice with respect to x, supposing y constant, thereby introducing two arbitrary functions of y; to integrate the second equation alternately with respect to x and y, thereby introducing two more arbitrary functions, the one of y and the other of x; and to integrate the third equation with respect to y twice, thereby introducing two more arbitrary functions of x. Finally, by a comparison of the solutions thus got, we should endeavour to determine the characters of the resultant arbitrary functions as far as possible.

It is obvious that our method of solution will apply to the system

$$\left.\begin{aligned} \frac{d^2z}{d\phi^2} &= f_1(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi), \\ \frac{d^2z}{d\phi\, d\psi} &= f_2(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi), \\ \frac{d^2z}{d\psi^2} &= f_3(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi), \end{aligned}\right\},$$

as well as to many others which readily suggest themselves.

5. It is indispensable that we should discuss an exceptional case, which will sometimes occur in the employment of the present, as of any other, method of integration.

This arises from the circumstance that the inverse process may generate an infinite coefficient, and can be illustrated by the partial differential equation

$$x\frac{dz}{dx} + y\frac{dz}{dy} - az = \Theta_m.$$

The solution of this equation, as given by our method, is

$$z = \frac{\Theta_m}{m-a} + u_a;$$

in which, when $a = m$, the first term becomes *infinite*.

To clear away this difficulty, assume, in the general solution,

$$u_a = v_a - \frac{\Theta_a}{m-a},$$

which gives

$$z = \frac{\Theta_m - \Theta_a}{m-a} + v_a.$$

This becomes indeterminate when $a = m$; therefore, differentiating with respect to m both numerator and denominator, and remembering that

$$x^m f\left(\frac{y}{x}\right) + y^m F\left(\frac{x}{y}\right)$$

[redacted]

Again, the solution of

$$a\frac{dz}{dx}+b\frac{dz}{dy}=c$$

is
$$z=\frac{c}{2}\left(\frac{x}{a}+\frac{y}{b}\right)\phi_0(e^{\frac{x}{a}}, e^{\frac{y}{b}}).$$

The integral of this equation, as given by Gregory, is incomplete; and the same remark will apply to his solutions of the equations numbered in the *Examples* (7), (9), (26). Such a result might indeed be anticipated from the method of integration employed. It is, of course, *a priori* obvious that, as these partial differential equations are more or less symmetrical in x and y, their solutions should possess the same character.

Particular instances of the occurrence of such exceptional cases are furnished by the ordinary differential equations

$$\frac{dy}{d\theta}-ay=ce^{m\theta},\quad \text{or}\quad x\frac{d}{dx}-ay=cx^m,$$

and $\dfrac{d^2y}{d\theta^2}+a^2y=\cos m\theta,\quad$ or $\left(x\dfrac{d}{dx}\right)^2y+a^2y=\frac{1}{2}(x^{m\sqrt{(-1)}}+x^{-m\sqrt{(-1)}}),$

when $a=m$.

As regards this latter example, it may be observed that the partial differential equation corresponding is

$$\frac{d^2z}{d\theta^2}+2\frac{d^2z}{d\theta d\phi}+\frac{d^2z}{d\phi^2}+a^2z=\cos(m\theta+n\phi),$$

and its solution

$$z=\frac{\cos(m\theta+n\phi)}{a^2-(m+n)^2}+\psi_{a\sqrt{(-1)}}(e^\theta, e^\phi)+\psi_{-a\sqrt{(-1)}}(e^\theta, e^\phi).$$

The partial differential equation corresponding to the exceptional case is

$$\frac{d^2z}{d\theta^2}+2\frac{d^2z}{d\theta d\phi}+\frac{d^2z}{d\phi^2}+(m+n)^2z=\cos(m\theta+n\phi),$$

and its solution

$$z=\frac{\sin(m\theta+n\phi)}{2(m+n)}\cdot\frac{\theta+\phi}{2}+\psi_{(m+n)\sqrt{(-1)}}(e^\theta, e^\phi)+\psi_{-(m+n)\sqrt{(-1)}}(e^\theta, e^\phi).$$

The methods of integration hitherto in use seem inadequate to contend with the difficulties which the integration of such an equation would present.

Trinity College, Dublin,
October 1852.

SOLUTIONS OF TWO PROBLEMS.

By Arthur Cohen, Magdalene College, Cambridge.

The following problem is not without historical interest, and the solution given in Gregory's *Examples* is rather complicated.

"To find a point within a given triangle, such that the sum of the distances of the point from the three vertices of the triangle may be a minimum."

Let (a_1b_1), (a_2b_2), (a_3b_3) be the coordinates of the three vertices A_1, A_2, A_3 of the triangle. Let x, y be the coordinates of the required point X. Let $XA_1 = r_1$, $XA_2 = r_2$, $XA_3 = r_3$.

Let θ_1, θ_2, θ_3 be the angles made by r_1, r_2, r_3, respectively, with the axis of x; and let, finally, ϕ_1, ϕ_2, ϕ_3 denote the angles between (r_2r_3), (r_3r_1), (r_1r_2), respectively, then

$$u=\sqrt{\{(x-a_1^2)+(y-b_1^2)\}}+\sqrt{\{(x-a_2^2)+(y-b_2^2)\}}+\sqrt{\{(x-a_3^2)+(y-b_3^2)\}}$$

is to be a minimum, therefore

$$\frac{du}{dx}=0, \quad \frac{du}{dy}=0,$$

The theorem that the geometrical mean of n positive quantities is less than their arithmetical mean has been proved in *Liouville's Journal* by means of the Differential Calculus. If, however, the Differential Calculus be employed, the following proof seems to be shorter than the one given in the French journal.

It may be easily proved that the *absolute* maximum value of $u = (x_1 \ldots x_n)^{\frac{1}{n}}$ with the condition that $x_1 + \ldots + x_n = c = a_1 + \ldots + a_n$ is found by putting $x_1 = x_2 \ldots = x_n$, and it is therefore $\frac{a_1 + \ldots + a_n}{n}$, and any other value being less than the absolute maximum value, we have $(a_1 \ldots a_n)^{\frac{1}{n}}$ is less than $\frac{a_1 + \ldots + a_n}{n}$.

MATHEMATICAL NOTE.

By Professor De Morgan.

In a Letter of Newton to Collins, dated Nov. 8, 1676, there is so remarkable an assertion relative to the extent to which Newton had carried the integral calculus, that the little notice it has received is to be wondered at. Looking at the evidences which the *Principia* offers of the possession of more methods than were ever published, such an assertion is not to be lightly passed over, extraordinary as it may be.

The following extract was published by William Jones, in his *Analysis per Quantitatum Series, &c.*, Lond. 4to. 1711. The most interesting part of this work was reprinted in 1712 in the *Commercium Epistolicum*, so that it fell into comparative neglect. How what follows came to be omitted in the *Com. Epist.* it is difficult to say. Both in date and matter it would have been much to the purpose: it may be that Newton had subsequent doubts as to the correctness of the assertion in all its extent. The whole Letter is in the Macclesfield Collection (vol. II. pp. 403-5), to which the editor has added the following words: "An extract from

this Letter is published in the *Analysis per Quant. Series*, but with interpolations." This is not correct: the extract made by Jones (from which the following is taken) agrees to a syllable with what is printed in the Macclesfield Collection.

"There is no Curve-line express'd by any Equation of three terms, tho' the unknown quantities affect one another in it, or the Indices of their Dignities be surd quantities (suppose $ax^\lambda + bx^\mu y^\sigma + cy^\tau = 0$, where x signifies the Base, y the Ordinate, λ, μ, σ, τ the Indices of the Dignities of x and y, and a, b, c known quantities with their signs + or −,) I say, there is no such Curve-line, but I can, in less than half a quarter of an hour, tell whether it may be Squar'd, or what are the simplest Figures it may be compared with, be those Figures Conic Sections, or others. And then by a direct and short way (I dare say the shortest the nature of the thing admits of for a general one,) I can compare them. This may seem a bold assertion, because it's hard to say a Figure may, or may not, be Squar'd, or Compar'd with another; but it's plain to me by the fountain I draw it from, tho' I will not undertake to prove it to others. The same Method extends to Equations of four Terms, and others also, but not so generally."

Nov. 26, 1851.

4. A perfectly rough sphere is laid upon a perfectly rough plane, inclined at a given angle to the horizon. This plane is then made to revolve uniformly about an axis perpendicular to itself. Determine the motion of the sphere.

5. (*a*) What would be the density and pressure of air at an immense distance from the earth, if the earth were at rest in a space of constant temperature?

(*b*) If the earth and moon were both at rest, at immense distances from one another, in a space of constant temperature, what would be the pressure and density at the surface of the moon? [Work out numerically, taking 0° centrigrade as the constant temperature; 2114 times the weight of a pound at the earth's surface, as the atmospheric pressure at the earth's surface; $\frac{1}{12 \cdot 383}$ lb. the mass of a cubic foot of air at 0° and under that pressure; 7912 miles the diameter of the earth; 2160 miles the diameter of the moon; ·163 the force of gravity at the moon's surface as compared with that at the surface of the earth.]

6. Find the attraction of a solid sphere of which the density varies inversely as the fifth power of the distance from an external point, on any point external or internal; the mutual attraction between two particles varying inversely as the square of their distance.

7. If a ball weighing W be shot from an air-gun, the volume of the barrel of which is U, by means of the expansion of a quantity of air which occupies the space V under the pressure P at the commencement of the motion, the mass of the air being very small compared with that of the ball, and the mass of the ball very small compared with that of the gun; shew that, provided the cooling effect of the expansion be not, during the motion of the ball through the barrel, sensibly compensated by the communication of any heat to the air from the matter round it, and provided there be no sensible loss of effect by friction, the velocity of the ball on leaving the gun is

$$\left\{\frac{2g}{W}\left[PV\frac{1}{k-1}\left\{1-\left(\frac{V}{U}\right)^{k-1}\right\}-\Pi U\right]\right\}^{\frac{1}{2}},$$

where Π denotes the atmospheric pressure, and k the ratio of the specific heat of air under constant pressure to the

specific heat of air in constant volume, which may be taken as a constant quantity. Find the proportion which the work spent in communicating the motion to the ball bears to that spent in producing noise and in overcoming fluid friction near the mouth of the gun.—*St. Peter's College Examination Papers.* Third Year. June 1852.

8. (*a*) If an infinite number of perfectly elastic material points equally distributed through a hollow sphere, be set in motion each with any velocity, shew that the resulting continuous pressure (referred to a unit of area) on the internal surface is equal to one-third of the *vis viva* of the particles divided by the volume of the sphere.—*St. Peter's College Examination Papers.* June 1852.

(*b*) Prove the same proposition for a hollow space of any form.

9. If in the case of homoloidal spaces, we denote the volume formed in space of r dimensions, from $r+1$ next inferior spaces by V; and through each of their $r+1$ points of intersection draw parallel lines meeting the opposite spaces in points, the volume formed from which is V_1; then

$$V_1 = (-1)^{r-1} r V.$$

If the lines pass through the same finite point, give the corresponding formula.

ON THE RATIONALISATION OF CERTAIN ALGEBRAICAL EQUATIONS.

By Arthur Cayley.

Suppose $\quad x + y = 0, \quad x^2 = a, \quad y^2 = b;$

then if we multiply the first equation by 1, xy, and reduce by the two others, we have

$$x + y = 0,$$
$$bx + ay = 0,$$

from which, eliminating x, y,

$$\begin{vmatrix} 1, & 1 \\ b, & a \end{vmatrix} = 0;$$

which is the equation between a and b, Or, considering x, y as quadratic radicals, the rational equation between x, y. So if the original equation be multiplied by x, y, we have

$$a + xy = 0,$$
$$b + xy = 0.$$

Or, eliminating 1, xy,

$$\begin{vmatrix} a, & 1 \\ b, & 1 \end{vmatrix} = 0,$$

which may be in like manner considered as the rational equation between x, y.

The preceding results are of course self-evident, but by applying the same process to the equations

$$x + y + z = 0, \quad x^2 = a, \quad y^2 = b, \quad z^2 + c,$$

we have results of some elegance. Multiply the equation first by 1, yz, zx, xy, reduce and eliminate the quantities x, y, z, xyz, we have the rational equation

$$\left|\begin{array}{cccc} & 1 & 1 & 1 \\ 1 & . & c & b \\ 1 & c & . & a \\ 1 & b & a & . \end{array}\right| = 0.$$

Again, multiply the equation by x, y, z, xyz, reduce and eliminate the quantities 1, yz, zx, xy, the result is

$$\left|\begin{array}{cccc} & a & b & c \\ a & . & 1 & 1 \\ b & 1 & . & 1 \\ c & 1 & 1 & . \end{array}\right| = 0.$$

which is of course equivalent to the preceding one (the two determinants are in fact identical in value), but the form is essentially different. The former of the two forms is that given in my paper "On a theorem in the Geometry of Position," (Old Series, vol. II. p. 270): it was only very recently that I perceived that a similar process led to the latter of the two forms.

Similarly, if we have the equations

$$x + y + z + w = 0, \quad x^2 = a, \quad y^2 = b, \quad z^2 = c, \quad w^2 = d,$$

then multiplying by 1, yz, zx, xy, xw, yw, zw, $xyzw$, reducing and eliminating the quantities,

we have the result

$$\begin{array}{c} \begin{array}{cccccccc} x, & y, & z, & w, & yzw, & zwx, & wxy, & xyz \end{array} \\ \left|\begin{array}{ccc|c|ccc|c} 1 & 1 & 1 & 1 & . & . & . & . \\ \hline . & c & b & . & 1 & . & . & 1 \\ c & . & a & . & . & 1 & . & 1 \\ b & a & . & . & . & . & 1 & 1 \\ \hline d & . & . & a & . & 1 & 1 & \\ . & d & . & b & 1 & . & 1 & \\ . & . & d & c & 1 & 1 & . & \\ \hline . & . & . & . & a & b & c & d \end{array}\right| = 0. \end{array}$$

So if we multiply the equations by x, y, z, w, yzw, zwx, wxy, and xyz, reduce and eliminate the quantities,

[redacted]

I was indebted to Mr. Sylvester for the remark that the above process applies to radicals of a higher order than the second. To take the simplest case, suppose

$$x + y = 0, \quad x^3 = a, \quad y^3 = b.$$

And multiply first by $1, x^2y, xy^2$; this gives

$$\begin{array}{llll} x + y & & & = 0 \\ \cdot & ay & + x^2y^2 & = 0 \\ bx & \cdot & + x^2y^2 & = 0\,; \end{array}$$

or, eliminating,

$$\begin{vmatrix} 1 & 1 & \cdot \\ \cdot & a & 1 \\ b & \cdot & 1 \end{vmatrix} = 0.$$

Next multiply by x, y, x^2y^2; this gives

$$\begin{array}{llll} x^2 & \cdot & + xy & = 0 \\ \cdot & y^2 & + xy & = 0 \\ bx^2 & + ay^2 & \cdot & = 0\,; \end{array}$$

or, eliminating,

$$\begin{vmatrix} 1 & \cdot & 1 \\ \cdot & 1 & 1 \\ b & a & \cdot \end{vmatrix} = 0.$$

And lastly, multiply by x^2, y^2, xy; this gives

$$\begin{array}{llll} a & + x^2y & \cdot & = 0 \\ b & \cdot & + xy^2 & = 0 \\ \cdot & x^2y & + xy^2 & = 0\,; \end{array}$$

or, eliminating,

$$\begin{vmatrix} a & 1 & \cdot \\ b & \cdot & 1 \\ \cdot & 1 & 1 \end{vmatrix} = 0.$$

And it is proper to remark that the second and third forms are not essentially distinct, since the one may be derived from the other by the interchange of lines and columns.

Apply the preceding process to the system

$$x + y + z = 0, \quad x^3 = a, \quad y^3 = b, \quad z^3 = c.$$

First multiply by 1, xyz, $x^2y^2z^2$, x^2z, y^2x, z^2y, x^2y, y^2z, z^2x, reduce and eliminate the quantities,

the result is

x,	y,	z,	y^2z^2,	x^2yz,	y^2zx,	z^2xy,	z^2x^2,	x^2y^2
1	1	1						
			1	1	1			
						a	b	c
.	.	a	1	.	.	.	1	.
b	.	.	.	1	.	.	.	1
.	c	.	.	.	1	1	.	.
.	a	.	1	.	.	.	.	1
.	.	b	.	1	.	1	.	.
c	.	.	.	.	1	.	1	.

$= 0.$

Next multiply by x, y, z, y^2z^2, z^2x^2, x^2y^2, x^2yz, y^2zx, z^2xy, reduce and eliminate the quantities,

the result is

x^2,	y^2,	z^2,	yz,	zx,	xy,	xy^2z^2,	yz^2x^2,	zx^2y^2
1	.	.	.	1	1	.	.	.
.	1	.	1	.	1	.	.	.
.	.	1	1	1	.	.	.	.
.	c	b	.	.	.	1	.	.
c	.	a	.	.	.	.	1	.

$= 0.$

[redacted]

where, as in the case of two cubic radicals, two forms, viz. the first and third forms of the rational equation, are not essentially distinct, but may be derived from each other by interchanging lines and columns.

And in general, whatever be the number of cubic radicals, two of the three forms are not essentially distinct, but may be derived from each other by interchanging lines and columns.

2, *Stone Buildings*,
Dec. 28, 1852.

NOTE ON THE DOCTRINE OF IMPOSSIBLES.

By William Walton.

In a note in page 47 of the last Number of the *Journal*, Mr. Salmon has made certain observations on the question of Geometrical Impossibles, in reply to my paper on the subject in the previous Number. I subjoin in separate paragraphs, for convenience of reference, all his remarks except one, which does not seem to me to bear upon the question at issue.

(1). "I see no reason why we should not close our controversy on the terms of arbitration proposed by Professor De Morgan, namely that Mr. Gregory's conventions shall be banished from the regions of Algebraic Geometry to those of Geometrical Algebra."

(2). "Mr. Walton does not deny the only point for which I am anxious to contend, viz. that the curvilinear loci obtained by Mr. Gregory's rules have *no geometrical connection* with plane curves represented by the same equations."

(3). "And if this be so, they cannot be expected to throw any light on any difficulty, real or supposed, in the theory of plane curves."

(4). "I have only to add that I believe Mr. Walton was hasty in asserting (vol. VII. p. 239) that if $f(x, y) = 0$ be transcendental, a conjugate point, not double but single, may easily present itself."

(5). "And that the case of a conjugate point appearing to have a real tangent is explained by observing that such a point results from the union of two or more ordinary conjugate points."

(1). I quite concur with Mr. Salmon in thinking that we may close our controversy on the terms of arbitration proposed by Professor De Morgan, whose excellent term *Geometrical Algebra* so exactly accords with the distinction, on which I myself most carefully insisted in my own paper, between the two provinces of reasoning.

(2). I neither assert nor deny that there is any *geometrical connection* between the possible and the impossible branches of the locus of an equation involving x and y; conceiving that the one or the other view might be adopted according to the precise sense attached to the phrase.

(3). I am quite ready to admit that all the properties of the possible branches may be thoroughly discussed without any reference to the impossible ones, even where I might not myself see any obvious explanation of an apparent difficulty; just as I should believe it possible to examine adequately the nature of the impossible ones without considering the possible.

(4). Take the equation

$$x + y = x^2\{\alpha^{x\sqrt{(-1)}} + \alpha_1^{x\sqrt{(-1)}} + \alpha_2^{x\sqrt{(-1)}} + \dots + \alpha_n^{x\sqrt{(-1)}}\}.$$

There is a *single* conjugate point at the origin, the equation to the tangent of the one branch through it being $x + y = 0$.

The only objections which, as far as I am aware, could

explained by the idea of the union of two or more conjugate points, as it is always I think interesting, in philosophical inquiries, to contemplate the same phenomenon from different points of view.

Cambridge, Jan. 15, 1853.

ON CERTAIN GEOMETRICAL THEOREMS.

By W. Spottiswoode.

Theorem I. *When two systems of four points taken upon two straight lines, and corresponding each to each, have their anharmonic ratios equal, if the two straight lines be so placed that two homologous points coincide, the three straight lines, which join the three other points of the first system to the three homologous points of the second respectively, will meet in a point.*

Let $0, x_1, x_2, x_3$ be the distances of the four points on the first line, and $0, y_1, y_2, y_3$ those on the second, measured from the point of intersection; then the equivalence of the anharmonic ratios of the two systems will be thus expressed:

$$\begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & 1 \\ \frac{1}{x_2} & \frac{1}{y_2} & 1 \\ \frac{1}{x_3} & \frac{1}{y_3} & 1 \end{vmatrix} = 0.$$

But if the two straight lines be taken as coordinate axes, this equation expresses the condition that the three straight lines

$$\frac{x}{x_1} + \frac{y}{y_1} - 1 = 0,$$

$$\frac{x}{x_2} + \frac{y}{y_2} - 1 = 0,$$

$$\frac{x}{x_3} + \frac{y}{y_3} - 1 = 0,$$

shall meet in a point.

But these lines pass through the pairs of points

$$0,\ x_1;\ 0,\ y_1,$$
$$0,\ x_2;\ 0,\ y_2,$$
$$0,\ x_3;\ 0,\ y_3,$$

respectively. Hence the theorem above enunciated.

THEOREM II. *When two systems of four straight lines, which correspond each to each respectively, have their anharmonic ratios equal, if they be so placed that two corresponding straight lines coincide in direction, the three other straight lines of the first pencil will meet the corresponding lines of the second respectively in three points situated on the same straight line.*

If $\omega, \alpha, \beta, \gamma$ be the four straight lines forming the first pencil, and $\omega, \alpha_1, \beta_1, \gamma_1$ those forming the second, the equivalence of the anharmonic ratios of the two pencils will be thus expressed:

$$\begin{vmatrix} \cot\omega\alpha, & \cot\omega\alpha_1, & 1 \\ \cot\omega\beta, & \cot\omega\beta_1, & 1 \\ \cot\omega\gamma, & \cot\omega\gamma_1, & 1 \end{vmatrix} = 0.$$

But if the common straight line ω be taken as the axis of x, and the centre of one pencil as the origin; then, a being the distance between the two centres, and $x_1, y_1;\ x_2, y_2;\ x_3, y_3$

[redacted]

ON CERTAIN GEOMETRICAL RELATIONS BETWEEN A SURFACE OF THE SECOND DEGREE AND A TETRAHEDRON WHOSE EDGES TOUCH THE SURFACE.

By THOMAS WEDDLE.

LET $t=0$, $u=0$, $v=0$, $w=0$, be the equations to the faces of a tetrahedron; then if t, u, v, w have been multiplied by arbitrary constants, the equation to any surface of the second degree may be denoted by

$$t^2+u^2+v^2+w^2+2\lambda tu+2\mu tv+2\nu tw+2\rho uv+2\sigma uw+2\tau vw=0.$$

If the surface which this equation denotes touch the edge (vw) of the tetrahedron, then, putting $v=w=0$ in the preceding equation, the result $t^2+u^2+2\lambda tu=0$, must be a complete square, and this requires $\lambda=\pm 1$. Similarly, that the surface may touch the other edges, we must have $\mu=\pm 1$, $\nu=\pm 1$, $\rho=\pm 1$, $\sigma=\pm 1$, and $\tau=\pm 1$; and the equation to the surface becomes

$$t^2+u^2+v^2+w^2\pm 2tu\pm 2tv\pm 2tw\pm 2uv\pm 2uw\pm 2vw=0 \ldots (1).$$

Now the double signs in this equation, being independent of each other, may be combined in $(2^6=)$ 64 different ways; but it will on examination be found that there are eight ways in which the signs of (1) may be taken so that the left-hand member shall be a complete square, and the equation will then denote a plane; also there are 48 ways in which the signs of (1) may be combined so that it shall denote a cone having its vertex on one of the edges of the tetrahedron, (thus $t^2+u^2+v^2+w^2+2tu-2tv-2tw-2uv-2uw-2vw=0$, or $(-t-u+v+w)^2+(v-w)^2=(v+w)^2$, denotes a cone having its vertex at the point $t+u=v=w=0$, on the edge (vw)); hence these $(8+48=)$ 56 combinations of signs must be rejected, and there remain $(64-56=)$ 8 combinations to be considered. One of these gives

$$t^2+u^2+v^2+w^2-2tu-2tv-2tw-2uv-2uw-2vw=0 \ldots (2);$$

and if this be denoted by $f(t,u,v,w)=0$, it will be found that the other seven combinations are

$$f(-t,-u,v,w)=0,\ f(-t,u,-v,w)=0,\ f(-t,u,v,-w)=0,$$
$$f(-t,u,v,w)=0,\quad f(t,-u,v,w)=0,\quad f(t,u,-v,w)=0,$$
$$\text{and } f(t,u,v,-w)=0:$$

but since each of these may, by changing the signs of one or more of the quantities t, u, v, w, be reduced to $f(t,u,v,w)=0$,

we conclude that (2) is the most general form of the equation to surfaces of the second degree touching the edges of the tetrahedron. It must be remembered that since t, u, v and w have been supposed multiplied by arbitrary constants, (2) implicitly contains three arbitrary constants, as it evidently ought.*

Let A, B, C and D be the angular points of the tetrahedron, the faces BCD, CDA, DAB, and ABC being denoted by $t=0$, $u=0$, $v=0$, and $w=0$, respectively; also let E, F, G, e, f, g be the points of contact of the surface (2) with the edges AB, AC, AD, CD, BD, and BC, respectively; and e_1, f_1, g_1, E_1, F_1, G_1, the points in which tangent planes to (2) that pass through these edges intersect the opposite edges; so that E and E_1 are on the edge AB, &c.

To find the equations to the point E, put $v=w=0$ in (2), and we have $t^2+u^2-2tu=0$, or $t-u=0$. In a similar manner we shall find that the points E, F, G, e, f, g are denoted as follows:

$$E,\quad v=w=t-u=0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$F,\quad w=u=t-v=0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$G,\quad u=v=t-w=0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

* If the surface, instead of touching all the edges of the tetrahedron, only touch the four edges (vw), (wt), (tu), (uv); that is, the sides AB, BC, CD, DA, of a twisted quadrilateral $ABCD$, its equation will be

[redacted]

$$\text{e}, \quad t = u = v - w = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6),$$
$$\text{f}, \quad t = v = w - u = 0 \ldots\ldots\ldots\ldots\ldots\ldots(7),$$
$$\text{g}, \quad t = w = u - v = 0 \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Hence the equations to the planes drawn through the edges of the tetrahedron and the points of contact of the opposite edges are as under.

Equation to the plane CDE, $t - u = 0 \ldots\ldots\ldots\ldots\ldots\ldots(9)$,
BDF, $t - v = 0 \ldots\ldots\ldots\ldots\ldots\ldots(10)$,
BCG, $t - w = 0 \ldots\ldots\ldots\ldots\ldots\ldots(11)$,
$AB\text{e}$, $v - w = 0 \ldots\ldots\ldots\ldots\ldots\ldots(12)$,
$AC\text{f}$, $w - u = 0 \ldots\ldots\ldots\ldots\ldots\ldots(13)$,
$AD\text{g}$, $u - v = 0 \ldots\ldots\ldots\ldots\ldots\ldots(14)$.

Again, since the equation (2) may be put under the form

$$(t-u)^2 + (v-w)^2 = 2(t+u)(v+w) \ldots\ldots\ldots(\gamma),$$

it appears that $t + u = 0$, and $v + w = 0$, denote tangent planes, and they evidently pass through the edges CD and AB respectively. In a similar manner the tangent planes through the other edges may be obtained as under.

The equation to the tangent plane through the edge

AB touching at E is $v + w = 0 \ldots\ldots\ldots(15)$,
$AC \ldots\ldots\ldots\ldots\ldots F \ldots$ $w + u = 0 \ldots\ldots\ldots(16)$,
$AD \ldots\ldots\ldots\ldots\ldots G \ldots$ $u + v = 0 \ldots\ldots\ldots(17)$,
$CD \ldots\ldots\ldots\ldots\ldots$ e $\ldots$ $t + u = 0 \ldots\ldots\ldots(18)$,
$BD \ldots\ldots\ldots\ldots\ldots$ f $\ldots$ $t + v = 0 \ldots\ldots\ldots(19)$,
$BC \ldots\ldots\ldots\ldots\ldots$ g $\ldots$ $t + w = 0 \ldots\ldots\ldots(20)$.

Hence the points E_1, F_1, &c. are denoted as follows:

$$E_1, \quad v = w = t + u = 0 \ldots\ldots\ldots\ldots\ldots(21),$$
$$F_1, \quad w = u = t + v = 0 \ldots\ldots\ldots\ldots\ldots(22),$$
$$G_1, \quad u = v = t + w = 0 \ldots\ldots\ldots\ldots\ldots(23),$$
$$\text{e}_1, \quad t = u = v + w = 0 \ldots\ldots\ldots\ldots\ldots(24),$$
$$\text{f}_1, \quad t = v = w + u = 0 \ldots\ldots\ldots\ldots\ldots(25),$$
$$\text{g}_1, \quad t = w = u + v = 0 \ldots\ldots\ldots\ldots\ldots(26).$$

It is evident from equation (γ) that the tangent plane $v + w = 0$ meets the surface (γ); that is, (2), in *one* point only; hence the surface cannot be either an hyperboloid

of one sheet, an hyperbolic paraboloid, a cone, or a cylinder, but must be either an ellipsoid, an hyperboloid of two sheets, or an elliptic paraboloid. In other words,

I. *No ruled or developable* surface of the second degree can touch* ALL *the edges of a tetrahedron, so that every surface of the second degree touching all the edges is necessarily umbilical.*

In my first memoir "On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*Journal*, new series, vol. IV. pp. 30 and 33), I have also shewn that no ruled surface of the second degree can touch all the edges of either a hexahedron or an octahedron. I have not as yet met with an instance in which a ruled surface of the second degree touches all the edges of a solid figure.

It appears from what has been already said, that (1) denotes 64 surfaces; 8 of which are planes, 8 umbilical surfaces, and 48 cones; and it is easily seen that the whole of these surfaces meet the edges of the tetrahedron in the points E, E_1; e, e_1; F, F_1; f, f_1; G, G_1; g, g_1 only—each surface meeting the edges in some six of these points. Now, taking one point on each edge, the preceding six pairs of points can be combined, six in each combination, in ($2^6 =$) 64 different ways (and no more), and therefore these different ways correspond to the different surfaces just mentioned. Hence

II. *If any six of the points* E, E_1; e, e_1; F, F_1; f, f_1; G, G_1; g, g_1*—one point on each edge—be taken, then*

$$E_1F_1G_1e_1f_1g_1, \quad t+u+v+w=0 \text{............}(31),$$
$$E_1FGe_1fg, \quad -t-u+v+w=0 \text{............}(32),$$
$$EF_1Gef_1g, \quad -t+u-v+w=0 \text{............}(33),$$
$$EFG_1efg_1, \quad -t+u+v-w=0 \text{............}(34).$$

It will be observed that the four planes (27), (28), (29), and (30), pass through the points of contact on the contiguous edges of the tetrahedron.

Put the left-hand members of (27),...(34) equal to $2t'$, $2u'$, $2v'$, $2w'$, $2p$, $2q$, $2r$, and $2s$, respectively, and it will readily be found that we have the following identities:

$$\left.\begin{array}{lll}
2p = & t+u+v+w & = t'+u'+v'+w' \\
2q = & -t-u+v+w & = t'+u'-v'-w' \\
2r = & -t+u-v+w & = t'-u'+v'-w' \\
2s = & -t+u+v-w & = t'-u'-v'+w' \\
2t' = & -t+u+v+w & = p+q+r+s \\
2u' = & t-u+v+w & = p+q-r-s \\
2v' = & t+u-v+w & = p-q+r-s \\
2w' = & t+u+v-w & = p-q-r+s \\
2t = & -t'+u'+v'+w' & = p-q-r-s \\
2u = & t'-u'+v'+w' & = p-q+r+s \\
2v = & t'+u'-v'+w' & = p+q-r+s \\
2w = & t'+u'+v'-w' & = p+q+r-s
\end{array}\right\} \text{...}(35).$$

The equation (2) to the surface touching the edges of the tetrahedron may also be written in any of the three following forms:

$$tt'+uu'+vv'+ww'=0 \text{..............}(36),$$
$$t'^2+u'^2+v'^2+w'^2-2t'u'-2t'v'-2t'w'-2u'v'-2u'w'-2v'w'=0 \text{...}(37),$$
$$q^2+r^2+s^2=p^2 \text{.................}(38).*$$

* If $$ap+bq+cr+es=0 \text{..........................}(\alpha)$$
denote a tangent plane to (38), then will
$$b^2+c^2+e^2=a^2 \text{.............................}(\beta),$$
and the point of contact will be determined by the equations
$$-\frac{p}{a}=\frac{q}{b}=\frac{r}{c}=\frac{s}{e} \text{...........................}(\gamma).$$
And conversely, if (γ) denote a point in the surface (38), so that we have the condition (β), then will (α) denote the tangent plane at that point.

The planes t', u', v', w' will be the faces of a second tetrahedron $A'B'C'D'$; and the planes p, q, r, s those of a third tetrahedron $OO'O''O'''$; the faces $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$, $O'O''O'''$, $O''O'''O$, $O'''OO'$, and $OO'O''$, being those denoted by $t'=0$, $u'=0$, $v'=0$, $w'=0$, $p=0$, $q=0$, $r=0$, and $s=0$, respectively. I shall refer to the tetrahedra $ABCD$, $A'B'C'D'$, and $OO'O''O''$, as the given, second, and third tetrahedra respectively; and when two tetrahedra are referred to without specifying which, I mean the first two. Omitting the third tetrahedron for the present, I proceed to the consideration of properties connected with the other two.

At the point A' we have $u'=v'=w'=0$, or $-t=u=v=w$; and at O''' we have $p=q=r=0$, or $-t=u=v=-w$, and so on; hence the angular points of the second and third tetrahedra are denoted as follows:

$$\left.\begin{array}{lrrrr} A', & -t= & u= & v= & w \\ B', & t= & -u= & v= & w \\ C', & t= & u= & -v= & w \\ D', & t= & u= & v= & -w \end{array}\right\} \text{............(39)},$$

$$\left.\begin{array}{lrrrr} O, & t= & u= & v= & w \\ O', & -t= & -u= & v= & w \\ O'', & -t= & u= & -v= & w \end{array}\right\} \text{............(40).}$$

III. *If a surface of the second degree touch the edges of a tetrahedron, the tangent planes passing through the opposite edges will intersect in three straight lines in one plane.*

Moreover, since the six planes (9...14) all pass through the point $t = u = v = w$, we infer that

IV. *The six planes which pass through the edges of the given tetrahedron and the points of contact of the opposite edges, intersect in one point;* or, which is the same thing, *the three straight lines joining the points of contact of opposite edges pass through one point.*

Since the planes $u = 0$, $t = 0$, $t - u = 0$, $t + u = 0$, form a harmonic system, and pass through the points A, B, E, E_1, on the edge AB, it appears that

V. *The two faces and the tangent plane through any edge, together with the plane passing through the same edge and the point of contact of the opposite edge, form a harmonic system; and hence any edge is divided harmonically by its point of contact and the point in which it is intersected by the tangent plane through the opposite edge.**

Again, (27...30) and (35), the straight lines (tt'), (uu'), (vv') and (ww') are all situated in the plane $t + u + v + w = 0$; hence

VI. *If (four) planes be drawn through the points of contact of every three contiguous edges of the given tetrahedron, these planes will intersect the corresponding faces of the tetrahedron in four straight lines in one plane.* This amounts to saying that, *the corresponding faces of the two tetrahedra $ABCD$, $A'B'C'D'$ intersect in four straight lines in one plane.*

The equations to the straight lines AA', BB', CC', and DD' are, (39),

$$
\begin{aligned}
&u = v = w \\
&v = w = t \\
&w = t = u \\
\text{and}\quad &t = u = v,
\end{aligned}
$$

and these pass through the point $t = u = v = w$ or O. Consequently

---

* Hence when the points of contact are given, the tangent planes passing through the edges may be constructed.

It is evident, however, that the six points of contact are not all independent, but that if three of them (which must not be all in the same face of the tetrahedron) be given, the other three can be found. This may be done in various ways, one of which is derived from the well-known plane theorem, that "If a triangle be circumscribed about a conic, the three straight lines joining the points of contact to the opposite angles intersect in a point," so that when two of the points of contact are given, the third may be readily found. We have only to apply the construction here indicated to three of the faces of the tetrahedron, to find the points of contact of the other three edges.

VII. *The straight lines joining the corresponding angles of the tetrahedra $ABCD$, $A'B'C'D'$ intersect in one point.**

On account of the complete reciprocity of t, u, v, w and t', u', v', w' (see 2, 37, 35), the following is evident:

VIII. *The edges of the tetrahedron $A'B'C'D'$ touch the surface* (2), *and the points of contact coincide with those of the edges of the tetrahedron $ABCD$.*

Since the equations to the edge $C'D'$ are $t' = u' = 0$, or (35) $t - u = v + w = 0$, it follows that the edge $C'D'$ intersects the edge CD in $t = u = v + w = 0$, or the point e_1; and likewise the edge AB in $v = w = t - u = 0$, or the point E. Hence

IX. *The edges $A'B'$, $A'C'$, $A'D'$, $C'D'$, $D'B'$, $B'C'$ intersect the edges AB, AC, AD, CD, DB, BC, respectively, in the points E_1, F_1, G_1, e_1, f_1, g_1; and the former edges likewise intersect the edges CD, DB, BC, AB, AC, AD, respectively, in the points of contact* e, f, g, E, F, G. *The former six pairs of edges lie in the planes* (9...14), *and the latter six pairs in the tangent planes* (15...20).

The equations to the lines $B\mathrm{e}$, $C\mathrm{f}$, $D\mathrm{g}$ are, (6, 7, 8), $t = 0$, combined with $v - w = 0$, $w - u = 0$, and $u - v = 0$, respectively; and hence these lines pass through the point in which the face BCD (whose equation is $t = 0$) is intersected by the line AA' (whose equations are $u = v = w$). In a similar manner it may be shewn that the lines $B'E$, $C'F$, and $D'G$ intersect in the point in which the same line AA' intersects the face

(*Journal*, new series, vol. VI. p. 123). So likewise is the following: "If a surface of the second order be tangential to three planes, the planes passing through the mutual intersections of every two of them, and the point of contact of the third tangent plane, will intersect in the same straight line."* The preceding are not the only analogous properties however. (VII.) may be viewed as an analogue, if we present the plane theorem in this form: "If a triangle be circumscribed about a conic and another be inscribed in the same, having its angles at the points of contact of the sides of the former, then shall the straight lines joining the corresponding angular points intersect in one point." I dare say the analogy between (VII.) and this theorem will not be denied by most mathematicians, but in case there should be any doubt, I give one way of establishing an analogy. If a plane figure be circumscribed about a conic, and the points of contact of the contiguous sides be joined, a plane figure will be inscribed in the conic. Now if the edges of a solid figure touch a surface of the second degree, and planes be drawn through the points of contact of the contiguous edges, these planes will form the faces of another solid figure whose edges will also touch the surface; and the two solids will be such that the faces of each pass through the points of contact of the contiguous edges of the other. It hence follows that two solids so related bear some analogy to two plane figures, one inscribed in, and the other circumscribed about a conic, the sides of the latter figure touching at the angular points of the former.

I wish to state here that theorem (IV.), and the general equation (2) to surfaces of the second degree touching the edges of a tetrahedron, were discovered by the late G. W. Hearn and myself, independently of each other, and about the same time. It also appears (from letters in my possession) that Mr. Hearn had also investigated many of the preceding equations and theorems (and possibly some of those that follow), but he never pointed out which of my results coincided with his own.

Equation (2) may be put under the form

$$(-t+u+v+w)^2 = 4(vw+wu+uv);$$

* A very elegant demonstration of this theorem, by the late G. W. Hearn, will be found at p. 55 of his *Researches on Curves of the Second Order*. It is however only a particular case of a more general theorem, due it would seem to M. Chasles, for which see *Journal*, new series, vol. VI. p. 130, theorem (XIX.).

hence $vw + wu + uv = 0$ is the equation to the cone which has its vertex at the point (uvw) or A, and which envelopes the surface (2). In like manner we shall obtain the equations of the cones which have their vertices at the other angles of the tetrahedron $ABCD$, and which envelope the surface (2); and collecting the whole we have

$$\left.\begin{aligned} vw + wu + uv &= 0 \\ wt + tv + vw &= 0 \\ tu + uw + wt &= 0 \\ uv + vt + tu &= 0 \end{aligned}\right\}\text{..................(41).}$$

These I shall denominate the *circumscribed* cones of the given tetrahedron.

Again, if in (2) we put $t = 0$, $u = 0$, $v = 0$, and $w = 0$, in succession, we shall get

$$\left.\begin{aligned} u^2 + v^2 + w^2 - 2uv - 2uw - 2vw &= 0 \\ t^2 + v^2 + w^2 - 2tv - 2tw - 2vw &= 0 \\ t^2 + u^2 + w^2 - 2tu - 2tw - 2uw &= 0 \\ t^2 + u^2 + v^2 - 2tu - 2tv - 2uv &= 0 \end{aligned}\right\}\text{......(42);}$$

and these are evidently the equations to the cones whose vertices are at the angular points of the given tetrahedron, and whose directors are the conics in which the opposite faces of the tetrahedron intersect the surface. These may

them coincide with those denoted by (9)...(14), and the equations to the other six are as follows:

$$\left.\begin{aligned} t+u &= 2(v+w) \\ t+v &= 2(u+w) \\ t+w &= 2(u+v) \\ v+w &= 2(t+u) \\ u+w &= 2(t+v) \\ u+v &= 2(t+w) \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (43).$$

Now the first and fourth of these equations are satisfied by $t+u=v+w=0$; the second and fifth by $t+v=u+w=0$; and the third and sixth by $t+w=u+v=0$. Hence, (15...20) and (III.),

XII. *The inscribed cones intersect each other, two and two, in twelve planes—which consist of a system of six planes that intersect in a point, and of a system of six planes, the corresponding pairs of which intersect in three straight lines in the same plane, and these straight lines coincide with the intersections of the three pairs of tangent planes to the surface* (2) *that pass through the opposite edges of the tetrahedron.*

It is evident from a comparison of (2) and (37), that if we accent t, u, v, and w in all the equations from (41) inclusive, we shall get the equations to the inscribed and circumscribed cones of the second tetrahedron $A'B'C'D'$, and their mutual intersections. Now, (35), $t'-u'=u-t$, &c., $t'+u'=v+w$, &c. identically; hence the following theorem is evident:

XIII. *The* $\left\{\begin{matrix}\textit{inscribed} \\ \textit{circumscribed}\end{matrix}\right\}$ *cones of the second tetrahedron intersect each, two and two, in the same planes as the* $\left\{\begin{matrix}\textit{inscribed} \\ \textit{circumscribed}\end{matrix}\right\}$ *cones of the given tetrahedron.*

Again, considering the circumscribed cones of both tetrahedra, each pair of corresponding cones will intersect in the plane $t+u+v+w=0$; besides which there are *second* planes of intersection which are different for each pair of cones; their equations are,

$$\left.\begin{aligned} -3t+u+v+w &= 0 \\ -3u+t+v+w &= 0 \\ -3v+t+u+w &= 0 \\ -3w+t+u+v &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots (44);$$

and each of these passes through the point O, or $t=u=v=w$. Hence

XIV. *The corresponding circumscribed cones of the two tetrahedra intersect each other in five planes—which consist of the single plane that passes through the four straight lines in which the corresponding faces of the two tetrahedra intersect; and of a system of four planes which pass respectively through these lines and which intersect in the point O.*

The corresponding inscribed cones will be found to intersect each other in the preceding five planes; so that

XV. *The corresponding inscribed cones of the two tetrahedra intersect each other in five planes which coincide with those in which the corresponding circumscribed cones intersect each other.*

Besides the faces of the given tetrahedron, the inscribed cones of the given tetrahedron intersect the corresponding circumscribed cones of the second tetrahedron in the planes

$$\left.\begin{array}{l} \tfrac{3}{2}t + u + v + w = 0 \\ \tfrac{3}{2}u + t + v + w = 0 \\ \tfrac{3}{2}v + t + u + w = 0 \\ \tfrac{3}{2}w + t + u + v = 0 \end{array}\right\} \text{..................} (45).$$

Also, besides the faces of the second tetrahedron, the circumscribed cones of the given tetrahedron intersect the corresponding inscribed cones of the second tetrahedron in the planes

$$\tfrac{3}{2}t + u + v + w = 0$$

[redacted]

Also, besides the faces of the second tetrahedron, the inscribed cones of the same intersect the surface (2) in the four planes

$$\left.\begin{array}{l} 7t + u + v + w = 0 \\ 7u + t + v + w = 0 \\ 7v + t + u + w = 0 \\ 7w + t + u + v = 0 \end{array}\right\} \dots\dots\dots\dots\dots (48).$$

Hence, recollecting that the former and latter cones also intersect the given surface in the faces of the first and second tetrahedra respectively, we may say that

XVII. *The sixteen planes in which the inscribed cones of the two tetrahedra intersect the surface* (2) *pass, four and four, through four straight lines that are in one plane.*

We have thus a great number of planes, (see 44, 45, 46, 47, 48), which pass through the four lines (tt'), (uu'), (vv'), (ww'), in which the corresponding faces of the two tetrahedra intersect.

Several harmonic relations between the faces of the tetrahedra and the planes just referred to, might easily be deduced, but I shall omit them.

From (44) inclusive, I have only written down the equations to the planes of intersection in order to save space. The investigations are extremely easy, and I shall, as an example, merely investigate the equations for (xv.).

The equation to the inscribed cone of the second tetrahedron having its vertex at the point A' is of course

$$u'^2 + v'^2 + w'^2 - 2u'v' - 2u'w' - 2v'w' = 0,$$

which, when expressed in terms of t, u, v, and w, becomes

$$-3t^2 + 5u^2 + 5v^2 + 5w^2 - 2tu - 2tv - 2tw - 6uv - 6uw - 6vw = 0;$$

from this equation deduct four times

$$u^2 + v^2 + w^2 - 2uv - 2uw - 2vw = 0,$$

(the equation to the corresponding inscribed cone of the given tetrahedron), and we get

$$-3t^2 - 2t(u + v + w) + (u + v + w)^2 = 0,$$

or $$(t + u + v + w)(-3t + u + v + w) = 0;$$

so that the two cones intersect in two plane curves situated in the planes $t + u + v + w = 0$ and $-3t + u + v + w = 0$.

It may also be observed, that having obtained (45) and (47), we shall get (46) and (48) by accenting the letters in (45) and (47), and then expressing the resulting equations in terms of t, u, v, and w by means of (35).

Putting $w = 0$ in the first equation of (47), we have $t = 2(u + v)$, and these values of t and w being substituted in (2), there results $(u - v)^2 = 0$, or $u - v = 0$; hence the straight line in which the face w and the first plane of (47) intersect, touches the given surface. Similarly, it may be shewn that any of the planes (47) and a *non-corresponding* face of the given tetrahedron intersect in a straight line which touches the surface (2). Also, since the equations (48) are what (47) become when t, u, v, and w are accented, it follows that the same is true of the planes (48) and the faces of the second tetrahedron. Hence

XVIII. *The twelve straight lines in which the four planes* (47) *intersect the non-corresponding faces of the given tetrahedron, and the twelve straight lines in which the planes* (48) *intersect the non-corresponding faces of the second tetrahedron, touch the given surface* (2 *or* 37).

It will be observed that the faces of the given tetrahedron and the planes (47) form the faces of a duodecangular octahedron whose edges touch the surface (2) of the second degree; but as the consideration of this subject here might be too great a digression, I shall postpone it to the end of this paper.

The equation to any surface of the second degree circumscribed about the given tetrahedron is evidently

$$ftu + gtv + htw + f'vw + g'uw + h'uv = 0.$$

The general equation to such a surface is

$$f(tu + vw) + g(tv + uw) + h(tw + uv) = 0 \ldots\ldots (49),$$

f, g, and h being arbitrary constants.

The last equation may be presented in a neat form in terms of p, q, r, and s. By (35) we have

$$tu + vw = \tfrac{1}{2}(p^2 + q^2 - r^2 - s^2), \text{ \&c.},$$

hence (49) becomes

$$(f+g+h)p^2 + (f-g-h)q^2 + (-f+g-h)r^2 + (-f-g+h)s^2 = 0.$$

Put $k = f+g+h$, $l = f-g-h$, $m = -f+g-h$, and $n = -f-g+h$, so that $k+l+m+n = 0$. Hence the equation to surfaces of the second degree circumscribed about the given and second tetrahedra may be written in this form:

$$\left.\begin{aligned} kp^2 + lq^2 + mr^2 + ns^2 &= 0 \\ \text{where} \qquad k + l + m + n &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (50).$$

XX. *Any surface of the second degree touching seven of the faces of the two tetrahedra, will touch the eighth face.*

This theorem may be established as follows:

In the *Mathematician*, vol. III. p. 278, I have shewn that the general equation to surfaces of the second degree touching the faces t, u, v, w of the given tetrahedron is

$$l(at + bu - cv - ew)^2 + m(at - bu + cv - ew)^2 + n(at - bu - cv + ew)^2 = (at + bu + cv + ew)^2,$$

where $\qquad l^{-1} + m^{-1} + n^{-1} = 1.$

Substitute $-k^{-1}l$, $-k^{-1}m$, and $-k^{-1}n$, for l, m, and n, and these equations become

$$k(at + bu + cv + ew)^2 + l(at + bu - cv - ew)^2 + m(at - bu + cv - ew)^2 + n(at - bu - cv + ew)^2 = 0 \ldots (1').^*$$

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it follows that (XIX.) is only a particular case of theorem (VIII.) in my second memoir "On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*Journal*, new series, vol. v. p. 65). But indeed the latter theorem itself is only a particular case of the following: *Every surface of the second degree passing through seven of the eight points of intersection of three surfaces of the second degree, will necessarily pass through the eighth.*

I may as well observe here that, since the faces of the two tetrahedra form those of an octahedron, the next theorem (XX.) in the text is only a particular case of theorem (IX.) (*ib.* p. 66), and this again is only a particular case of the following: *Every surface of the second degree touching seven of the eight common tangent planes to three surfaces of the second degree, will necessarily touch the eighth.*

* If we change the sign of the arbitrary constant a in this equation, we shall get the more symmetrical form

where $$k^{-1}+l^{-1}+m^{-1}+n^{-1}=0 \quad\text{.....................(2').}$$

Put
$$\begin{aligned} at+bu+cv+ew&=2P,\\ at+bu-cv-ew&=2Q,\\ at-bu+cv-ew&=2R,\\ at-bu-cv+ew&=2S; \end{aligned}$$

also put
$$\left.\begin{aligned} -a^{-1}+b^{-1}+c^{-1}+e^{-1}&=2\alpha\\ a^{-1}-b^{-1}+c^{-1}+e^{-1}&=2\beta\\ a^{-1}+b^{-1}-c^{-1}+e^{-1}&=2\gamma\\ a^{-1}+b^{-1}+c^{-1}-e^{-1}&=2\delta \end{aligned}\right\}\text{............(3').}$$

We thus readily find
$$\left.\begin{aligned} -2t'&=-\alpha P+\beta Q+\gamma R+\delta S\\ 2u'&=\beta P-\alpha Q+\delta R+\gamma S\\ 2v'&=\gamma P+\delta Q-\alpha R+\beta S\\ 2w'&=\delta P+\gamma Q+\beta R-\alpha S \end{aligned}\right\}\text{.......(4');}$$

and the equation (1′) to the tangent surface becomes
$$kP^2+lQ^2+mR^2+nS^2=0\text{...............(5').}$$

Hence, observing (4′) and the foot-note at p. 110, we see that if $t'=0$, $u'=0$, $v'=0$, and $w'=0$, denote tangent planes to (1′) or (5′), we must have

Add the first two equations of (6′), recollecting that $k^{-1} + l^{-1} = -(m^{-1} + n^{-1})$, therefore

$$(k^{-1} + l^{-1})(\alpha^2 + \beta^2 - \gamma^2 - \delta^2) = 0;$$

therefore either $k^{-1} + l^{-1} = 0$, or $\alpha^2 + \beta^2 = \gamma^2 + \delta^2$:

but, (3′), $\alpha^2 + \beta^2 = \gamma^2 + \delta^2$ is equivalent to $ab = ce$,

therefore either $k^{-1} + l^{-1} = 0$, or $ab = ce$.

Similarly, $k^{-1} + m^{-1} = 0$, or $ac = be$,

and $k^{-1} + n^{-1} = 0$, or $ae = bc$.

Now, since the equations $k^{-1} + l^{-1} = 0$, $k^{-1} + m^{-1} = 0$, and $k^{-1} + n^{-1} = 0$, cannot exist simultaneously (2′), we shall have three cases to consider:

Firstly, $ab = ce$, $ac = be$, and $ac = bc$.

Secondly, such as

$$k^{-1} + l^{-1} = 0, \quad ac = be, \quad \text{and } ae = bc.$$

Thirdly, such as

$$k^{-1} + l^{-1} = 0, \quad k^{-1} + m^{-1} = 0, \quad \text{and } ae = bc.$$

Taking the first of these, we must have

$$\text{either } a = b = c = e,$$

$$\text{or, such as } a = b = -c = -e.$$

Substituting the former of these in (1′), we find the required equation to be

$$k(t + u + v + w)^2 + l(t + u - v - w)^2 + m(t - u + v - w)^2$$
$$+ n(t - u - v + w)^2 = 0 \ldots\ldots (7').$$

If we were to take $a = b = -c = -e$ in (1′), and then interchange k and l, and m and n, (which does not affect (2′)), we should also get (7′).

Secondly, let $k^{-1} + l^{-1} = 0$, $ac = be$, and $ae = bc$; hence, (2′), $l = -k$ and $n = -m$. Also either $a = b$ and $c = e$; or $a = -b$ and $c = -e$. Taking $l = -k$, $n = -m$, $b = a$, and $e = c$, in (1′), we get

$$k(t + u)(v + w) + m(t - u)(v - w) = 0;$$

also taking $l = -k$, $n = -m$, $b = -a$, and $c = -e$ in (1′), we get

$$k(t - u)(v - w) + m(t + u)(v + w) = 0,$$

and this coincides with the former when k and m are interchanged; but the former equation is what (7′) becomes when $l = -k$ and $n = -m$, so that both these equations may be rejected as being only cases of (7′).

Lastly, let $k^{-1} + l^{-1} = 0$, $k^{-1} + m^{-1} = 0$, and $ae = bc$; hence (2′), $k = -l = -m = n$; and (1′) is reduced to

$$aetw + bcuv = 0:$$

but since $ae = bc$, this becomes $tw + uv = 0$, and this equation may also be rejected, seeing that it results from putting $k = -l = -m = n$ in (7′).

From the preceding discussion it follows that the general equation to surfaces of the second degree touching the faces of the given and second tetrahedra may, (7′), be written

$$k(t+u+v+w)^2 + l(-t-u+v+w)^2 + m(-t+u-v+w)^2 + n(-t+u+v-w)^2 = 0 \ldots (51),$$

where k, l, m, n are arbitrary constants, subject however to the condition

$$k^{-1} + l^{-1} + m^{-1} + n^{-1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (52).$$

The equation (51) may also be written

$$kp^2 + lq^2 + mr^2 + ns^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (53)^*,$$

also of course with the condition (52).

Since (2) may be written in the form

$$(t-u)^2 + (v-w)^2 - 2(t+u)(v+w) = 0,$$

we see that the equation to any other surface of the second degree touching the edges of the given tetrahedron is

a complete square; this requires that

$$(a-b)(c+e) = \pm (a+b)(c-e);$$

which gives $\quad ac = be \;$ or $\; ae = bc$.

Proceeding in this way, we find the conditions required in order that the surface (α) should touch the various edges of the second tetrahedron to be as follows:

If the surface (α) touch $A'B'$ or $C'D'$, then must

$$\text{either } ac = be \text{ or } ae = bc \ldots\ldots\ldots\ldots\ldots (\beta),$$

$$\text{if } A'C' \text{ or } B'D', \text{ either } ab = ce \text{ or } ae = bc \ldots\ldots\ldots\ldots\ldots (\gamma),$$

$$\text{if } A'D' \text{ or } B'C', \text{ either } ab = ce \text{ or } ac = be \ldots\ldots\ldots\ldots\ldots (\delta).$$

Hence, if the surface (α) touch any edge of the second tetrahedron, it must necessarily touch the opposite edge; also if the surface touch any edge, say $A'B'$, so that one of the conditions (β) is satisfied, then one of the conditions either of (γ) or of (δ) will be satisfied, so that the surface will touch other two opposite edges. Hence this theorem:

XXI. *Every surface of the second degree touching all the edges of the given tetrahedron, and also one of the edges of the second tetrahedron, will necessarily touch the opposite edge, and likewise one or other of the two remaining pairs of opposite edges.*

In order that the surface (α) should also touch all the edges of the second tetrahedron, we must satisfy (β), (γ), and (δ), and this may be effected in three ways:

$$\text{either } ac = be \text{ and } ae = bc \ldots\ldots\ldots\ldots\ldots\ldots (\varepsilon),$$

$$\text{or } ab = ce \text{ and } ae = bc \ldots\ldots\ldots\ldots\ldots\ldots (\eta),$$

$$\text{or } ab = ce \text{ and } ac = be \ldots\ldots\ldots\ldots\ldots\ldots (\zeta).$$

$$(\varepsilon) \text{ gives } b = \pm a \text{ and } e = \pm c,$$

$$(\eta) \text{ gives } c = \pm a \text{ and } b = \pm e,$$

$$(\zeta) \text{ gives } e = \pm a \text{ and } c = \pm b.$$

Substituting these values successively in (α), we readily find the resulting equations to be

$$a^2(t \mp u)^2 + c^2(v \mp w)^2 = 2ac(t \pm u)(v \pm w) \ldots\ldots (54),$$

$$a^2(t \mp v)^2 + e^2(w \mp u)^2 = 2ae(t \pm v)(w \pm u) \ldots\ldots (55),$$

$$a^2(t \mp w)^2 + b^2(u \mp v)^2 = 2ab(t \pm w)(u \pm v) \ldots\ldots (56),$$

where a, b, c, and e are arbitrary constants, and each equation, on account of the double signs, is equivalent to two.

Hence, every surface of the second degree touching all the edges of both tetrahedra, must have an equation of some

of the above forms. But each of the surfaces (54), (55), and (56), whatever may be the values of the constants, always touches one pair of edges of each tetrahedron at their points of intersection: thus (54), if the upper signs be taken, touches the edges CD and $A'B'$ at their point of intersection e, and the edges AB and $C'D'$ at their point of intersection E; while, if the lower signs be taken, (54) touches the edges CD and $C'D'$ at their point of intersection e_1, and the edges AB and $A'B'$ at their point of intersection E_1. Hence

XXII. *A surface of the second degree that touches all the edges of both tetrahedra must touch one pair of opposite edges of the given tetrahedron, and the corresponding pair of opposite edges of the second tetrahedron at the points of intersection of these edges; but it does not necessarily touch the other eight edges at their points of intersection.*

The two tetrahedra possess other properties of a similar kind, but I deem it advisable to omit them.

The equation $tu + vw = 0$ denotes a ruled hyperboloid, which being satisfied by $t = v = 0$, $t = w = 0$, $u = v = 0$, and $u = w = 0$, respectively, passes through two pairs of opposite edges of the given tetrahedron. Now $tu + vw = t'u' + v'w'$ identically, and hence the hyperboloid also passes through the corresponding two pairs of edges of the second tetrahedron. Hence

[redacted]

XXIV. *Each of the hyperboloids just mentioned touches the given surface* (2) *in a plane. Also the plane of contact of any of these surfaces and the tangent planes to the given surface that pass through the third pairs of opposite edges intersect in a straight line.*

Equation (54) may be written in either of the forms

$$\{a(t \mp u) + c(v \mp w)\}^2 = 4ac(tv + uw),$$

or

$$\{a(t \mp u) - c(v \mp w)\}^2 = \pm\, 4ac(tw + uv);$$

and (55) and (56) may be thrown into similar forms. Hence

XXV. *Every surface of the second degree touching all the edges of both tetrahedra has necessarily plane contact with two of the three hyperboloids* (57).

In the *Mathematician*, vol. III., p. 278, I have shewn that the general equation to surfaces of the second degree touching the faces of the tetrahedron ($tuvw$), and which moreover are such that the straight lines joining the points of contact and the opposite angles shall intersect in a point, is

$$a^2t^2 + b^2u^2 + c^2v^2 + e^2w^2 - abtu - actv - aetw - bcuv - beuw - cevw = 0,$$

and the point through which the said lines pass is denoted by

$$at = bu = cv = ew;$$

if this point coincide with Q or $t = u = v = w$ we must have $a = b = c = e$, and then the equation to the surface becomes

$$t^2 + u^2 + v^2 + w^2 - tu - tv - tw - uv - uw - vw = 0 \ldots (59).$$

Hence (59) is the equation to the surface which touches the faces of the given tetrahedron in the points in which they are intersected by the lines joining the corresponding angular points of the given and second tetrahedra. But since (59) still denotes the same surface when t', u', v', and w' are written for t, u, v, and w, it follows that the surface (59) also touches the faces of the second tetrahedron in the points in which these faces are intersected by the lines joining the corresponding angles of the two tetrahedra.

If we put $t = 0$ in (59), the resulting equation may be written

$$(-2u + v + w)^2 + 3(v - w)^2 = 0,$$

which requires $-2u + v + w = v - w = 0$, so that the tangent plane $t = 0$ meets the surface (59) in one point only.

XXVI. *An umbilical surface of the second degree can be described to touch the faces of the two tetrahedra in the points in which the said faces are intersected by the straight lines joining the corresponding angles of the two tetrahedra.*

Suppose every inscribed as well as every circumscribed cone of each tetrahedron to be limited by the corresponding face (of the same tetrahedron) which will thus be the *base* of the cone. A surface will be circumscribed about one of these cones when it passes through its vertex and the perimeter of its base; and will be inscribed in the same when it touches the base and the curved surface of the cone in a curve.

The equation (59) may be written

$$(-2t+u+v+w)^2+3(u^2+v^2+w^2-2uv-2uw-2vw)=0;$$

hence the inscribed cone, $u^2+v^2+w^2-2uv-2uw-2vw=0$, touches the surface (59) in the conic in which it is intersected by the plane $-2t+u+v+w=0$, and we already know, (XXVI.), that the base of the cone touches (59); hence the surface (59) is inscribed in this cone. Similarly, putting (59) under the form

$$t'^2+u'^2+v'^2+w'^2-t'u'-t'v'-t'w'-u'v'-u'w'-v'w'=0,$$

it may be shewn that the surface is inscribed in the inscribed cone of the second tetrahedron which has its vertex at the point A'. Hence

XXVII. *An umbilical surface of the second degree may be inscribed in the eight inscribed cones of the two tetrahedra.*

Again, the surface whose equation is

XXIX. *A ruled hyperboloid may be described so as to touch two corresponding circumscribed cones of the two tetrahedra along the perimeters of their bases.*

The equations to these four surfaces are

$$\left.\begin{array}{l} t^2 + uv + uw + vw = 0 \\ u^2 + tv + tw + vw = 0 \\ v^2 + tu + tw + uw = 0 \\ w^2 + tu + tv + uv = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(63).$$

For the surface $t^2 + uv + uw + vw = 0$ touches the cone $uv + uw + vw = 0$, in the plane $t = 0$; and since $t^2 + uv + uw + vw = t'^2 + u'v' + u'w' + v'w'$, identically, the same surface touches the cone $u'v' + u'w' + v'w' = 0$ in the plane $t' = 0$. Also $t^2 + uv + uw + vw = (t + u)(t - u) + (u + v)(u + w)$, hence $t^2 + uv + uw + vw = 0$ denotes a ruled hyperboloid.

We have already seen that the two tetrahedra $ABCD$ and $A'B'C'D'$ are copolar, that is, the four lines joining the corresponding angles A and A', B and B', C and C', and D and D' intersect in a point—the *pole* of the tetrahedra. There are, however, altogether *four* ways in which the tetrahedra are copolar, the angular points of the tetrahedron $OO'O''O'''$ coinciding with their poles. If we write the corresponding angular points of the tetrahedra in the same order, so that if, for instance, $ABCD$ and $B'A'D'C'$ be the tetrahedra, the corresponding angles are A and B', B and A', C and D', and D and C'; then the four ways in which the tetrahedra are copolar are as under:—

1. The tetrahedra $ABCD$ and $A'B'C'D'$ have O for pole; their edges AB and $C'D'$, AC and $B'D'$, AD and $B'C'$, CD and $A'B'$, BD and $A'C'$, BC and $A'D'$, intersect in the points E, F, G, e, f, g, respectively; and the surface of the second degree whose equation is

$$q^2 + r^2 + s^2 = p^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(64),$$

touches the edges of both tetrahedra at these points.

2. The tetrahedra $ABCD$ and $B'A'D'C'$ have O' for pole; their edges AB and $D'C'$, AC and $A'C'$, AD and $A'D'$, CD and $B'A'$, BD and $B'D'$, BC and $B'C'$, intersect in the points E, F_1, G_1, e, f_1, g_1; and the surface of the second degree whose equation is

$$r^2 + s^2 + p^2 = q^2 \ldots\ldots\ldots\ldots\ldots\ldots(65),$$

touches the edges of both tetrahedra at these points.

3. The tetrahedra $ABCD$ and $C'D'A'B'$ have O'' for pole; their edges AB and $A'B'$, AC and $D'B'$, AD and $A'D'$, CD and $C'D'$, BD and $A'C'$, BC and $B'C'$, intersect in the points E_1, F, G_1, e_1, f, g_1, respectively; and the surface

$$s^2 + p^2 + q^2 = r^2 \quad\text{..................}(66),$$

touches the edges of both tetrahedra at these points.

4. The tetrahedra $ABCD$ and $D'C'B'A'$ have O''' for pole; their edges AB and $A'B'$, AC and $A'C'$, AD and $B'C'$, CD and $C'D'$, BD and $B'D'$, BC and $A'D'$, intersect in the points E_1, F_1, G, e_1, f_1, g, respectively; and the surface

$$p^2 + q^2 + r^2 = s^2 \quad\text{..................}(67),$$

touches the edges of both tetrahedra at these points.

Moreover the two tetrahedra $ABCD$ and $OO'O''O'''$ are copolar in four ways.

1′. The tetrahedra $ABCD$ and $OO'O''O'''$ have A' for pole; their edges AB and $O''O'''$, AC and $O'O'''$, AD and $O'O''$, CD and OO', BD and OO'', BC and OO''', intersect in the points E_1, F_1, G_1, e, f, g, respectively; and the surface

$$u'^2 + v'^2 + w'^2 = t'^2 \quad\text{..................}(68),$$

touches the edges of both tetrahedra at these points.

2′. The tetrahedra $ABCD$ and $O'OO'''O''$ have B' for pole; their edges AB and $O''O'''$, AC and OO'', AD and OO''', CD and OO', BD and $O'O'''$, BC and $O'O''$, intersect in the points E_1, F, G, e, f_1, g_1, respectively; and the surface

$$v'^2 + w'^2 + t'^2 = u'^2 \quad\text{..................}(69),$$

touches the edges of both tetrahedra at these points.

3′. The tetrahedra $ABCD$ and $O''O'''OO'$ have C' for pole; their edges AB and OO', AC and $O'O'''$, AD and OO''', CD and $O''O'''$, BD and OO'', BC and $O'O''$, intersect in the points E, F_1, G, e_1, f, g_1, respectively; and the surface

$$w'^2 + t'^2 + u'^2 = v'^2 \quad\text{..................}(70),$$

touches the edges of both tetrahedra at these points.

4′. The tetrahedra $ABCD$ and $O'''O''O'O$ have D' for pole; their edges AB and OO', AC and OO'', AD and $O'O''$, CD and $O''O'''$, BD and $O'O'''$, BC and OO''', intersect in the points E, F, G_1, e_1, f_1, g, respectively; and the surface

$$t'^2 + u'^2 + v'^2 = w'^2 \quad\text{..................}(71),$$

touches the edges of both tetrahedra at these points.

Lastly, the two tetrahedra $A'B'C'D'$ and $OO'O''O'''$ are copolar in four different ways.

1″. The tetrahedra $A'B'C'D'$ and $OO'O''O'''$ have A for pole; their edges $A'B'$ and $O''O'''$, $A'C'$ and $O'O'''$, $A'D'$ and $O'O''$, $C'D'$ and OO', $B'D'$ and OO'', $B'C'$ and OO''', intersect in the points E_1, F_1, G_1, E, F, G, respectively; and the surface

$$u^2 + v^2 + w^2 = t^2 \quad \text{.................. (72)},$$

touches the edges of both tetrahedra at these points.

2″. The tetrahedra $A'B'C'D'$ and $O'OO'''O''$ have B for pole; their edges $A'B'$ and $O''O'''$, $A'C'$ and OO'', $A'D'$ and OO''', $C'D'$ and OO', $B'D'$ and $O'O'''$, $B'C'$ and $O'O''$, intersect in the points E_1, f, g, E, f_1, g_1, respectively; and the surface

$$v^2 + w^2 + t^2 = u^2 \quad \text{.................. (73)},$$

touches the edges of both tetrahedra at these points.

3″. The tetrahedra $A'B'C'D'$ and $O''O'''OO'$ have C for pole; their edges $A'B'$ and OO', $A'C'$ and $O'O'''$, $A'D'$ and OO''', $C'D'$ and $O''O'''$, $B'D'$ and OO'', $B'C'$ and $O'O''$, intersect in the points e, F_1, g, e_1, F, g_1, respectively; and the surface

$$w^2 + t^2 + u^2 = v^2 \quad \text{.....................(74)},$$

touches the edges of both tetrahedra at these points.

4″. The tetrahedra $A'B'C'D'$ and $O'''O''O'O$ have D for pole; their edges $A'B'$ and OO', $A'C'$ and OO'', $A'D'$ and $O'O''$, $C'D'$ and $O''O'''$, $B'D'$ and $O'O'''$, $B'C'$ and OO''', intersect in the points e, f, G_1, e_1, f_1, G, respectively; and the surface

$$t^2 + u^2 + v^2 = w^2 \quad \text{.....................(75)},$$

touches the edges of both tetrahedra at these points.

Hence the following theorem:

XXX. *Any two of the three tetrahedra ABCD, $A'B'C'D'$ and $OO'O''O'''$ are copolar in four different ways; and the poles of any two coincide with the angular points of the third tetrahedron. Also through the six points in which the edges of any two of these copolar tetrahedra intersect, an umbilical surface of the second degree may be described so as to touch all the twelve edges.*

There are, as we have already seen, twelve such surfaces, and it may be noted that those eight which touch the edges of the tetrahedron $ABCD$ are the eight umbilical surfaces mentioned in (II.).

Each of the following sets of planes forms a harmonic system: v, w, $v+w$, $v-w$; $v+w$, $t+u$, $2p=(v+w)+(t+u)$, $2q=(v+w)-(t+u)$; $u-t$, $v-w$, $2s=(u-t)+(v-w)$,

$2r = (u-t)-(v-w)$; $v+w$, $u-t$, $2t' = (v+w)+(u-t)$, $2u' = (v+w)-(u-t)$; &c., &c.; hence the following theorem is evident.

XXXI. *Some two of the twelve planes,* (3...8) *and* (15...20), *pass through each of the eighteen edges of the three tetrahedra* $ABCD$, $A'B'C'D'$, *and* $OO'O''O'''$; *also the two planes and the two faces that pass through any edge form a harmonic system.*

It is evident from the symmetry of the equations (35) that any property which the given and second tetrahedra may possess, will also be true (with suitable modifications) of *any* two of the three tetrahedra. Thus, on substituting $t'u'v'w'$ instead of $pqrs$ in (50) or (53), we shall have the general equation to surfaces of the second degree circumscribed about, or inscribed in, the two tetrahedra $ABCD$ and $OO'O''O'''$; and if $tuvw$ be written for $pqrs$ in the same, we shall get the equations for the tetrahedra $A'B'C'D'$ and $OO'O''O'''$.

From this and the equations (64...75) it appears that

XXXII. *The angular points of any one of the three tetrahedra* $ABCD$, $A'B'C'D'$, *and* $OO'O''O'''$ *are the poles of the opposite faces (of the same tetrahedron), with respect to any of the following surfaces of the second degree:*

1. *Any surface circumscribed about the other two tetrahedra;*
2. *Any surface inscribed in the same; and*
3. *Each of the four surfaces touching the edges of the other two*

*diagrammatic octahedron whose diagrammatic planes are the faces of the remaining tetrahedron.**

I now come to consider some formulas and properties relative to the centres of gravity of the tetrahedra, and the centres of various surfaces and curves of the second degree.

Let a, b, c, and e be such constants that

$$at + bu + cv + ew = 1 \text{ ....} (76),$$

identically. I shall occasionally refer to this as the *equation of identity.*

Let

$$\left.\begin{aligned} 2a' &= -a + b + c + e \\ 2b' &= a - b + c + e \\ 2c' &= a + b - c + e \\ 2e' &= a + b + c - e \end{aligned}\right\} \text{ ....} (77),$$

also

$$\left.\begin{aligned} 2a'' &= a + b + c + e \\ 2b'' &= -a - b + c + e \\ 2c'' &= -a + b - c + e \\ 2e'' &= -a + b + c - e \end{aligned}\right\} \text{ ....} (78).$$

Hence, (35), the equation of identity is equivalent to either of the following equations:

$$a't' + b'u' + c'v' + e'w' = 1 \text{ ....} (79),$$

$$a''p + b''q + c''r + e''s = 1 \text{ ....} (80).$$

When $u = v = w = 0$, we have $at = 1$ identically, (76); hence the plane, $at = 1$, passes through the point A, and it is parallel to the face BCD: but t is proportional to the perpendicular from A on BCD; consequently the equation to the plane, parallel to the face BCD and at a distance from it equal to one-fourth of this perpendicular, is $at = \frac{1}{4}$; and this plane of course contains the centre of gravity of the tetrahedron $ABCD$. Similarly, the planes $bu = \frac{1}{4}$, $cv = \frac{1}{4}$, and $ew = \frac{1}{4}$, all contain the centre of gravity; hence this centre is denoted by

$$at = bu = cv = ew = \tfrac{1}{4} \text{ ....} (81).\dagger$$

* For the definitions of 'complete syngrammatic octangle' and 'complete diagrammatic octahedron,' see the memoir just referred to. (*Journal*, new series, vol. VII. pp. 255 and 257.)

† The neat equations (81) to the centre of gravity of a tetrahedron were discovered by the late Mr. Hearn and myself independently, the demonstration given above being however Mr. Hearn's. Equations (82) and (83) are due to myself.

Let K denote the centre of gravity of the tetrahedron $ABCD$; K_1, K_2, K_3, K_4 those of its faces BCD, ACD, ABD, ABC, respectively; and L, M, N, l, m, n, the middle points of the edges AB, AC, AD, CD, BD, BC, respectively. Also let these letters accented apply to the tetrahedron $ABCD$; and the same letters with two accents apply to the tetrahedron $OO'O''O'''$.

By (81) the equations to the line AK which intersects the face BCD in K_1 are $bu = cv = ew$; hence, when $t = 0$, we have, (76), $bu = cv = ew = \frac{1}{3}$. The centres of gravity of the faces of the tetrahedron $ABCD$ are therefore denoted as follows:

$$\left.\begin{array}{lll} K_1, & t = 0, & bu = cv = ew = \frac{1}{3} \\ K_2, & u = 0, & at = cv = ew = \frac{1}{3} \\ K_3, & v = 0, & at = bu = ew = \frac{1}{3} \\ K_4, & w = 0, & at = bu = cv = \frac{1}{3} \end{array}\right\} \quad \ldots\ldots\ldots (82).$$

Again, from these equations we see that the plane, $at = bu$, contains the points K_3K_4, and it passes through the edge CD; consequently this plane bisects the edge AB, and therefore the middle point (L) of AB is denoted by $v = w = 0$ and $at = bu = \frac{1}{2}$, by (76). Hence the middle points of the edges of the tetrahedron $ABCD$ are denoted as follows:

$$L, \quad v = w = 0, \quad at = bu = \tfrac{1}{2}$$

third tetrahedron $OO'O''O'''$. It is unnecessary therefore to write down the formulas for these two tetrahedra.

The preceding equations (81), (82), and (83) may be viewed somewhat more generally. If instead of $at+bu+cv+ew$ being $=1$ identically (so that $at+bu+cv+ew=0$ is the equation of the plane at infinity), we assume

$$at + bu + cv + ew = 0$$

to be the equation to any plane; then (omitting '$=\frac{1}{2}$') (83) will denote the harmonic conjugates of the points in which the plane cuts the edges of the tetrahedron $(tuvw)$, the harmonic conjugate of each point being taken with respect to the two angles of the tetrahedron that are on the same edge. If the points (*i.e.* the aforesaid harmonic conjugates) in each face be joined to the opposite angles in that face, they will intersect in points denoted by (82), ('$=\frac{1}{3}$' being omitted); and, finally, if these latter points be joined to the opposite angles of the tetrahedron, the four straight lines thus drawn will intersect in the point denoted by (81), (omitting '$=\frac{1}{4}$').

Let $\phi = \varepsilon t^2 + \zeta u^2 + \eta v^2 + \xi w^2 + 2\lambda tu + 2\mu tv + 2\nu tw$
$$+ 2\rho uv + 2\sigma uw + 2\tau vw = 0 \ldots (84),$$
be the equation to any surface of the second degree. Multiplying (84) by ε, it may be put under the form

$$(\varepsilon t + \lambda u + \mu v + \nu w)^2 = (\lambda^2 - \varepsilon\zeta)\, u^2 \ldots + 2\,(\mu\nu - \varepsilon\tau)\, vw\,;$$

from which we see that $\varepsilon t + \lambda u + \mu v + \nu w = 0$ is the polar plane of the point $u = v = w = 0$: but

$$\frac{1}{2}\cdot\frac{d\phi}{dt} = \varepsilon t + \lambda u + \mu v + \nu w,$$

so that $\dfrac{d\phi}{dt} = 0$ is the polar plane of the point $u = v = w = 0$.

Next, to find the polar plane of any point

$$\frac{t}{\alpha} = \frac{u}{\beta} = \frac{v}{\gamma} = \frac{w}{\delta} \;\ldots\ldots\ldots\ldots\ldots\ldots (85),$$

put $\qquad \dfrac{U}{\beta} = \dfrac{u}{\beta} - \dfrac{t}{\alpha},\quad \dfrac{V}{\gamma} = \dfrac{v}{\gamma} - \dfrac{t}{\alpha}$ and $\dfrac{W}{\delta} = \dfrac{w}{\delta} - \dfrac{t}{\alpha}$;

or $\qquad u = U + \dfrac{\beta t}{\alpha},\quad v = V + \dfrac{\gamma t}{\alpha},$ and $w = W + \dfrac{\delta t}{\alpha}$;

so that the point (85) is denoted by $U = V = W = 0$: hence, if we write $U + \dfrac{\beta t}{\alpha}$, $V + \dfrac{\gamma t}{\alpha}$, and $W + \dfrac{\delta t}{\alpha}$ for u, v, and w

in (84), and then differentiate with respect to t, we shall have the polar plane required. Now the result of this, after multiplying by α, is easily seen to be

$$\alpha . \frac{d\phi}{dt} + \beta . \frac{d\phi}{du} + \gamma . \frac{d\phi}{dv} + \delta . \frac{d\phi}{dw} = 0 \ldots\ldots\ldots (86).$$

Hence (86) is the polar plane of the point (85) with respect to the surface (84).

Now
$$\frac{1}{2} . \frac{d\phi}{dt} = \varepsilon t + \lambda u + \mu v + \nu w,$$
$$\frac{1}{2} . \frac{d\phi}{du} = \zeta u + \lambda t + \rho v + \sigma w,$$
$$\frac{1}{2} . \frac{d\phi}{dv} = \eta v + \mu t + \rho u + \tau w,$$
and
$$\frac{1}{2} . \frac{d\phi}{dw} = \xi w + \nu t + \sigma u + \tau v;$$
also put
$$\alpha' = \varepsilon\alpha + \lambda\beta + \mu\gamma + \nu\delta,$$
$$\beta' = \zeta\beta + \lambda\alpha + \rho\gamma + \sigma\delta,$$
$$\gamma' = \eta\gamma + \mu\alpha + \rho\beta + \tau\delta,$$
and
$$\delta' = \xi\delta + \nu\alpha + \sigma\beta + \tau\gamma.$$

Substitute for $\frac{d\phi}{du}$, &c. their values just given, and (86)

[redacted]

These equations enable us to find the centre of the surface (84); for the centre is the pole of the plane at infinity, and, (76), the equation to the latter being

$$at + bu + cv + ew = 0,$$

we have only to write $abce$ for $\alpha\beta\gamma\delta$ in (88), and we get

$$\frac{1}{a} \cdot \frac{d\phi}{dt} = \frac{1}{b} \cdot \frac{d\phi}{du} = \frac{1}{c} \cdot \frac{d\phi}{dv} = \frac{1}{e} \cdot \frac{d\phi}{dw} \text{.........} (89),$$

for the equations to the centre of the surface (84).

In a similar manner it may be shewn that if

$$\psi = \epsilon t^2 + \zeta u^2 + \eta v^2 + 2\lambda tu + 2\mu tv + 2\nu vw = 0 \ldots (90),$$

be the equation to a cone, and

$$\frac{t}{\alpha} = \frac{u}{\beta} = \frac{v}{\gamma} \text{.......................} (91),$$

those of any straight line through the vertex; then the polar plane of (91) with respect to the cone (90) is denoted by

$$\alpha \cdot \frac{d\psi}{dt} + \beta \cdot \frac{d\psi}{du} + \gamma \cdot \frac{d\psi}{dv} = 0 \text{ ....} (92).$$

Also, if

$$at + \beta u + \gamma v = 0 \text{ ....} (93),$$

be the equation of any plane through the vertex of (90), its polar line with respect to the cone (90) is denoted by

$$\frac{1}{\alpha} \cdot \frac{d\psi}{dt} = \frac{1}{\beta} \cdot \frac{d\psi}{du} = \frac{1}{\gamma} \cdot \frac{d\psi}{dv} \text{...............} (94).$$

Since (90), (91) and (92), also (90), (94) and (93), are cut by any plane (not passing through the vertex) in a conic, a point and a straight line respectively, such that the point is the pole of the straight line relative to the conic, it follows that if a plane cut the cone (90) in a conic, we can find the equation to the polar line of any given point in that plane relative to the conic; and conversely.

Hence also we can find the equations to the centre of the conic in which a plane, $w = 0$, cuts the cone (90). From (76) we have

$$at + bu + cv = 1 - ew, \text{ identically},$$

so that the plane $at + bu + cv = 0$, (or $ew - 1 = 0$), which passes through the vertex of the cone, is parallel to the plane $w = 0$ of the conic, and therefore the intersection of these planes is at infinity; consequently the pole of this line with respect to the conic is its centre. Hence, writing

abc for $\alpha\beta\gamma$ in (94), we see that the equations to the centre of the conic in which the plane $w=0$ cuts the cone (90) are

$$w=0,\quad \frac{1}{a}\cdot\frac{d\psi}{dt}=\frac{1}{b}\cdot\frac{d\psi}{du}=\frac{1}{c}\cdot\frac{d\psi}{dv}\;\text{.........(95).*}$$

Putting $w=0$ in (2), we see that the equations to the conic in which the face *ABC* is intersected by the surface (2) touching edges, are

$$w=0,\ \text{and}\ t^2+u^2+v^2-2tu-2tv-2uv=0.$$

Hence, (95), the centre of the conic is denoted by $w=0$, combined with

$$\frac{t-u-v}{a}=\frac{u-t-v}{b}=\frac{v-t-u}{c},$$

which are equivalent to

$$\frac{t}{b+c}=\frac{u}{a+c}=\frac{v}{a+b},$$

and these are evidently the equations to the straight line drawn from the angle *D* or (*tuv*) to the centre of the conic.

Proceeding in a similar manner, we find that the following are the equations to the straight lines drawn from the angular points of the tetrahedron to the centres of the conics in which the opposite faces are intersected by the surface (2).

$$\left.\begin{aligned}
&\frac{u}{c+e}=\frac{v}{b+e}=\frac{w}{b+c}\\
&\frac{t}{c+e}=\frac{v}{a+e}=\frac{w}{a+c}\\
&\frac{t}{b+e}=\frac{u}{a+e}=\frac{w}{a+b}\\
&\frac{t}{b+c}=\frac{u}{a+c}=\frac{v}{a+b}
\end{aligned}\right\}\;\text{.................(96).}$$

* This evidently amounts to saying that the plane whose equation is

$$at+bu+cv=1,$$

will cut the cone (90) in a conic whose centre is denoted by

$$\frac{1}{a}\cdot\frac{d\psi}{dt}=\frac{1}{b}\cdot\frac{d\psi}{du}=\frac{1}{c}\cdot\frac{d\psi}{dv},$$

combined of course with $at+bu+cv=1$.

Now these lines evidently lie on the hyperboloid

$$\{(b+e)(a+c)-(b+c)(a+e)\}\{(a-b)tu-(c-e)vw\}$$
$$+\{(b+c)(a+e)-(a+b)(c+e)\}\{(a-c)tv-(b-e)uw\}$$
$$+\{(a+b)(c+e)-(b+e)(a+c)\}\{(a-e)tw-(b-c)uv\}=0 \ldots (97);$$

and this hyperboloid contains the point $t=u=v=w$, or the point O. Hence the following theorem:

XXXIV. *Let a surface of the second degree touch the edges of a tetrahedron, and cut the faces in conics; the four straight lines drawn from the centres of the conics to the opposite angular points of the tetrahedron lie in a ruled hyperboloid and belong to the same system of generators; also the point in which intersect the three straight lines joining the points of contact of opposite edges, is a point in this hyperboloid.*

By using (94) instead of (95) it is easily seen that this theorem is still true when instead of the centres of the conics we substitute the poles of the lines in which the faces are intersected by any plane, the pole of each line being taken with respect to the conic in the same face of the tetrahedron. It is worthy of observation too that, though the hyperboloid will vary for different positions of the cutting plane, yet it (the hyperboloid) always passes through the fixed point in which intersect the three straight lines joining the points of contact of opposite edges.

Again, the polar plane of the point

$$\frac{t}{\alpha}=\frac{u}{\beta}=\frac{v}{\gamma}=\frac{w}{\delta} \ldots\ldots\ldots\ldots\ldots\ldots (98),$$

that is, of the straight line

$$\frac{t}{\alpha}=\frac{u}{\beta}=\frac{v}{\gamma},$$

with respect to the circumscribed cone (of the given tetrahedron) $$tu+tv+uv=0$$

is, (92), $\quad \alpha(u+v)+\beta(t+v)+\gamma(t+u)=0;$

that is, $\quad (\beta+\gamma)\,t+(\alpha+\gamma)\,u+(\alpha+\beta)\,v=0.$

In a similar manner the polar planes of the point (98) with respect to the other circumscribed cones may be obtained, and tabulating the whole, we have

$$\left.\begin{aligned}(\gamma+\delta)\,u+(\beta+\delta)\,v+(\beta+\gamma)\,w&=0\\(\gamma+\delta)\,t\,+(\alpha+\delta)\,v+(\alpha+\gamma)\,w&=0\\(\beta+\delta)\,t\,+(\alpha+\delta)\,u+(\alpha+\beta)\,w&=0\\(\beta+\gamma)\,t\,+(\alpha+\gamma)\,u+(\alpha+\beta)\,v&=0\end{aligned}\right\}\ldots\ldots (99).$$

The four straight lines in which these planes intersect the faces t, u, v, w respectively of the tetrahedron evidently lie in the hyperboloid whose equation is

$$\begin{aligned}
&(\gamma+\delta)(\beta+\delta)(\beta+\gamma)\,t^2+(\gamma+\delta)(\alpha+\delta)(\alpha+\gamma)\,u^2\\
&+(\beta+\delta)(\alpha+\delta)(\alpha+\beta)\,v^2+(\beta+\gamma)(\alpha+\gamma)(\alpha+\beta)\,w^2\\
&+\{(\beta+\delta)(\alpha+\gamma)+(\beta+\gamma)(\alpha+\delta)\}\,.\,\{(\gamma+\delta)\,tu+(\alpha+\beta)\,vw\}\\
&+\{(\gamma+\delta)(\alpha+\beta)+(\beta+\gamma)(\alpha+\delta)\}.\{(\beta+\delta)\,tv+(\alpha+\gamma)\,uw\}\\
&+\{(\gamma+\delta)(\alpha+\beta)+(\beta+\delta)(\alpha+\gamma)\}.\{(\beta+\gamma)\,tw+(\alpha+\delta)\,uv\}=0\ldots(100).
\end{aligned}$$

If we put

$$X=(-\alpha+\beta+\gamma+\delta)\,t+(\gamma+\delta)\,u+(\beta+\delta)\,v+(\beta+\gamma)\,w$$

and

$$Y=(\alpha^2+\beta\gamma+\beta\delta+\gamma\delta)\,t+(\alpha+\gamma)(\alpha+\delta)\,u+(\alpha+\beta)(\alpha+\delta)\,v+(\alpha+\beta)(\alpha+\gamma)\,w,$$

it will be found that (100) may be made to assume the form

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\,(t+u+v+w)\,t+XY=0,$$

and hence the hyperboloid (100) always touches the plane $t+u+v+w=0$, whatever be the values of $\alpha\beta\gamma\delta$, that is, whatever be the position of the point (98). Hence the following theorem:

XXXV. *Let a surface of the second degree touch the edges of a tetrahedron, and the four circumscribed cones be drawn; also let the polar plane of any point be taken with respect to each of these*

[redacted]

By (89) the equations to the centre of the surface (2) are

$$\frac{t-u-v-w}{a}=\frac{u-t-v-w}{b}=\frac{v-t-u-w}{c}=\frac{w-t-u-v}{e},$$

which, (77), are equivalent to either system of equations

$$\frac{t}{a'}=\frac{u}{b'}=\frac{v}{c'}=\frac{w}{e'} \text{ ....(101),}$$

or

$$\frac{t'}{a}=\frac{u'}{b}=\frac{v'}{c}=\frac{w'}{e} \text{ .... (102).}$$

The centre of the surface (2) or (38) is also, (80, 89), denoted by

$$-\frac{p}{a''}=\frac{q}{b''}=\frac{r}{c''}=\frac{s}{e''} \text{ ....(103).}$$

By (96) the equations to the centre of the conic in which the face t' cuts the surface (2) are

$$t'=0, \text{ and } \frac{u'}{c'+e'}=\frac{v'}{b'+e'}=\frac{w'}{b'+c'},$$

which are equivalent to

$$t'=0, \text{ and } \frac{t'-u'+v'+w'}{b'}=\frac{t'+u'-v'+w'}{c'}=\frac{t'+u'+v'-w'}{e'};$$

that is,

$$t'=0, \text{ and } \frac{u}{b'}=\frac{v}{c'}=\frac{w}{e'}.$$

Now these equations, the first of (96), equations (101), the equations $u=v=w=0$, and the first of (39), each satisfy the equation

$$(c-e)\,u+(e-b)\,v+(b-c)\,w=0,$$

or, (which is the same thing, (77),)

$$(e'-c')\,u+(b'-e')\,v+(c'-b')\,w=0,$$

and hence we have the following theorem (which is due to the late Mr. Hearn),

XXXVI. *The centres of the two conics in which the surface* (2) *cuts any two corresponding faces* (*as BCD, B'C'D'*) *of the two tetrahedra, the centre of the surface itself, and the two corresponding angular points* (*A, A'*), *are in one plane.*

By (89) the centres of the surfaces (61) and (62) are denoted by

$$-\frac{p}{3a''}=\frac{q}{b''}=\frac{r}{c''}=\frac{s}{e''},$$

and

$$-\frac{3p}{a''}=\frac{q}{b''}=\frac{r}{c''}=\frac{s}{e''}, \text{ respectively.}$$

Now each of these systems of equations, as well as (103), and $q = r = s = 0$ (which denote the point O), satisfies the equations

$$\frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''}.$$

Hence

XXXVII. *The point of intersection of straight lines joining the corresponding angles of the two tetrahedra, and the centres of the surface* (2) *touching the edges, of the surface* (61) *inscribed in the inscribed cones, and of the surface* (62) *circumscribed about the circumscribed cones, are in a straight line.*

The equations to the straight line joining the middle points of the edges AB and CD are, (83),

$$at = bu \text{ and } cv = ew:$$

also the equations to that joining the middle points of $A'B'$ and $C'D'$ are

$$a't' = b'u' \text{ and } c'v' = e'w';$$

that is,

$$(-a+b+c+e)(-t+u+v+w) = (a-b+c+e)(t-u+v+w),$$

$$\text{and } (a+b-c+e)(t+u-v+w) = (a+b+c-e)(t+u+v-w).$$

Now the equations to both these straight lines are satisfied by the equations

The centres of the surfaces (64), (65), (66) and (67) are denoted as follows:

$$-\frac{p}{a''} = \frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''},$$

$$\frac{p}{a''} = -\frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''},$$

$$\frac{p}{a''} = \frac{q}{b''} = -\frac{r}{c''} = \frac{s}{e''},$$

and
$$\frac{p}{a''} = \frac{q}{b''} = \frac{r}{c''} = -\frac{s}{e''} \text{ respectively;}$$

hence the straight lines drawn from these points to the corresponding angles of the tetrahedron $OO'O''O'''$ are

$$\frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''},$$

$$\frac{p}{a''} = \frac{r}{c''} = \frac{s}{e''},$$

$$\frac{p}{a''} = \frac{q}{b''} = \frac{s}{e''},$$

and
$$\frac{p}{a''} = \frac{q}{b''} = \frac{r}{c''} \text{ respectively;}$$

and these intersect in the point

$$\frac{p}{a''} = \frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''}.$$

Hence

XXXIX. *The straight lines joining the centres of the surfaces* (64), (65), (66) *and* (67) (*which touch the edges of the two tetrahedra* $ABCD$ *and* $A'B'C'D'$ *at the points of intersection of these edges*), *to the corresponding angles of the third tetrahedron* $OO'O''O'''$ *intersect in a point.*

It is scarcely necessary to observe that these theorems admit of being generalized (by substituting *poles* for *centres*, &c.)

I come, finally, to the subject alluded to at page 118.

In my third memoir 'On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane,' (*Journal*, new series, vol. VI. p. 132), I have shewn under what conditions, at once necessary and sufficient, a duodecangular octahedron

will be inscriptible in, and an octangular dodecahedron* circumscriptible about, a surface of the second degree, namely, in the former case that the four straight lines in which the opposite faces intersect, and in the latter that the four straight lines joining opposite angles, shall lie in a ruled hyperboloid and belong to the same system of generators. We cannot, of course, inquire under what conditions a duodecangular octahedron will be circumscriptible about, or an octangular dodecahedron inscriptible in, a surface of the second degree, because innumerable surfaces may evidently be always inscribed in the former, and circumscribed about the latter :† but we may inquire under what conditions, at once necessary and sufficient, a surface of the second degree may touch the edges of these figures; and these conditions it is my present object to ascertain.

I begin by finding the equations to the faces of a duodecangular octahedron whose edges touch a surface of the second degree.

Let $t = 0$, $u = 0$, $v = 0$, $w = 0$, be the equations to the hexagonal faces of a duodecangular octahedron; then, supposing t, u, v, and w to have been multiplied by the proper constants, the equation to the surface of the second degree touching the edges of the duodecangular octahedron may be denoted by

$$t^2 + u^2 + v^2 + w^2 - 2tu - 2tv - 2tw - 2uv - 2uw - 2vw = 0 \ldots (a),$$

for the edges of the tetrahedron ($tuvw$) are edges of the octahedron.

Let $$T = t + \beta u + \gamma v + \delta w = 0$$

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* To avoid unnecessary reference, I copy the definitions of these two solid figures from the page cited in the text.

1. A *duodecangular octahedron* is a solid figure generated by taking a tetrahedron, and cutting off a portion towards each angular point by a plane.

2. An *octangular dodecahedron* is a solid figure generated by taking four tetrahedra whose *bases* are respectively equal to the faces of a fifth tetrahedron, and applying the bases of the former to the faces equal to them of the latter.

It is obvious that the octahedron has four hexagonal and four triangular faces, and the dodecahedron four hexahedral and four trihedral angles.

† We may, however, ask under what conditions the faces of duodecangular octahedron are the eight common tangent planes, and the angular points of an octangular dodecahedron the eight points of intersection, of three surfaces of the second degree. It is easy by the first four theorems of my second memoir 'On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane,' (*Journal*, new series, vol. v. pp. 58 and 60) to give these conditions, which, however, may be said to belong rather to the system of planes or points than to the solid figure.

denote the triangular face of the octahedron opposite to the hexagonal face t. Since the edge (uT) or $u = t + \gamma v + \delta w = 0$ touches the surface (a), putting $u = 0$ and $t = -\gamma v - \delta w$ in (a), the resulting equation

$$(\gamma + 1)^2 v^2 + (\delta + 1)^2 w^2 + 2\{(\gamma + 1)(\delta + 1) - 2\} vw = 0$$

must be a complete square. This requires that

$$(\gamma + 1)^2 (\delta + 1)^2 = \{(\gamma + 1)(\delta + 1) - 2\}^2,$$

or

$$(\gamma + 1)(\delta + 1) = 1.$$

Proceeding in a similar manner, we find that

$$(\beta + 1)(\delta + 1) = 1,$$

and

$$(\beta + 1)(\gamma + 1) = 1,$$

if the edges (vT) and (wT) touch the surface (a).

These three equations being solved give

either

$$\beta = \gamma = \delta = 0,$$

or

$$\beta = \gamma = \delta = -2,$$

the former of which being inadmissible, we must take the latter, and then we have

$$-T = -t + 2u + 2v + 2w,$$

so that

$$-\tfrac{1}{2}t + u + v + w = 0$$

is the equation to the triangular face opposite to the hexagonal face t. In a similar manner the equations to the other triangular faces may be found, and collecting the whole, we have

$$\left.\begin{aligned} -\tfrac{1}{2}t + u + v + w &= 0 \\ -\tfrac{1}{2}u + t + v + w &= 0 \\ -\tfrac{1}{2}v + t + u + w &= 0 \\ -\tfrac{1}{2}w + t + u + v &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (104).$$

Hence, if the edges of a duodecangular octahedron touch a surface of the second degree, its hexagonal faces may be denoted by $t = 0$, $u = 0$, $v = 0$ and $w = 0$, its triangular faces by (104), and the surface itself by (2) or (a).

Now it is evident at a glance that the opposite faces intersect in straight lines in the plane $t + u + v + w = 0$, but this implies only *five* conditions; whereas, that a surface of the second degree should touch the edges (eighteen in number) of a duodecangular octahedron requires *nine* conditions. The four conditions wanting can, however, be supplied as follows. It is evident that each hexagonal force is circumscribed about

a conic, so that the diagonals joining the opposite angles of each will intersect in a point (by Brianchon's theorem); and since there are four hexagonal faces, this will supply four conditions. We have, therefore, the following theorem:

XL. *If a surface of the second degree touch the edges of a duodecangular octahedron, the opposite faces will intersect in four straight lines in one plane, and the three diagonals joining the opposite angles of each hexagonal face will intersect in a point.*

It remains to be seen whether the conditions found are sufficient as well as necessary; or, in other words, whether the preceding theorem is convertible. I shall first enunciate the converse theorem, and then shew that it is true.

XLI. *If the opposite faces of a duodecangular octahedron intersect in four straight lines in one plane; and if, moreover, the three diagonals joining the opposite angles of each hexagonal face intersect in a point; the edges of the octahedron will touch a surface of the second degree.*

Let $t = 0$, $u = 0$, $v = 0$, and $w = 0$, be the equations to the hexagonal faces of the octahedron; then, supposing t, u, v, and w to have been multiplied by the proper constants, the equation to the plane in which the opposite faces intersect may be denoted by

$$t + u + v + w = 0;$$

hence the triangular faces will be denoted as follows:

[redacted]

Proceeding in a similar manner with the other hexagonal faces, we obtain

$$1 - \gamma\delta - \alpha\gamma - \alpha\delta + 2\alpha\gamma\delta = 0 \ldots\ldots\ldots (e),$$

$$1 - \beta\delta - \alpha\beta - \alpha\delta + 2\alpha\beta\delta = 0,$$

and $$1 - \beta\gamma - \alpha\beta - \alpha\gamma + 2\alpha\beta\gamma = 0.$$

Deduct (e) from (c); therefore

$$(\alpha - \beta)(\gamma + \delta - 2\gamma\delta) = 0,$$

hence, either $\alpha = \beta$ or $\gamma + \delta - 2\gamma\delta = 0$.

Taking the latter, multiply it by β and add the product to (c), therefore $$1 - \gamma\delta = 0;$$

but the two equations $\gamma + \delta - 2\gamma\delta = 0$ and $1 - \gamma\delta = 0$ give $\gamma = \delta = 1$, which are inadmissible values; hence we must have $\alpha = \beta$. Treating every two of the four equations in the same way, we find

$$\alpha = \beta = \gamma = \delta,$$

and (c) is reduced to

$$1 - 3\alpha^2 + 2\alpha^3 = 0.$$

The three roots of this equation are 1, 1, and $-\frac{1}{2}$, the last only being an available root, so that

$$\alpha = \beta = \gamma = \delta = -\tfrac{1}{2}.$$

Hence the equations (b) coincide with the equations (104), and all the edges of the duodecangular octahedron therefore touch the surface (2) or (a).

It remains to consider this subject for the octangular dodecahedron. Here it would be shortest to employ the method of reciprocal polars, but I shall give an independent investigation.

Let $t = 0$, $u = 0$, $v = 0$, $w = 0$, be the equations to those diagonal planes of the octangular dodecahedron that pass through the four hexahedral angles;* then, if the edges of the dodecahedron touch a surface of the second degree, the equation to the surface may be denoted by (a), for the edges of the tetrahedron $(tuvw)$ are edges of the dodecahedron. Also, since the faces intersecting in that trihedral angle which is opposite to the hexahedral angle (uvw) pass through the

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* In other words, $t=0$, $u=0$, $v=0$, $w=0$, denote the faces of the 'fifth tetrahedron' mentioned in the definition at the foot of p. 142.

edges (tu), (tv), and (tw) respectively, their equations may be denoted by

$$u = at, \quad v = \beta t, \quad \text{and} \quad w = \gamma t.$$

Since the edge in which the faces $u = at$ and $v = \beta t$ intersect touches the surface (a), if we write at and βt in (a) for u and v respectively, the resulting equation

$$(1 + a^2 + \beta^2 - 2a - 2\beta - 2a\beta)\, t^2 + w^2 - 2(1 + a + \beta)\, tw = 0,$$

must be a complete square. This requires

$$1 + a^2 + \beta^2 - 2a - 2\beta - 2a\beta = (1 + a + \beta)^2,$$

which reduces to

$$(a + 1)(\beta + 1) = 1.$$

In a similar manner, we get

$$(a + 1)(\gamma + 1) = 1,$$

and
$$(\beta + 1)(\gamma + 1) = 1.$$

These equations being solved, we have

either
$$a = \beta = \gamma = 0,$$

or
$$a = \beta = \gamma = -2.$$

The former values being clearly inadmissible, we must take the latter and the equations to the three faces become $u + 2t = 0$, $v + 2t = 0$, and $w + 2t = 0$. In a similar manner the equations to the other faces may be obtained, and collecting the whole we see that if the edges of an octangular

Also the equations to the faces that intersect in one of the hexahedral angles, as (uvw), are

$$u + 2v = 0, \quad u + 2w = 0,$$
$$v + 2w = 0, \quad v + 2u = 0,$$

and
$$w + 2u = 0, \quad w + 2v = 0,$$

where the equations to the opposite faces are placed in the same horizontal line. Hence, the opposite faces intersect in straight lines in the plane $u + v + w = 0$; so that the following theorem is established.

XLII. *If a surface of the second degree touch the edges of an octangular dodecahedron, the four straight lines joining the opposite angles will intersect in one point, and the three straight lines in which the opposite faces of each hexahedral angle intersect, will be in one plane.*

Conversely,

XLIII. *If the straight lines joining the opposite angles of an octangular dodecahedron intersect in a point, and if besides the three straight lines in which the opposite faces of each hexahedral angle intersect, lie in a plane; then shall the edges of the octangular dodecahedron touch a surface of the second degree.*

To prove this theorem, let as before $t = 0$, $u = 0$, $v = 0$, $w = 0$, denote the diagonal planes passing through every three of the hexahedral angular points: also let

$$t = u = v = w$$

be the equations to the point in which intersect the straight lines joining the opposite angles. One of these lines (namely that passing through the hexahedral angle (uvw)) is denoted by

$$u = v = w;$$

so that if α be properly assumed,

$$\frac{t}{\alpha} = u = v = w$$

will denote the trihedral angular point on this line, and the equations to the three faces intersecting in this point will be $t = \alpha u$, $t = \alpha v$, and $t = \alpha w$. In like manner the equations to the other faces will be obtained, and collecting the whole we have

$$\left.\begin{array}{lll} t = \alpha u, & t = \alpha v, & t = \alpha w \\ u = \beta t, & u = \beta v, & u = \beta w \\ v = \gamma t, & v = \gamma u, & v = \gamma w \\ w = \delta t, & w = \delta u, & w = \delta v \end{array}\right\} \ldots\ldots\ldots (f).$$

Since none of the angles of the dodecahedron must coincide with the point $t = u = v = w$, it is clear that none of the quantities α, β, γ, or δ can $= 1$.

Now the faces that intersect in the hexahedral angle (uvw) are

$$v = \gamma u, \quad w = \delta u,$$

$$w = \delta v, \quad u = \beta v,$$

and

$$u = \beta w, \quad v = \gamma w,$$

where the equations to the opposite faces are placed in the same horizontal line. Hence, these opposite faces intersect in the straight lines

$$u = \frac{v}{\gamma} = \frac{w}{\delta},$$

$$v = \frac{u}{\beta} = \frac{w}{\delta},$$

and

$$w = \frac{u}{\beta} = \frac{v}{\gamma};$$

and if these straight lines are in the plane $lu + mv + nw = 0$, we must have

$$l + \gamma m + \delta n = 0,$$

$$\beta l + m + \delta n = 0,$$

and

$$\beta l + \gamma m + n = 0;$$

[redacted]

ON A PROBLEM IN COMBINATIONS.

By R. R. Anstice.

(Continued from Vol. vii. p. 292*).*

To complete the discussion of the problem in combinations which I have begun, I will now give the general expression, a particular case of which was considered in my former paper, and add a sketch of the proof.

Let n be an odd number; let $3.2^{s+1}.n+1$ be a prime; and let r be a primitive root thereto. Let ρ, ρ_1 be any two different roots of the equivalence

$$x^{3n}+1\equiv 0 \quad (\text{modulus } 3.2^{s+1}.n+1).$$

(This same modulus will be understood in all the equivalences in this paper where no other is expressed.)

Let α be any odd number less than 2^{s+1}. Also, as before, let k be the constant term, and P_0P_1, &c., Q_0Q_1, &c. the successive members of the two cycles. Then primary arrangement

$$= kP_0Q_0$$

$$+\Sigma\Sigma P_{\frac{\rho_1-\rho}{\rho_1-1}r^{2^{s+1}i+\alpha i'}},\quad Q_{r^{2^{s+1}i+\alpha i'}},\quad Q_{\rho r^{2^{s+1}i+\alpha i'}}$$

from $i=0$ to $i=3n-1$, and from $i'=0$ to $i'=2^s-1$,

$$+\Sigma\Sigma P_{\frac{\rho_1-\rho}{\rho_1-1}r^{2^s(2i+1)+\alpha i'}},\quad P_{\frac{\rho_1-\rho}{\rho_1-1}r^{2^s(2i+2n+1)+\alpha i'}},\quad P_{\frac{\rho_1-\rho}{\rho_1-1}r^{2^s(2i+4n+1)+\alpha i'}}$$

from $i=0$ to $i=n-1$, and from $i'=0$ to $i'=2^s-1$.

First, to the letter P all the subscripts are annexed, *i.e.* none are repeated. These subscripts are the different terms of the series

$$\Sigma\Sigma\frac{\rho_1-\rho}{\rho_1-\rho}r^{2^s j+\alpha i'}$$

from $j=0$ to $j=6n-1$, and from $i'=0$ to $i'=2^s-1$.

If possible then, let

$$\frac{\rho_1-\rho}{\rho_1-1}r^{2^s j+\alpha i'}\equiv\frac{\rho_1-\rho}{\rho_1-1}r^{2^s J+\alpha I'},$$

where J and I' are comprised between the same limits as j and i'.

Therefore $$r^{2^s(J-j)+\alpha(I'-i')}\equiv 1,$$
and, consequently,

$$2^s(J-j)+\alpha(I'-i')\equiv 3.2^{s+1}.n\lambda.$$

λ being some integer. Therefore $I' - i'$ must be divisible by 2^s. But the difference $I' - i'$ lies between the limits (inclusive) $\pm(2^s - 1)$, and therefore cannot be divisible by 2^s unless it vanishes.

Let then $I' - i' = 0$, and divide both sides by 2^s. Therefore $J - j = 6n\lambda$. Here the value $\lambda = 0$ is inadmissible, since we cannot have $J = j$ and $I' = i'$ together. And so is any other value; for the difference $J - j$ lies between the limits (inclusive) $\pm(6n - 1)$. So the above equation is impossible.

Next, to the letter Q all the subscripts are annexed: *i.e.* none are repeated. The possibility of the repetition of any of the subscripts of Q is easily seen to depend on the possibility of any of the three equivalences

$$\left.\begin{array}{rcl} r^{2^{s+1}i+\alpha i'} & \equiv & r^{2^{s+1}I+\alpha I'} \\ \rho . r^{2^{s+1}i+\alpha i'} & \equiv & \rho . r^{2^{s+1}I+\alpha I'} \\ \rho . r^{2^{s+1}i+\alpha i'} & \equiv & r^{2^{s+1}I+\alpha I'} \end{array}\right\}.$$

It being given that both i and I are less than $3n$; that both i' and I' are less than 2^s; and further, that we cannot have simultaneously $i = I$ and $i' = I'$.

The two first of these equivalences are manifestly impossible from the same reasoning as before. From the third we get

$$\rho \equiv r^{2^{s+1}(I-i)+\alpha(I'-i')}.$$

But, by the definition of ρ, its most general value, expressed

If we take the upper sign, we must have

$$2^{r+1}(I-i)+\alpha(I'-i'')\equiv 3.2^{r+1}.n\lambda,$$

λ being some integer. Impossible (as before) unless $I'-i''=0$. Let then $I'-i''=0$ and divide both sides by 2^{r+1}.

Therefore $I-i=3n\lambda$, which is impossible, since the difference $I-i$ lies between the limits (inclusive) $\pm(3n-1)$, and the value $\lambda=0$ is inadmissible, as before.

If we take the lower sign, we must have

$$2^{r+1}(I-i)+\alpha(I'-i'')=3.2^{r}n+3.2^{r+1}.n\lambda,$$

λ being some integer. Impossible (as before) unless $I'-i''=0$. And if this is the case, still impossible, since then we must have

$$2(I-i)=3n+6n\lambda,$$

i.e. an even number equal to an odd one.

The rest of the proof may be easily inferred.

The number of distinct species remains to be considered.

Now in the general expression, as it stands, we give to the variables i and i'' successive integral values, beginning from 0 in each case. But in proving the efficiency of the cycles, we are only concerned with the *differences* $I-i$ and $I'-i''$. Therefore we might have taken for i or for i'' successive integers, commencing from *any* integer as origin, and the cycles would still have been efficient. Now what effect does this change of origin of i or of i'' have on the form of the general expression?

First, suppose the origin of i is changed. This will have no effect whatever. It is true that i is summed in one case between limits 0 and $3n-1$, and in another between limits 0 and $n-1$. But we might have used in this case the same limits as in the former and divided the result by 3, as we should only have reproduced the same triads. Now if we attribute to i any value equal to or greater than $3n$, only its *remainder* when divided by $3n$ will have any effect, since i is always multiplied by 2^{r+1} and $r^{3.2^{r+1}n}\equiv 1$.

Next, suppose the origin of i'' is changed. Suppose that successive integral values are to be substituted for i'' beginning from λ instead of from 0. This will clearly be the same as if we wrote $i''+\lambda$ in place of i'' in the general expression, and then measured the i'''s from origin 0 as before. This certainly may change the form of the expression not the system generated, since it is the same cha would be wrought by multiplying all the subscripts

same factor $r^{\alpha\lambda}$. Consequently we do not extend the expression by altering the origin of i'.

Again, the efficiency of the cycles would have been demonstrated in just the same way, if we had taken for α any odd number whatever. What effect then is produced on the form of the general expression by taking for α odd numbers greater than 2^{s+1}?

In place of α write $2^{s+1}\lambda + \alpha$; and the same effect is produced as by writing in place of i, $i + \lambda i'$; that is, it has the effect only of changing the origin of i, and therefore produces no alteration in the general expression.

Or again, if we take another primitive root; if we write r^{β} in place of r, where β is any number prime to $6n$, it is easily seen (though the explanation would be rather tedious), that though we may change the form of the primary arrangement, it would be only for another included in the same expression.

Lastly, what effect is produced on the form of the primary arrangement by multiplying all the subscripts by any the same number prime to the modulus, an operation which will not alter the system generated? Such a number must always be equivalent to some power of r. Let it $\equiv r^{\mu}$.

(1). Let μ be divisible by 2^{s+1}. Then the origin of i only will be altered and no effect result.

As an example, let us apply the formula to 27 symbols. Here $n = 1$, $s = 1$, $3.2^{s+1}n + 1 = 13$ and is prime. A primitive root of 13 is 6. Taking then $r = 6$ throughout, we can construct six forms of the primary arrangement which will generate six distinct systems, thus:

I. $(\rho \equiv r^2 \equiv 10,\ \rho_1 = -1,\ \alpha = 1)$

$$\begin{aligned} kp_0q_0 &+ p_{12}q_1q_{10} + p_4q_9q_{12} + p_{10}q_3q_4 \\ &+ p_7q_6q_8 + p_{11}q_2q_7 + p_8q_5q_{11} \\ &+ p_3p_1p_9 + p_5p_6p_2. \end{aligned}$$

II. $(\rho = 10,\ \rho_1 = -1,\ \alpha = 3)$

$$\begin{aligned} kp_0q_0 &+ p_{12}q_1q_{10} + p_4q_9q_{12} + p_{10}q_3q_4 \\ &+ p_5q_8q_2 + p_6q_7q_5 + p_2q_{11}q_6 \\ &+ p_3p_1p_9 + p_{11}p_8p_7. \end{aligned}$$

III. $(\rho = 10,\ \rho_1 \equiv \dfrac{1}{\rho} \equiv 4,\ \alpha = 1)$

$$\begin{aligned} kp_0q_0 &+ p_{11}q_1q_{10} + p_8q_9q_{12} + p_7q_3q_4 \\ &+ p_1q_6q_8 + p_9q_2q_7 + p_3q_5q_{11} \\ &+ p_6p_2p_5 + p_{10}p_{12}p_4. \end{aligned}$$

IV. $(\rho = 10,\ \rho_1 = 4,\ \alpha = 3)$

$$\begin{aligned} kp_0q_0 &+ p_{11}q_1q_{10} + p_8q_9q_{12} + p_7q_3q_4 \\ &+ p_{10}q_8q_2 + p_{12}q_7q_5 + p_4q_{11}q_6 \\ &+ p_6p_2p_5 + p_9p_3p_1. \end{aligned}$$

V. $(\rho = -1,\ \rho_1 = 10,\ \alpha = 1)$

$$\begin{aligned} kp_0q_0 &+ p_7q_1q_{12} + p_{11}q_9q_4 + p_8q_3q_{10} \\ &+ p_3q_6q_7 + p_1q_2q_{11} + p_9q_5q_8 \\ &+ p_5p_6p_2 + p_4p_{10}p_{12}. \end{aligned}$$

VI. $(\rho = -1,\ \rho_1 = 10,\ \alpha = 3)$

$$\begin{aligned} kp_0q_0 &+ p_7q_1q_{12} + p_{11}q_9q_4 + p_8q_3q_{10} \\ &+ p_4q_8q_5 + p_{10}q_7q_6 + p_{12}q_{11}q_2 \\ &+ p_5p_6p_2 + p_1p_9p_3. \end{aligned}$$

We might, indeed, give other forms of the primary arrangement than these, but they would all generate other of these systems. These are all, I mean, result from the formula given: for the method of

and Mr. Kirkman, I should imagine, would generate distinct systems from these.

November 20, 1852.

Note.—I have determined, I believe, in this and my former paper, the number of distinct species comprised in the formulæ given. But I now find there are more general formulæ, including the formulæ given as particular cases: and three formulæ in particular, each comprising many distinct species, which are applicable even when the modulus is composite, provided all its factors are of the form $6n+1$.

April 13, 1853.

ELEMENTARY INVESTIGATION OF THE FORMULÆ FOR THE VARIATIONS OF THE INCLINATION AND LONGITUDE OF THE LINE OF NODES.

By R. Townsend.

The following elementary but rigorous investigation of the formulæ for the variations of the inclination and longitude of the line of nodes may be acceptable to the younger class of students entering upon the subject of the planetary perturbations.

Round the sun, as centre, let a sphere of unit radius be conceived described intersecting the fixed plane of reference

connecting $q'o$, and producing the connecting great circle to meet aa and bb at x' and y' respectively, the plane of the great circle $x'oy'$ will be that of the new instantaneous orbit at the end of the time dt, and xx' and yy' will be the momentary changes in the longitude of the line of nodes and in the inclination of the orbit which it is our object to calculate in terms of the ordinary quantities usually employed for the purpose.

Letting fall the small perpendicular $x'x''$ from x' on xy, and denoting by ϕ the argument of latitude or the arc ox, we have

$$di = yy' \text{ and } d\omega = xx' = \frac{x'x''}{\sin i};$$

but, from the proportions

$$qq' : x'x'' :: \tfrac{1}{2}pq : \sin\phi \quad \text{and} \quad qq' : yy' :: \tfrac{1}{2}pq : \cos\phi,$$

we have $\quad yy' = \dfrac{2qq'}{pq} \cdot \cos\phi \quad$ and $\quad x'x'' = \dfrac{2qq'}{pq} \cdot \sin\phi.$

Let P be the perpendicular component of the disturbing force, that which alone causes the plane of the orbit to change position; then, since from its action alone continued during the time dt, the body has been forced out of that plane to a distance $= r.qq'$, r being the radius vector of the planet, we have

$$P = \frac{2r.qq'}{dt^2}, \text{ and therefore } 2qq' = \frac{Pdt^2}{r};$$

so that $\quad yy' = \dfrac{P.dt^2}{r.pq} \cdot \cos\phi \quad$ and $\quad x'x'' = \dfrac{P.dt^2}{r.pq} \cdot \sin\phi;$

but $r^2.pq = Hdt$, H being the momentary elementary area of the planet in the plane of its orbit; hence, substituting from this for pq, and dividing by dt, we have finally

$$\frac{di}{dt} = \frac{P}{H} \cdot r\sin\phi \quad \text{and} \quad \frac{d\omega}{dt} = \frac{P}{H} \cdot \frac{r\cos\phi}{\sin i} \quad \ldots\ldots (1),$$

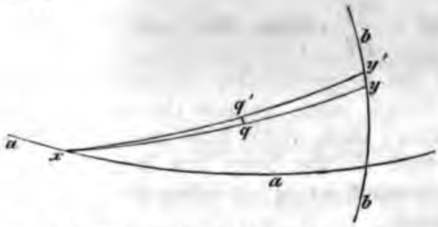
the most general expressions for the required variations, whatever be the inclination of the orbit or the nature of the curve described by the body.

But these formulæ not being practically applicable in their present form, as containing explicitly the actual position of the planet, the force P may be easily expressed in terms of the partial differential coefficient of the ordinary distur[illegible] function R with respect to either of the variables i or ω

thus the simplest form of the ordinary formulæ may be readily obtained.

Let R and $R + dR$ be the values of the disturbing function for the positions q and q' respectively; then, from the characteristic property of that function, the disturbing force along the line joining those positions, that is the force P, $= -\dfrac{dR}{r.qq'}$, the distance $r.qq'$ being considered positive, the actual direction of the force being of course from q towards q', and R being supposed to have the same sign as in Airy and Pratt.

Now the change of position from q to q' might be conceived to take place by either of the two variables i or ω receiving a small change, the other in common with all the remaining elements of the orbit remaining unaltered, and we should have



In one case, i alone varying, (fig. 2)

first and second of formulæ (1) respectively, we get

$$\frac{di}{dt} = \frac{1}{H\sin i} \cdot \frac{dR}{d\omega} \quad \text{and} \quad \frac{d\omega}{dt} = -\frac{1}{H\sin i} \cdot \frac{dR}{di} \;\ldots\ldots(2);$$

the ordinary formulæ in their simplest form expressed like those for all the other variations in terms of the elements of the instantaneous orbit, partial differential coefficients of R with respect to the elements, and the time.

In an excellent article on the present subject by Mr. Blackburne, in the first volume of this *Journal* (new series), the formulæ (1) and (2) are obtained in a different way, and some important remarks are added in explanation of the proper method of applying them. See pp. 37 to 45.

Trinity College, Dublin,
Dec. 31, 1852.

ON DEFINITE INTEGRALS SUGGESTED BY THE THEORY OF HEAT.

By W. H. L. Russell, Esq., B.A., Shepperton, Middlesex.

Poisson has considered in the tenth chapter of his *Théorie de la Chaleur*, the case of the distribution of heat in a sphere of very great radius situated in a medium of variable temperature. In the course of his investigation he remarks that a certain part of the temperature of the interior of the sphere arising from a supposed increment of the exterior temperature vanishes with the time, and thus finds

$$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\frac{1}{2}\alpha}}\left\{(p+\sqrt{\tfrac{1}{2}\alpha})\cos x\sqrt{\tfrac{1}{2}\alpha} - \sqrt{\tfrac{1}{2}\alpha}\sin x\sqrt{\tfrac{1}{2}\alpha}\right\} d\alpha}{(\alpha'^2 - \alpha^2)(\alpha + p\sqrt{2\alpha} + p^2)}$$

$$= \frac{\pi\varepsilon^{-x\sqrt{\frac{1}{2}\alpha'}}\left\{(p+\sqrt{\tfrac{1}{2}\alpha'})\sin x\sqrt{\tfrac{1}{2}\alpha'} + \sqrt{\tfrac{1}{2}\alpha'}\cos x\sqrt{\tfrac{1}{2}\alpha'}\right\}}{2\alpha'(\alpha' + p\sqrt{2\alpha'} + p^2)}.$$

He then says, that this process "nous fait connaitre la valeur d'une integrale definie que l'on n'obtiendrait par aucun procedé direct, mais qui n'en est pas moins certaine, puis quelle est une consequence necessaire de notre analyse."

Now I am going to shew that this is not the case, but that the integral can be obtained without any reference to the Theory of Heat and by a direct process. I shall also investigate the values of some other integrals suggested by this process.

Let $u = \int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 - \alpha^2}$, whence

$$\frac{d^4u}{dx^4} + 4\alpha'^2 u = 4\int_0^\infty \varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha = 0,$$

therefore

$$u = C_1\varepsilon^{-x\sqrt{\alpha'}}\cos x\sqrt{\alpha'} + C_2\varepsilon^{-x\sqrt{\alpha'}}\sin x\sqrt{\alpha'} + C_3\varepsilon^{x\sqrt{\alpha'}}\cos x\sqrt{\alpha'} + C_4\varepsilon^{x\sqrt{\alpha'}}\sin x\sqrt{\alpha'}.$$

Now the integral cannot go on perpetually increasing with x, consequently $C_3 = C_4 = 0$. It remains to determine C_1 and C_2.

Let $x = 0$, then $C_1 = \int_0^\infty \frac{d\alpha}{\alpha'^2 - \alpha^2} = 0$.

Differentiate with regard to (x) and put $x = 0$, then we have

$$-\int_0^\infty \frac{d\alpha\sqrt{\alpha}}{\alpha'^2 - \alpha^2} = -\int_0^\infty \frac{d\sqrt{\alpha}}{\alpha' - \alpha} + \int_0^\infty \frac{d\sqrt{\alpha}}{\alpha' + \alpha} = C_2\sqrt{\alpha'}, \quad \therefore\ C_2 = \frac{\pi}{2\alpha'}.$$

Hence we have

$$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 - \alpha^2} = \frac{\pi}{2\alpha'}\, \varepsilon^{-x\sqrt{\alpha'}} \sin x\sqrt{\alpha'}.$$

Put $\frac{1}{2}\alpha$ for α and $\frac{1}{2}\alpha'$ for α', and multiply by ε^{-px}, and we find

$$\int_0^\infty \frac{\varepsilon^{-x(p+\sqrt{\frac{1}{2}\alpha})} \cos x\sqrt{\frac{1}{2}\alpha}\, d\alpha}{\alpha'^2 - \alpha^2} = \frac{\pi}{2\alpha'}\, \varepsilon^{-x(p+\sqrt{\frac{1}{2}\alpha'})} \sin x\sqrt{\tfrac{1}{2}\alpha'}.$$

therefore

$$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^4 + \alpha^4}$$

$$= \frac{\pi}{2\alpha'^3\sqrt{2}}\, \varepsilon^{-x\sqrt{\alpha'}\frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}}} \left\{\sin x\sqrt{\alpha'}\, \frac{\sqrt{(2-\sqrt{2})}}{\sqrt{2}} + \cos x\sqrt{\alpha'}\, \frac{\sqrt{(2-\sqrt{2})}}{\sqrt{2}}\right\}.$$

If $u = \int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 + \alpha^2}$, we find $\frac{d^4u}{dx^4} - 4\alpha'^2 u = 0$,

therefore $u = C_1\, \varepsilon^{-x\sqrt{2\alpha'}} + C_2 \varepsilon^{x\sqrt{2\alpha'}} + C_3 \cos x\sqrt{2\alpha'} + C_4 \sin x\sqrt{2\alpha'}$.

The integral must continually decrease as x increases, therefore $C_2 = C_3 = C_4 = 0$.

If we put $x = 0$, we have

$$C_1 = \int_0^\infty \frac{d\alpha}{\alpha'^2 + \alpha^2} = \frac{\pi}{2\alpha'},$$

therefore
$$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 + \alpha^2} = \frac{\pi}{2\alpha'}\, \varepsilon^{-x\sqrt{2\alpha'}}.$$

Put $\frac{1}{2}\alpha$ for α, and $\frac{1}{2}\alpha'$ for α', and multiply by ε^{-px}, then

$$\int_0^\infty \frac{\varepsilon^{-(p+\sqrt{\frac{1}{2}\alpha})x} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 + \alpha^2} = \frac{\pi}{2\alpha'}\, \varepsilon^{-(p+\sqrt{\alpha'})x},$$

$$\therefore \int_0^\infty \frac{\varepsilon^{-x\sqrt{\frac{1}{2}\alpha}}\{(p + \sqrt{\tfrac{1}{2}\alpha}) \cos x\sqrt{\tfrac{1}{2}\alpha} - \sqrt{\tfrac{1}{2}\alpha} \sin x\sqrt{\tfrac{1}{2}\alpha}\}}{(\alpha'^2 + \alpha^2)(\alpha + p\sqrt{2\alpha} + p^2)} = \frac{\pi}{2\alpha'}\, \frac{\varepsilon^{-x\sqrt{\alpha'}}}{p + \sqrt{\alpha'}}.$$

NOTES ON MOLECULAR MECHANICS.

By the Rev. Samuel Haughton.

No. 2.—*Propagation of Plane Waves.*

The equations of small motions of elastic media are as follows:

$$\left.\begin{aligned}
-\varepsilon \frac{d^2\xi}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\alpha_1} + \frac{d}{dy}\cdot\frac{dV}{d\alpha_2} + \frac{d}{dz}\cdot\frac{dV}{d\alpha_3} \\
-\varepsilon \frac{d^2\eta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\beta_1} + \frac{d}{dy}\cdot\frac{dV}{d\beta_2} + \frac{d}{dz}\cdot\frac{dV}{d\beta_3} \\
-\varepsilon \frac{d^2\zeta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\gamma_1} + \frac{d}{dy}\cdot\frac{dV}{d\gamma_2} + \frac{d}{dz}\cdot\frac{dV}{d\gamma_3}
\end{aligned}\right\} \ldots (4).*$$

* Vide vol. IV. p. 174 and p. 173, for the meaning of α_1, α_2, &c.

Let V be the most general function of the second order of the differential coefficients α_1, α_2, &c.; then

$$-2V = (a_1^2)\,\alpha_1^2 + (a_2^2)\,\alpha_2^2 + \&c. + (b_1^2)\,\beta_1^2 + \&c.$$
$$+ (a_1b_2)\,\alpha_1\beta_2 + \&c.\ (c_2a_3)\,\gamma_2\alpha_3 + \&c.$$

Introducing this value of V into equations (4), we find

$$\left.\begin{aligned}
-\varepsilon\frac{d^2\xi}{dt^2} &= (a_1^2)\frac{d^2\xi}{dx^2} + (a_2^2)\frac{d^2\xi}{dy^2} + (a_3^2)\frac{d^2\xi}{dz^2}\\
&\quad + 2(a_2a_3)\frac{d^2\xi}{dy\,dz} + 2(a_1a_3)\frac{d^2\xi}{dx\,dz} + 2(a_1a_2)\frac{d\xi^2}{dx\,dy}\\
&\quad + (a_1b_1)\frac{d^2\eta}{dx^2} + (a_2b_2)\frac{d^2\eta}{dy^2} + (a_3b_3)\frac{d^2\eta}{dz^2}\\
&+ (a_2b_3 + a_3b_2)\frac{d^2\eta}{dy\,dz} + (a_1b_3 + a_3b_1)\frac{d^2\eta}{dx\,dz} + (a_1b_2 + a_2b_1)\frac{d^2\eta}{dx\,dy}\\
&\quad + (a_1c_1)\frac{d^2\zeta}{dx^2} + (a_2c_2)\frac{d^2\zeta}{dy^2} + (a_3c_3)\frac{d^2\zeta}{dz^2}\\
&+ (a_2c_3 + a_3c_2)\frac{d^2\zeta}{dy\,dz} + (a_1c_3 + a_3c_1)\frac{d^2\zeta}{dx\,dz} + (a_1c_2 + a_2c_1)\frac{d^2\zeta}{dx\,dy}
\end{aligned}\right\} \ldots\ldots (9),$$

$$-\varepsilon\frac{d^2\eta}{dt^2} = \&c.$$

$$-\varepsilon\frac{d^2\zeta}{dt^2} = \&c.$$

in which

$$P' = (a_1^2)l^2 + (a_2^2)m^2 + (a_3^2)n^2 + 2(a_2a_3)mn + 2(a_1a_3)ln + 2(a_1a_2)lm,$$
$$Q' = (b_1^2)l^2 + (b_2^2)m^2 + (b_3^2)n^2 + 2(b_2b_3)mn + 2(b_1b_3)ln + 2(b_1b_2)lm,$$
$$R' = (c_1^2)l^2 + (c_2^2)m^2 + (c_3^2)n^2 + 2(c_2c_3)mn + 2(c_1c_3)ln + 2(c_1c_2)lm,$$
$$F' = (b_1c_1)l^2 + (b_2c_2)m^2 + (b_3c_3)n^2 + (b_2c_3 + b_3c_2)mn + (b_1c_3 + b_3c_1)ln + (b_1c_2 + b_2c_1)lm,$$
$$G' = (a_1c_1)l^2 + (a_2c_2)m^2 + (a_3c_3)n^2 + (a_2c_3 + a_3c_2)mn + (a_1c_3 + a_3c_1)ln + (a_1c_2 + a_2c_1)lm,$$
$$H' = (a_1b_1)l^2 + (a_2b_2)m^2 + (a_3b_3)n^2 + (a_2b_3 + a_3b_2)mn + (a_1b_3 + a_3b_1)ln + (a_1b_2 + a_2b_1)lm.$$

Equations (10) are the well-known equations which determine the axes of the ellipsoid

$$P'x^2 + Q'y^2 + R'z^2 + 2F'yz + 2G'xz + 2H'xy = 1 \ldots (11);$$

there are, therefore, three possible directions of molecular vibration for a given direction of wave plane; and there will be three parallel waves moving with velocities determined by the magnitude of the axes of the ellipsoid, the direction of vibration in each plane wave being parallel to one of the axes.

The construction of the direction of molecular vibration just found was given by M. Cauchy* for a system of attracting and repelling molecules; it is here shewn to be a necessary consequence of the assumption of a function such as V, to represent the effects of molecular force. Hence we may state the following theorem:

If the sum of the molecular moments of an elastic body can be represented by the variation of a single function, the directions of molecular vibration, corresponding to a given direction of wave plane, must be at right angles to each other.

The converse of this theorem in molecular mechanics has been proved by Professor Jellett.†

The cubic equation whose roots are the squares of the reciprocals of the axes of the ellipsoid (11) is

$$(P' - s)(Q' - s)(R' - s) - F'^2(P' - s) - G'^2(Q' - s) - H'^2(R' - s) + 2F'G'H' = 0,$$

where $s = \varepsilon v^2$; hence, if P, Q, R, F, G, H be the same

* *Exercises des Mathematiques*, tom. v. p. 32.

† *Transactions of Royal Irish Academy*, vol. XXII. p. 195.

functions of x, y, z, that P', Q', R', F', G', H' are of l, m, n; it can easily be shewn that the surface of *wave-slowness* is

$$(P-1)(Q-1)(R-1)-F^2(P-1)-G^2(Q-1)-H^2(R-1) + 2FGH = 0. \ldots (12.)$$

It is natural to inquire whether a series of plane waves may not be propagated, with a continually decreasing amplitude of vibration: to ascertain the possibility of this, let us assume

$$\left.\begin{aligned} \xi &= p\cos\alpha \,.\, e^{-\frac{2\pi}{\lambda}\rho(lx+my+nz)} \sin\frac{2\pi}{\lambda}(lx+my+nz-vt) \\ \eta &= p\cos\beta \text{ \&c.} \\ \zeta &= p\cos\gamma \text{ \&c.} \end{aligned}\right\} \ldots (13).$$

This is equivalent to assuming that the amplitude of vibration diminishes as the plane wave moves away from the plane $lx + my + nz = 0$, passing through the origin of coordinates; in fact, $lx + my + nz$ is equal to the perpendicular let fall from the point (x, y, z) upon that plane, and the vibration is here assumed to diminish in geometrical progression, as that perpendicular increases in arithmetical progression.

In order to simplify the introduction of (ξ, η, ζ) into the differential equations, we may use the following symbolical equation,
$$e^{-q}\sin p = \sin\{p + q\sqrt{(-1)}\} \ldots\ldots\ldots (14).$$

giving six equations of condition: but these equations cannot coexist unless $\rho = 0$, *i.e.* unless there be no diminution of intensity.

If we assume

$$\left.\begin{aligned}
&\xi = p\cos\alpha . e^{-\frac{2\pi}{\lambda}\{\rho(lx+my+nz)-Kvt\}}\sin\frac{2\pi}{\lambda}(lz+my+nz-vt)\\
&\eta = \&c.\\
&\zeta = \&c.
\end{aligned}\right\}\ldots(16),$$

so as to introduce another unknown constant K, we should find for the equations of condition

$$\left.\begin{aligned}
&\frac{1-K^2}{1-\rho^2}\,sv^2\cos\alpha = P'\cos\alpha + H'\cos\beta + G'\cos\gamma\\
&\frac{1-K^2}{1-\rho^2}\,sv^2\cos\beta = Q'\cos\beta + \&c.\\
&\frac{1-K^2}{1-\rho^2}\,sv^2\cos\gamma = R'\cos\gamma + \&c.\\
\text{and}\quad &\frac{K}{\rho}\,sv^2\cos\alpha = P'\cos\alpha + H'\cos\beta + G'\cos\gamma\\
&\frac{K}{\rho}\,sv^2\cos\beta = Q'\cos\beta + \&c.\\
&\frac{K}{\rho}\,sv^2\cos\gamma = R'\cos\gamma + \&c.
\end{aligned}\right\}\ldots(17).$$

These equations of condition can only coexist on the supposition that

$$\frac{1-K^2}{1-\rho^2} = \frac{K}{\rho}\ldots\ldots\ldots\ldots(18).$$

Equation (18), which is quadratic, if solved for K, will give

$$K_1 = \rho,\quad K_2 = -\frac{1}{\rho};$$

the latter is to be rejected, as it would give an imaginary velocity, and the former introduced into the expression (16) gives, if $\phi = \frac{2\pi}{\lambda}(lx + my + nz - vt)$,

$$\left.\begin{aligned}
\xi &= p\cos\alpha\, e^{-\rho\phi}\sin\phi\\
\eta &= p\cos\beta\, e^{-\rho\phi}\sin\phi\\
\zeta &= p\cos\gamma\, e^{-\rho\phi}\sin\phi
\end{aligned}\right\}\ldots\ldots(16)\text{ bis.}$$

These values of ξ, η, ζ are functions of the phase, but do not indicate a diminished intensity of vibration: this agrees with a statement of Mr. Rankine's, *Math. Jour.*, vol. VII. p. 218.

It is desirable, before concluding this note, to give some idea of the true nature of the differential equations which have been used. We have assumed that the density is very little altered during the motion of the particles: if we take account of the alteration of density, the equations will be different.

The equation of continuity is

$$\frac{d\varepsilon}{dt} + \frac{d}{dx}(\varepsilon u) + \frac{d}{dy}(\varepsilon v) + \frac{d}{dz}(\varepsilon w) = 0 \;\ldots\ldots\; (19),$$

where u, v, w, denote the velocities parallel to x, y, z. For small displacements this will become

$$\frac{d\varepsilon}{dt} + \varepsilon\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$

or

$$\frac{d\varepsilon}{dt} + \varepsilon\frac{d\omega}{dt} = 0 \;\ldots\ldots\ldots\ldots\; (20),$$

in which $\omega = \dfrac{d\xi}{dx} + \dfrac{d\eta}{dy} + \dfrac{d\zeta}{dz}$ = cubical compression.

Integrating (20), we obtain

$$\log\varepsilon + \omega = F(x, y, z),$$

incompressible, the equation of continuity would become

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0,$$

and equations (22) would coincide with (4); but the medium may be nearly incompressible and equations (4) used. In the case of plane waves the limitation may be thus formed,

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \frac{2\pi}{\lambda} p (l \cos\alpha + m \cos\beta + n \cos\gamma) \cos\phi\,;$$

or, if ϑ be the angle between the wave normal and direction of vibration, equation (21) will be

$$s = s_0 e^{-\frac{2\pi}{\lambda} p \cos\vartheta \cos\phi} \quad \ldots\ldots\ldots\ldots (23).$$

In order that s shall not differ much from s_0, it is, therefore, necessary to suppose that the vibration is nearly transversal, in which case $\cos\vartheta$ is very small; or, that if the vibration be nearly normal, λ must be very great; *i.e.* the velocity of the normal vibration must be very great.

In my next note I shall consider the case of vibrations, rigorously normal and transversal.

Trinity College, Dublin,
Dec. 30, 1852.

THEOREMS IN THE CALCULUS OF OPERATIONS.

By Robert Carmichael.

Those who have studied the Calculus of Operations in connexion with the Integral Calculus cannot but have felt a difficulty in the interpretation of the symbolic results at which they may have arrived. The farther the relation between these two subjects is prosecuted, whether in the solution of Differential Equations, the extension of Definite Integrals, or the reduction of equations in Finite Differences, the more imperative becomes the demand for such interpretation. In all these cases, so long as the solutions are symbolic and not completely evaluated, they are unsatisfactory to the advanced mathematician and perhaps calculated to lead the younger student to undervalue the utility of prosecuting these branches of analysis in conjunction.

The want here indicated can only be satisfied by the contributions of individuals. Much still is required, although

much has already been done. The present paper, with two others previously published in this *Journal,* are offered by me in furtherance of the object.

There is another matter to which it may be well to direct attention. When the questions to be investigated have a symmetrical character, not only should the results be symmetrical, but symmetrical methods should be employed for their deduction. A regard to this latter point might possibly have not only precluded some errors and many incomplete results, but also led the way to the discovery of higher and more elegant methods of analysis.

In conclusion, I would express my acknowledgments to the Rev. Professor Graves, without whose valuable suggestions these pages would never have been written. I have done little more than generalize and apply results communicated by him to the Royal Irish Academy in the month of April, 1852, an abstract of which is published in the Proceedings of that body.

1. Let it be proposed to investigate the value of the symbolic quantity

$$e^{\phi(x)\frac{d}{dx} + \psi(y)\frac{d}{dy} + \chi(z)\frac{d}{dz} + \&c.} U,$$

where $$U = f(x, y, z, \&c.).$$

Now if we put

we get

$$U = f\{\Phi^{-1}(\xi + c),\ \Psi^{-1}(\eta + d),\ X^{-1}(\zeta + e),\ \&c.\},$$

and therefore

$$e^{\frac{d}{d\xi}+\frac{d}{d\eta}+\&c.}\,U = f\{\Phi^{-1}(\xi + c + 1),\ \Psi^{-1}(\eta + d + 1),\ \&c.\};$$

whence, finally,

$$e^{\phi(x)\frac{d}{dx}+\psi(y)\frac{d}{dy}+\&c.}\,f(x, y, \&c.) = f\{\Phi^{-1}(\Phi x + 1), \Psi^{-1}(\Psi y + 1), \&c.\}.$$

In the practical application of this fundamental theorem the only difficulties with which we have to contend are, the deduction of the integrals

$$\int\frac{dx}{\phi(x)},\quad \frac{dy}{\psi(y)},\quad \&c.,$$

and the inversion of the functions Φ, Ψ, &c.

Ex. I. The simplest and most obvious illustration of this theorem is afforded by the suppositions

$$\phi(x) = x,\quad \psi(y) = y,\quad \chi(z) = z,\quad \&c.$$

In this case, in fact, the operative symbol

$$\phi(x)\frac{d}{dx} + \psi(y)\frac{d}{dy} + \chi(z)\frac{d}{dz} + \&c.$$

becomes the index symbol of homogeneous functions

$$x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} + \&c. = \nabla,$$

and therefore

$$e^{\nabla}.f(x, y, \&c.) = f\{\log^{-1}(1 + \log x), \log^{-1}(1 + \log y), \&c.\},$$

or $$e^{\nabla}.f(x, y, z, \&c.) = f(ex, ey, ez, \&c.).$$

If we break up f into sets of homogeneous terms, it is evident that this result is identical with that given in a previous paper, namely

$$e^{\nabla}U = u_0 + eu_1 + e^2u_2 + \&c. + e^nu_n.$$

Ex. II. More generally, let

$$\phi(x) = x^m,\quad \psi(y) = y^n,\quad \&c.,$$

and the result of the evaluation of

$$e^{x^m\frac{d}{dx}+y^n\frac{d}{dy}+\&c.}\,f(x, y, \&c.)$$

is

$$f\left[\frac{x}{\{1-(m-1)x^{m-1}\}^{\frac{1}{m-1}}}, \frac{y}{\{1-(n-1)y^{n-1}\}^{\frac{1}{n-1}}}, \&c.\right].$$

It is not difficult to verify this formula for the particular cases

$$m = n = \&c. = 0,$$

$$m = n = \&c. = 1.$$

In that in which

$$m = n = \&c. = 2,$$

we get the result

$$e^{x^2\frac{d}{dx}+y^2\frac{d}{dy}+\&c.}f(x, y, \&c.) = f\left(\frac{x}{1-x}, \frac{y}{1-y}, \&c.\right).$$

Ex. III. Let

$$\phi(x) = (a^2 - x^2)^{\frac{1}{2}}, \quad \psi(y) = (b^2 - y^2)^{\frac{1}{2}}, \quad \&c.$$

and the result of the evaluation of

$$e^{(a^2-x^2)^{\frac{1}{2}}\frac{d}{dx}+(b^2-y^2)^{\frac{1}{2}}\frac{d}{dy}+\&c.}f(x, y, \&c.)$$

is

$$f\{\sin(1+\sin^{-1}x), \ \sin(1+\sin^{-1}y), \ \&c.\}.$$

Ex. IV. Selecting now a particular form for the function operated on, we shall suppose, the simplest case, that it is linear in x, y, &c. Then

since it is obvious that

$$\phi(x)\frac{d}{dx} + \psi(y)\frac{d}{dy} + \&c.$$

satisfies the required condition.

2. If we operate on both sides of the fundamental theorem with

$$e^{\phi x \frac{d}{dx} + \psi y \frac{d}{dy} + \&c.}$$

we easily find that

$$e^{2\left(\phi x \frac{d}{dx} + \psi y \frac{d}{dy} + \&c.\right)} f(x, y, \&c.) = f\{\Phi^{-1}(\Phi x + 2),\ \Psi^{-1}(\Psi y + 2),\ \&c.\};$$

and hence, in general, that

$$e^{m\left(\phi x \frac{d}{dx} + \psi y \frac{d}{dy} + \&c.\right)} f(x, y, \&c.) = f\{\Phi^{-1}(\Phi x + m),\ \Psi^{-1}(\Psi y + m),\ \&c.\}$$

Thus, the form of f being supposed unknown, and those of Φ and Ψ given, the solution of the equation of finite differences, with constant coefficients,

$$\left.\begin{array}{c} Af\{\Phi^{-1}(\Phi x + a),\ \ \Psi^{-1}(\Psi y + a)\} \\ + \\ Bf\{\Phi^{-1}(\Phi x + b),\ \ \Psi^{-1}(\Psi y + b)\} \\ + \&c. \end{array}\right\} = 0,$$

is reduced to the solution of the symbolic partial differential equation

$$Ae^{a\left(\phi x \frac{d}{dx} + \psi y \frac{d}{dy}\right)} z + Be^{b\left(\phi x \frac{d}{dx} + \psi y \frac{d}{dy}\right)} z + \&c. = 0,$$

which may be written, for brevity,

$$F\left(e^{\phi x \frac{d}{dx} + \psi y \frac{d}{dy}}\right) z = 0;$$

or, by the previous transformation,

$$F\left(e^{\frac{d}{d\xi} + \frac{d}{d\eta}}\right) z = 0.$$

Now, if the roots of $\quad F(p) = 0$

be all real and unequal, the symbolic solution of this equation is

$$z = \left(e^{\frac{d}{d\xi} + \frac{d}{d\eta}} - m\right)^{-1}.\,0 + \left(e^{\frac{d}{d\xi} + \frac{d}{d\eta}} - n\right)^{-1}.\,0 + \&c.,$$

where m, n, &c. are the values of the roots.

But by a previous theorem (*Journal*, 1851)

$$\chi\left(\frac{d}{d\xi}+\frac{d}{d\eta}\right).f_m(e^{\xi}, e^{\eta}) = \chi(m).f_m(e^{\xi}, e^{\eta}) \ldots\ldots (2),$$

f_m being a homogeneous function of the m^{th} degree.

Hence the solution of the symbolic equation, and therefore the solution of the equation of finite differences is

$$z = u_{\log m}(e^{\xi}, e^{\eta}) + u_{\log n}(e^{\xi}, e^{\eta}) + \&c.,$$

where *the forms* of $u_{\log m}$, $u_{\log n}$, &c. are arbitrary, but their degrees given by the suffixes.

Finally, introducing the arbitrary constants c, d, &c., as is evidently legitimate, and then substituting their values for $\xi + c$, $\eta + d$, &c., we get the solution in the form

$$z = u_{\log m}(e^{\Phi x}, e^{\Psi y}) + u_{\log n}(e^{\Phi x}, e^{\Psi y}) + \&c.$$

If

$$F(p) = 0$$

contain pairs of imaginary roots, the solution assumes the form

$$z = u_{\log(m+n\sqrt{-1})}(e^{\Phi x}, e^{\Psi y}) + u_{\log(m-n\sqrt{-1})}(e^{\Phi x}, e^{\Psi y}) + \&c. + u_{\log p} + \&c.$$

Finally, if the same equation contain α equal roots, whose common value is m, the form of the solution is

p, q, &c. being positive or negative, integral or fractional, real or imaginary.

For, throwing the equation into the form

$$F(e^{\frac{d}{dx}+\frac{d}{dy}})\,\phi(x, y, \&c.) = \Sigma A_{p,\,q,\,\&c.}\, e^{px+qy+\&c.},$$

we have, by (2), the solution in the form

$$\phi = \Sigma A_{p,\,q,\,\&c.} \frac{e^{px+qy+\&c.}}{F(e^{p+q+\&c.})} + u_{\log m}(e^x, e^y, \&c.) + \&c.,$$

the roots of $F(p) = 0$ being supposed all real and unequal.

It is evident that the solution of the equation in finite differences, in which there is but a single variable, is but a particular case of the form now stated.

Trinity College, Dublin,
March 1853.

ON THE RELATION BETWEEN THE VOLUME OF A TETRAHEDRON AND THE PRODUCT OF THE *sixteen* ALGEBRAICAL VALUES OF ITS SUPERFICIES.

By J. J. SYLVESTER, F.R.S.

THE area of a triangle is related (as is well known) in a very simple manner to the 8 algebraical values of its perimeter: If we call the values of the squared sides of the triangle a, b, c, there will be nothing to distinguish the algebraical affections of sign of the simple lengths so as to entitle one to a preference over the other. The area of the triangle can only vanish by reason of the three vertices coming into a straight line; hence, according to the general doctrine of characteristics, we must have the Norm of

$$\sqrt{a} + \sqrt{b} + \sqrt{c},$$

containing as a factor some root or power of the expressions for the area of the triangle. The Norm in question being representable as $-N^2$ where N is the Norm of $a^{\frac{1}{2}} \pm b^{\frac{1}{2}} \pm c^{\frac{1}{2}}$, which is of 4 dimensions in the elements a, b, c, and undecomposable into rational factors, we infer that to a numerical factor *près* the square of the area must be identical with the Norm N, and thus, by a logical *coup-de-main*,

completely supersede all occasion for the ordinary geometrical demonstration given of this proposition, which in its turn, with certain superadded definitions, would admit of being adopted as the basis of an absolutely pure system of Analytical Trigonometry that should borrow nothing from the methods and results of sensuous or practical geometry. But into this speculation it is not my present purpose to enter: what I propose to do is to extend a similar mode of reasoning to space of three dimensions, and to point out a general theorem in determinants which is involved as a consequence in the generalization of the result of the inquiry when pushed forward into the regions of what may be termed Absolute or Universal Rational Space.

Let F, G, H, K be the four squared areas of the faces of a tetrahedron, and V the volume; then, since V only becomes zero in the case of the 4 vertices coming into the same plane, which is characterised by the equation

$$\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K} = 0$$

subsisting, we infer that N the Norm of

$$\sqrt{F} \pm \sqrt{G} \pm \sqrt{H} \pm \sqrt{K}$$

must contain a power of V as a rational factor. V^2 is rational and of 3 dimensions in the squared edges; the Norm above spoken of is of 8 dimensions in the same. Consequently there is a rational factor, say Q, remaining, which is of

Consequently Q is the quantity which characterises the fact of one or more of *the radii of the inscribed spheres becoming infinite.* For the triangle there exists no corresponding property; this we know *à priori*, and can explain also analytically from the fact that if we call P the product of the radii of the 4 inscribable circles, ν the Norm of the perimeter, and A the area, we have

$$P\nu = 2^4 A^4,$$

and
$$\nu = \frac{2^4 A^4}{P} = A^2,$$

which contains no denominator capable of becoming zero, so that as long as the sides remain finite the curvature of the inscribed circles is incapable of vanishing.

To determine N as a function of the edges, and then to discover by actual division the value of $\frac{N}{V^2}$, would be the direct but an excessively tedious and almost impracticably difficult process. I have ever felt a preference for the *à priori* method of discovering forms whose properties are known, and never yet have met with an instance where analysis has denied to gentle solicitation conclusions which she would be loth to grant to the application of force. The case before us offers no exception to the truth of this remark. Q is a function of 5 dimensions in terms of the squared edges: let us begin by finding the value of that part of Q in which at most a certain set of 4 of these edges make their appearance, and to find which consequently the other 2 edges may be supposed zero without affecting the result. We may make two distinct hypotheses concerning these 2 edges; we may suppose that they are opposite, *i.e.* non-intersecting edges, or that they are contiguous, *i.e.* intersecting edges.

To meet the first hypothesis suppose $ab = 0$, $ce = 0$.

For convenience sake, use F, G, H, K to denote 16 times the square of each area, instead of the simple square of the areas. Call

$$16(abc)^2 = K, \quad 16(abd)^2 = H, \quad 16(acd)^2 = G, \quad 16(bcd)^2 = F.$$

Then
$$-K = (ab)^4 + (ac)^4 + (bc)^4 - 2(ab)^2(ac)^2 - 2(ab)^2(bc)^2 - 2(ac)^2(bc)^2$$
$$= ac^4 + bc^4 - 2(ac)^2(bc)^2.$$

Similarly,
$$-H = ad^4 + bd^4 - 2(ad)^2(bd)^2,$$
$$-G = ca^4 + da^4 - 2ca^2\,da^2,$$
$$-F = cb^4 + db^4 - 2cb^2\,db^2.$$

Hence one value of $\sqrt{F}+\sqrt{G}+\sqrt{H}+\sqrt{K}$ will be

$$\sqrt{(-1)}\,\{(ac^2-bc^2)+(bd^2-ad^2)+(da^2-ac^2)+(bc^2-bd^2)\}=0.$$

Hence, on this first supposition, the Norm vanishes. But V^2 does not vanish when $ab=0$, $cd=0$, for it becomes, *saving* a numerical factor,

$$\begin{array}{ccccc} 0 & 0 & ac^2 & ad^2 & 1 \\ 0 & 0 & bc^2 & bd^2 & 1 \\ ca^2 & cb^2 & 0 & 0 & 1 \\ da^2 & db^2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & \end{array}$$

i.e.
$$\begin{aligned} &(ac^2.bd^2-ad^2.bc^2).(cb^2+ad^2-ca^2-bd^2) \\ &+(bc^2-ac^2)(ca^2.db^2-cb^2.da^2) \\ &+(ad^2-bd^2)(ca^2.db^2-cb^2.da^2) \\ =\;& 2\,(ac^2.bd^2-ad^2.bc^2).(ad^2+bc^2-ac^2-bd^2); \end{aligned}$$

and consequently, since N vanishes but V^2 does not vanish, (Q) vanishes, shewing that there is no term in (Q) but what contains one at least of any two opposite edges as a factor; or, in other words, there is no term in Q of which the product of the square of the product of all three sides of some one or other of the 4 faces does not form a constituent part.

that is

$$\{4ad^4 - 4ad^2(bd^2 + cd^2 + bc^2) + 4bc^2.bd^2 + 4bd^2.cd^2 + 4cd^2.bc^2\}$$
$$\times\ \{4ad^4 - 4ad^2(bd^2 + cd^2 - bc^2) + 4bd^2.cd^2\}.$$

Again, since $ac^2 = 0$ and $bc^2 = 0$, V^2 becomes

$$\begin{array}{ccccc} 0 & 0 & 0 & ad^2 & 1 \\ 0 & 0 & bc^2 & bd^2 & 1 \\ 0 & cb^2 & 0 & cd^2 & 1 \\ da^2 & db^2 & dc^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}$$

This is evidently equal to

$$2bc^2\left\{\begin{array}{cccc} 0 & 0 & ad^2 & 1 \\ 0 & cb^2 & cd^2 & 1 \\ da^2 & db^2 & 0 & 1 \\ 1 & 1 & 1 & \end{array}\right\} - bc^4\left\{\begin{array}{ccc} 0 & ad^2 & 1 \\ da^2 & 1 & 1 \\ 1 & 0 & \end{array}\right\}.$$

(the arrows being used to denote the directions of the positive diagonal sets of terms)

$$= 2bc^2\{2bc^2ad^2 + ad^4 - ad^2bd^2 - cd^2ad^2 + bd^2cd^2\}$$
$$- 2bc^4ad^2$$
$$= 2bc^2\{ad^4 - ad^2(bd^2 + cd^2 - bc^2) + bd^2.cd^2\}.$$

Hence, paying no attention to any mere numerical factor, we have found that when $ac = 0$ and $bc = 0$, Q or $\dfrac{N}{V^2}$ becomes

$$bc^2.bd^2.cd^2\{ad^4 - ad^2(bd^2 + cd^2 + bc^2) + bc^2.bd^2 + bd^2.cd^2 + cd^2.bc^2\}.$$

Hence, with the exception of the terms in which 5 out of the six edges enter, the complete value of Q will be

$$\Sigma(bc^2.bd^2.cd^2)\{ad^4 - ad^2(bd^2 + cd^2 + bc^2) + bc^2.bd^2 + bd^2.cd^2 + cd^2.bc^2\},$$

or more fully expressed, and still abstracting from terms containing 5 edges,

$$= \Sigma bc^2.bd^2.cd^2\{(ab^4 + ac^4 + ad^4) - (ab^2 + ac^2 + bc^2)(bd^2 + bc^2 + cd^2)$$
$$+ bc^2.bd^2 + bd^2.cd^2 + cd^2.bc^2\}.$$

It remains only to determine the value of the numerical coefficient affecting each of the 6 terms of the form

$$ab^2.ac^2.ad^2.bc^2.bd^2.$$

To find this, let

$$ab^2 = ac^2 = ad^2 = bc^2 = bd^2 = cd^2 = 1\,;$$

then evidently, since all the squared areas are equal, several of the factors of N will become zero, but V^2 evidently does not become zero for a regular tetrahedron; hence Q becomes zero: and if we call the numerical factor sought for λ, we must have (observing that the Σ includes 4 parts corresponding to each of the 4 faces)

$$4\{3-9+3\}+6\lambda=0,$$

therefore $\qquad -12+6\lambda=0, \text{ or } \lambda=2.$

Hence the complete value of Q is

$$\Sigma ab^2.bc^2.ca^2\,\{(da^4+db^4+dc^4)-(da^2+db^2+dc^2)(ab^2+bc^2+ca^2) + ab^2.bc^2+bc^2.ca^2+ca^2.ab^2\}, + 2\Sigma\,(ab^2.bc^2.cd^2.da^2.ac^2);$$

or, which is the same quantity somewhat differently and more simply arranged,

$$Q=\Sigma(ab^2.bc^2.ca^2)\,\{(da^4+db^4+dc^4+da^2.db^2+db^2.dc^2+dc^2.da) + (ab^2.bc^2+bc^2.ca^2+ca^2.ab^2)-(da^2+db^2+dc^2)(ab^2+bc^2+ca^2)\},$$

and this quantity equated to zero expresses the conditions of a radius of an inscribed sphere becoming infinite. The direct method would have involved, as the first step, the formation of the Norm of a numerator consisting of

$$\sqrt{F}+\sqrt{G}+\sqrt{H}+\sqrt{K}.$$

We may moreover remark that since $ab = 0$ and $cd = 0$ does not make V vanish, the perpendicular distance of ab from cd, which, multiplied by $ab \times cd$, gives 6 times the volumes, must on this supposition become infinite. When three edges lying in the same plane all vanish simultaneously, Q vanishes, since one edge at least in every face of the pyramid vanishes, and V also vanishes, as is evident from the expression for V^2, when $ab = 0$, $ac = 0$, $bc = 0$, becoming a multiple of

$$\begin{matrix} 0 & 0 & 0 & ad^2 & 1 \\ 0 & 0 & 0 & bd^2 & 1 \\ 0 & 0 & 0 & cd^2 & 1 \\ ad^2 & bd^2 & cd^2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{matrix}$$

which is evidently zero.

It appeared to me not unlikely, from the situation and look of Q (the characteristic of one of the inscribed spheres becoming infinite), that it might admit of being represented as a determinant, but I have not succeeded in throwing it under that form. I have a strong suspicion that if we take Q' a function corresponding to a tetrahedron $a'b'c'd'$, in the same way as Q corresponds to $abcd$, QQ', and not improbably $\sqrt{(QQ')}$, will be found to be (like as we know from Staud's Theorem of $\sqrt{(V^2 . V'^2)}$) a rational integral function of the squares of the distances of the points a, b, c, d, from the points a', b', c', d'.

That N should divide out by V^2 is in itself an analytical theorem relating to 6 arbitrary quantities ab^2, ac^2, ad^2, bc^2, bd^2, cd^2, which evidently admits of extension to any triangular number 10, 15, &c. of arbitrary quantities. Thus we may affirm, *à priori*, that the norm of

$$\sqrt{L} \pm \sqrt{M} \pm \sqrt{N} \pm \sqrt{P} \pm \sqrt{Q},$$

where (for the sake of symmetry, retaining double letters, as AB, AC, &c., to denote *simple* quantities)

$$Q = \begin{vmatrix} 0 & AB & AC & AD & 1 \\ AB & 0 & BC & BD & 1 \\ AC & BC & 0 & CD & 1 \\ AD & BD & CD & 0 & 1 \\ 1 & 1 & 1 & 1 & \end{vmatrix} \quad P = \begin{vmatrix} 0 & AB & AC & AE & 1 \\ AB & 0 & BC & BE & 1 \\ AC & BC & 0 & CE & 1 \\ AE & BE & CE & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

$$N = \&c., \quad M = \&c., \quad L = \&c.,$$

will contain as a factor the determinant

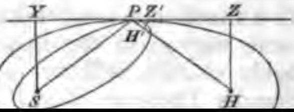
$$\begin{vmatrix} 0 & AB & AC & AD & AE & 1 \\ AB & 0 & BC & BD & BE & 1 \\ AC & BC & 0 & CD & CE & 1 \\ AD & BD & CD & 0 & DE & 1 \\ AE & BE & CE & DE & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

and a similar theorem may evidently be extended to the case of any $\frac{n.(n+1)}{2}$ arbitrary quantities whatever.

7, *New Square, Lincoln's Inn,*
March 29, 1853.

SOLUTIONS OF PROBLEMS.

1. Let S be the common focus of the two ellipses, H the second focus of the fixed, H' of the variable one, P their point of contact. Then, P, H', H will be in the same straight line. Let a, b be the semi-axes of the



and similarly situated to the given one, and having H for a focus.

2. Let a be the radius of the generating circle; the equation to the cycloid, meaning the arc from the vertex, will be

$$s^2 = 8ax.$$

Hence, it may easily be seen* that the equation to the tangent at any point is

$$s = mx + \frac{2a}{m},$$

m denoting the secant of its inclination to the axes of x.

This may be written

$$x.m^2 - s.m + 2a = 0:$$

this being a quadratic in m, shews that two tangents may be drawn to the cycloid through any point. If m_1, m_2 be the roots of the above equation, the condition of perpendicularity will be

$$\frac{1}{m_1^2} + \frac{1}{m_2^2} = 1;$$

whence, by the theory of equations,

$$\left(\frac{s}{2a}\right)^2 - 2\frac{x}{2a} = 1;$$

or

$$s^2 = 4a(x + a),$$

the equation to the required locus, which is evidently a cycloid of half the dimensions of, and similarly placed to the original one.

3. We know that

$$\int_0^\infty \frac{\varepsilon^{av} - \varepsilon^{-av}}{\varepsilon^{\pi v} - \varepsilon^{-\pi v}}\,dv = \tfrac{1}{2}\tan\tfrac{1}{2}a \,\ldots\ldots\ldots\ldots\ldots(1),$$

and that

$$\int_0^\infty \frac{\varepsilon^{av} - \varepsilon^{-av}}{\varepsilon^{\pi v} - \varepsilon^{-\pi v}}\cos rv\,dv = \frac{\sin a}{\varepsilon^r + 2\cos a + \varepsilon^{-r}} \ldots\ldots (2).$$

Hence, expanding $\cos rv$ in (2) and equating coefficients of $r^{2\mu}$, we have, by Sir John Herschel's theorem,

$$\int_0^\infty \frac{\varepsilon^{av} - \varepsilon^{-av}}{\varepsilon^{\pi v} - \varepsilon^{-\pi v}}\,v^{2\mu}\,dv = \frac{(-1)^\mu(1 + \Delta)\sin a}{(1 + \Delta)^2 + 2(1 + \Delta)\cos a + 1}\,0^{2\mu} \ldots (3),$$

* This will follow in precisely the same way as it is shewn that the equation to the tangent to the parabola $y^2 = kax$ is $y = mx + \dfrac{a}{m}$, where m denotes the tangent of its inclination to the axis of x.

Hence, writing $2x$ for α in (1), and differentiating 2μ times with respect to x, we have by (3)

$$\frac{d^{2\mu}}{dx^{2\mu}} \tan x = \frac{(-1)^{\mu}\, 2^{2\mu+2}\,(1+\Delta)\tan x}{b(1+\Delta)+\Delta^2(1+\tan^2 x)}\, 0^{2\mu}.$$

And in like manner, since

$$\int_0^\infty \frac{\varepsilon^{\alpha v}+\varepsilon^{-\alpha v}}{\varepsilon^{\pi v}-\varepsilon^{-\pi v}} \sin rv\, dv = \tfrac{1}{2}\, \frac{\varepsilon^{r}-\varepsilon^{-r}}{\varepsilon^{r}+2\cos\alpha+\varepsilon^{-r}},$$

we have, equating coefficients of r as before,

$$\int_0^\infty \frac{\varepsilon^{\alpha v}+\varepsilon^{-\alpha v}}{\varepsilon^{\pi v}-\varepsilon^{-\pi v}}\, v^{2\mu+1}\, dv = \frac{(-1)^{\mu}}{2}\; \frac{\Delta(\Delta+2)}{(1+\Delta)^2+2(1+\Delta)\cos x+1}\, 0^{2\mu+1}.$$

Hence, we have

$$\frac{d^{2\mu+1}}{dx^{2\mu+1}} \tan x = \frac{(-1)^{\mu}\, 2^{2\mu+1}\,\Delta(\Delta+2)(1+\tan^2 x)}{b(1+\Delta)+\Delta^2(1+\tan^2 x)}\, 0^{2\mu+1}.$$

4. Let ι be the inclination of the plane to the horizon, then the resolved part of the accelerating force of gravity parallel to the plane will be $g\sin\iota = f$ suppose. Let a be the radius of the sphere, M its mass, ω the angular velocity of the plane. Take the point where the axis of revolution meets the plane as origin of coordinates, let the plane itself be that of xy, and the axis of x be parallel to the horizon

of the point of the plane in contact with it. This gives us

$$\frac{dx}{dt} - \omega_y a = - \omega y \quad \text{.................. (5),}$$

$$\frac{dy}{dt} + \omega_x a = \omega x \quad \text{...................... (6).}$$

Differentiating (5), (6), and eliminating ω_x, ω_y, F_x, F_y by (1), (2), (3), (4), we get

$$\frac{d^2x}{dt^2} = - \frac{k^2}{k^2 + a^2}\,\omega\,\frac{dy}{dt} \quad \text{..................(7),}$$

$$\frac{d^2y}{dt^2} = \frac{k^2}{k^2 + a^2}\,\omega\,\frac{dx}{dt} - \frac{a^2}{k^2 + a^2} f \quad \text{......(8);}$$

whence
$$\frac{dx}{dt} = - \frac{k^2}{k^2 + a^2}\,\omega y + C \quad \text{...........................(9);}$$

therefore
$$\frac{d^2y}{dt^2} = - \frac{k^4}{(k^2 + a^2)^2}\,\omega^2 y + \frac{k^2}{k^2 + a^2}\,\omega C - \frac{a^2}{k^2 + a^2} f \text{...(10).}$$

To determine C, let x_0, y_0 be the initial values of x, y; and let I_x, I_y be the impulses of the friction when the sphere is just placed in the plane; $\left(\frac{dx}{dt}\right)_0$, $\left(\frac{dy}{dt}\right)_0$, $(\omega_x)_0$, $(\omega_y)_0$ the corresponding values of $\frac{dx}{dt}$, &c., then

$$M\left(\frac{dx}{dt}\right)_0 = I_x \quad \text{........................ (1'),}$$

$$M\left(\frac{dy}{dt}\right)_0 = I_y \quad \text{........................ (2'),}$$

$$Mk^2(\omega_x)_0 = I_y a \quad \text{........................(3'),}$$

$$Mk^2(\omega_y)_0 = - I_x a \quad \text{.....................(4'),}$$

and, as in (5), (6),

$$\left(\frac{dx}{dt}\right)_0 - (\omega_y)_0 a = - \omega y_0 \quad \text{..................(5'),}$$

$$\left(\frac{dy}{dt}\right)_0 + (\omega_x)_0 a = \omega x_0 \quad \text{.....................(6'),}$$

therefore, by (1'), (4'), (5'),

$$\left(1 + \frac{a^2}{k^2}\right)\left(\frac{dx}{dt}\right)_0 = - \omega y_0,$$

Similarly, by (2′), (3′), (6′),

$$\left(1+\frac{a^2}{k^2}\right)\left(\frac{dy}{dt}\right)_0 = \omega x_0.$$

Hence (9) gives $\quad C = 0,$

$$\therefore \text{ by (10)},\quad \frac{d^2y}{dt^2} = -\frac{k^2}{(k^2+a^2)^2}\,\omega^2 y - \frac{a^2}{k^2+a^2} f;$$

$$\text{whence}\quad y = -\frac{(k^2+a^2)\,a^2}{k^4}\frac{f}{\omega^2} + A\cos\left(\frac{k^2}{k^2+a^2}\,\omega t + B\right),$$

$$\therefore \left(\frac{dy}{dt}\right) = -\frac{k^2}{k^2+a^2}\,\omega A\sin\left(\frac{k^2}{k^2+a^2}\,\omega t + B\right).$$

To determine the constants, we have

$$\text{when } t=0,\ y=y_0,\quad \frac{dy}{dt} = \frac{k^2}{k^2+a^2}\,\omega x_0,$$

$$\therefore y_0 = -\frac{(k^2+a^2)\,a^2}{k^4}\frac{f}{\omega^2} + A\cos B,$$

$$\frac{k^2}{k^2+a^2}\,\omega x_0 = -\frac{k^2}{k^2+a^2}\,\omega A\sin B;$$

$$\therefore y = -\frac{(k^2+a^2)\,a^2}{k^4}\frac{f}{\omega^2} + \left\{y_0 + \frac{(k^2+a^2)\,a^2}{k^4}\frac{f}{\omega^2}\right\}$$

point from the given external point S. The expression for the density at any point Π, of which the distance from f is D, will be

$$\frac{kf^6}{D^5} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

If we take polar coordinates, ρ, ϑ, ϕ for the point Π, the corresponding expression for the volume of an element of the solid will be $\rho^2 \sin\vartheta\, d\phi\, d\vartheta\, d\rho$, and the quantity of matter which it contains will be

$$\frac{kf^6}{D^5} . \rho^2 \sin\vartheta\, d\phi\, d\vartheta\, d\rho \quad\ldots\ldots\ldots\ldots\ldots (2).$$

The potential due to this at any point P, at a distance Δ from it, will be

$$\frac{kf^6\, \rho^2 \sin\vartheta\, d\phi\, d\vartheta\, d\rho}{D^5 . \Delta} \quad\ldots\ldots\ldots\ldots\ldots (3),$$

and if r, θ, φ be the coordinates of P, we shall have

$$\Delta = [\rho^2 - 2\rho r \{\cos\vartheta \cos\theta + \sin\vartheta \sin\theta \cos(\phi - \varphi)\} + r^2]^{\frac{1}{2}} \ldots (4).$$

A similar expression might be used for D, but if we take S for the origin of polar coordinates, we have simply

$$D = \rho \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

If, farther, we take SC as the polar axis, we shall have for the equation of the surface of the sphere,

$$\rho^2 - 2f\rho \cos\vartheta + f \ = a^2 \quad\ldots\ldots\ldots\ldots\ (6),$$

where a denotes the radius.

If now we denote by V the potential at P due to the entire sphere, we deduce from the preceding expressions

$$V = kf^6 \iiint \frac{\sin\vartheta\, d\phi\, d\vartheta\, d\rho}{\rho^3 [\rho^2 - 2\rho r \{\cos\vartheta \cos\theta + \sin\vartheta \sin\theta \cos(\phi - \varphi)\} + r^2]} \ldots (7),$$

where the integration is to be made so as to include all values of ρ, ϑ, and ϕ subject to the condition

$$\rho^2 - 2f\rho \cos\vartheta + f^2 < a^2 \quad\ldots\ldots\ldots\ldots\ (8).$$

The evaluation of this integral may be effected by means of the following simple transformation:

Let
$$\rho = \frac{f^2 - a^2}{\rho'} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9);$$

then,
$$\frac{d\rho}{\rho^3} = -\frac{\rho' d\rho'}{(f^2 - a^2)^2} \quad\ldots\ldots\ldots\ldots\ldots (10),$$

if, besides, we assume $\quad r = \dfrac{f^2 - a^2}{r'}$ ....(11),

we shall have

$$\Delta = \frac{f^2 - a^2}{\rho' r'} [\rho'^2 - 2\rho' r' \{\cos\vartheta \cos\theta + \sin\vartheta \sin\theta \cos(\phi - \varphi)\} + r'^2]^{\frac{1}{2}} \ldots (12).$$

The expression for V then becomes

$$V = \frac{kf^5 r'}{(f^2 - a^2)^3} \iiint \frac{\rho'^2 \sin\vartheta \, d\phi \, d\vartheta \, d\rho'}{[\rho'^2 - 2\rho' r' \{\cos\vartheta \cos\theta + \sin\vartheta \sin\theta \cos(\phi - \varphi)\} + r'^2]^{\frac{1}{2}}} \ldots (13),$$

with the limiting condition

$$\frac{(f^2 - a^2)^2}{\rho'^2} - 2f \frac{f^2 - a^2}{\rho'} \cos\vartheta + f^2 < a^2 \ldots\ldots (14),$$

which is equivalent to

$$(f^2 - a^2)^2 - 2f\rho' \cos\vartheta \, (f^2 - a^2) + \rho'^2 (f^2 - a^2) < 0\,;$$

or to $\qquad f^2 - a^2 - 2f\rho' \cos\vartheta + \rho'^2 < 0\,;$

or, lastly, $\qquad \rho'^2 - 2f\rho' \cos\vartheta + f^2 < a^2$ ....(15).

Now the triple integral in the second members of (13) is, as we see immediately, the expression for the potential at a point (r', θ, φ), of a mass of the uniform density unity, occupying all the space over which the integration with reference to ρ', ϑ, ϕ, considered as polar coordinates, is

[redacted]

If now we recur to the original notation, by substituting $\frac{f^2-a^2}{r}$ for r' in these expressions, and modify the discriminating conditions as we have done above, in the case of (14), we obtain

$$(A)\ldots V = \frac{\frac{4\pi}{3}a^3kf^4}{(f^2-a^2)^2} \cdot \frac{1}{\left\{\left(\frac{f^2-a^2}{f^4}\right)^2 - 2\left(\frac{f^2-a^2}{f}\right) r\cos\theta + r^2\right\}^{\frac{1}{2}}};$$

when $$r^2 - 2fr\cos\theta + f^2 > a^2,$$

$$(B)\ldots V = \frac{\frac{4\pi}{3}kf^7}{(f^2-a^2)^2} \cdot \frac{\left(\frac{f^2-a^2}{f}\right)^2 - 2\left(\frac{f^2-a^2}{f}\right) r\cos\theta + r^2}{r^3};$$

when $$r^2 - 2fr\cos\theta + f^2 < a^2.$$

These expressions, for the potential without and within the sphere, contain the complete solution of the problem, since the two components, (F) along PS, and (G) perpendicular to PS in the plane PST, of the force at any point P, may be found from them by means of the equations

$$(C)\ldots F = \frac{dv}{dr}, \quad \text{and } G = \frac{1}{r}\frac{dv}{d\theta}.$$

If we put, for brevity,

$$\frac{\frac{4\pi}{3}a^3kf^4}{(f^2-a^2)} = m \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16),$$

and $$\frac{f^2-a^2}{f} = g \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

the expressions for the potential become simply,

(external point), $$V = \frac{m}{(g^2 - 2gr\cos\theta + r^2)^{\frac{1}{2}}} \;\ldots\ldots\ldots\ldots\ldots (18),$$

(internal point), $$V = \frac{mf^3}{a^3} \cdot \frac{g^2 - 2gr\cos\theta + r^2}{r^3} \;\ldots\ldots\ldots (19).$$

The denominator of the former expression being obviously the distance from P to a certain point I, taken in SC at a distance $g = \frac{f^2-a^2}{f}$ from S, we conclude that the resultant force is towards this point I, and inversely as the square of the distance of P from it.

7. When a gas expands without being allowed to take in any heat from without, its pressure varies as the m^{th} power of its density, provided only m is constant. Hence, if v be the volume, and p the pressure of the expanding air at any instant during the motion of the ball through the barrel, we have

$$p = P\left(\frac{V}{v}\right)^m.$$

Now the expanding air increases in volume during the motion of the ball from one end of the barrel to the other, by U, the volume of the barrel, and therefore the work done by it is $\int_V^{U+V} n\,dv$, or, as we find by integration from the preceding equation,

$$\frac{PV}{m-1}\left\{1-\left(\frac{V}{U+V}\right)^{m-1}\right\}.$$

According to the hypotheses, the whole of this work is spent in pushing out the ball, and produces only the two effects of overcoming the atmospheric pressure in front of the ball, and communicating motion to the ball. The mechanical value of the first of these effects is ΠU, and that of the second $\frac{1}{2}\frac{W}{g}q^2$, if q be the velocity acquired by the ball. Hence, by the principle of *vis viva*,

volume is U) at the instant when the ball leaves it,

$$u = V\left\{\left(\frac{P}{\Pi}\right)^{\frac{1}{m}} - \left(\frac{P}{Q}\right)^{\frac{1}{m}}\right\},$$

a quantity which will in general be positive since in all good practical arrangements Q must exceed Π. Now, if the volume of the barrel were $U + u$, there would be no appreciable proportion of the whole work spent in noise &c., since the mechanical value of the motion of the air in the barrel till the ball leaves it, is, according to the hypotheses, to be neglected in every case, and the only energy that remains to produce noise &c., would in this case be the motion of the air within the barrel. But in this case the whole work spent in communicating motion to the ball would be

$$\frac{PV}{m-1}\left\{1 - \left(\frac{V}{U+V+u}\right)^{m-1}\right\} - \Pi\,(U+u),$$

which exceeds the work spent in communicating motion to the ball in the actual case by

$$\frac{PV}{m-1}\left\{\left(\frac{V}{U+V}\right)^{m-1} - \left(\frac{V}{U+V+u}\right)^{m-1}\right\} - \Pi u,$$

or

$$\frac{PV}{m-1}\cdot\frac{Q^{\frac{m-1}{m}} - \Pi^{\frac{m-1}{m}}}{P^{\frac{m-1}{m}}} - \Pi V\left\{\left(\frac{P}{\Pi}\right)^{\frac{1}{m}} - \left(\frac{P}{Q}\right)^{\frac{1}{m}}\right\}.$$

Now the final effect, in lifting the atmosphere in the two cases is the same, (with the exception of small differences that may result from differences of temperature of the air near the mouth of the gun, which are neglected), and hence, the excess of work spent in communicating motion to the ball, in one case is equal to that wasted in noise and fluid friction after the ball leaves the gun in the other; and therefore the ratio required in the second part of the question is

$$\frac{\dfrac{PV}{m-1}\,\dfrac{Q^{\frac{m-1}{m}} - \Pi^{\frac{m-1}{m}}}{P^{\frac{m-1}{m}}} - \Pi V\left\{\left(\dfrac{P}{\Pi}\right)^{\frac{1}{m}} - \left(\dfrac{P}{Q}\right)^{\frac{1}{m}}\right\}}{\dfrac{PV}{m-1}\left\{1 - \left(\dfrac{Q}{P}\right)^{\frac{m-1}{m}}\right\} - \Pi V\left(\dfrac{P}{Q}\right)^{\frac{1}{m}}}.$$

Note—Solutions of problems (5), (8), (9), (10) will appear in the next number.

PROBLEMS.

1. Given two conics in the same plane such that the normal distance of the point of intersection of their transverse or major axes from each of the conics is one and the same pure imaginary quantity; shew that the conics may be projected into small circles of the same sphere.

2. If from a point in the circumference of a vertical circle two heavy particles be successively projected along the curve, then initial velocities being equal and either in the same or in opposite directions, the subsequent motion will be such that a straight line joining the particles at any instant will touch a given circle.

Note. The particles are supposed not to interfere with each other's motion.

3. A transparent medium is such that the path of a ray of light within it is a given circle, the index of refraction being a function of the distance from a given point in the plane of the circle.

Find the form of this function and shew that for light of the same refrangibility—

(1) The path of *every ray within the medium* is a circle.

(2) All the rays proceeding from any point in the medium

5. (*a*) What horse-power would be required to supply a building with 1 lb. of air per second, heated mechanically from 50° to 80° Fahrenheit? Compare the fuel that an engine producing this effect as $\frac{1}{10}$ of the equivalent of the heat of combustion would consume, with that which would be required to heat directly the same quantity of air.

(*b*) Explain how this effect may be produced with perfect economy by operating on the air itself to change its temperature, and give dimensions &c. of an apparatus that may be convenient for the purpose.

(*c*) Shew how the same apparatus may be adapted to give a supply of cooled air.

Ex. Let it be required to supply a building with 1 lb. of air per second, cooled from 80° to 50° Fahr. Determine the horse-power wanted to work the apparatus in this case.

6. Find the amount of "potential energy" (mechanical effect of such a kind as that of weights raised) that can be obtained by equalizing the temperature of two bodies given at different uniform temperatures, and determine the common temperature to which they are reduced.

Ex. 1. Let the bodies be of equal constant thermal capacities, and let their temperatures be 0° and 100° respectively.

Ex. 2. Let the bodies be masses W and W' of water, and let the temperatures at which they are given be 15° and 20° respectively.

7. If α, β, γ, be the trilinear coordinates of a point, a, b, c, the lengths of the sides of the triangle of reference, the equations to the greatest inscribed and least circumscribed ellipse will be respectively

$$(a\alpha)^{\frac{1}{2}} + (b\beta)^{\frac{1}{2}} + (c\gamma)^{\frac{1}{2}} = 0,$$

$$(a\alpha)^{-1} + (b\beta)^{-1} + (c\gamma)^{-1} = 0.$$

ON THE THIRD ELLIPTIC INTEGRAL.

By F. W. Newman, formerly Fellow of Balliol College, Oxford.

I. Following up the splendid discoveries of Jacobi, Legendre first, and since his labours were closed, Dr. Gudermann, have investigated series of an elevated kind for approximating to this integral. But the higher theory seems to have drawn off investigations unduly from what is more elementary; and the principal object of the present paper is to shew that the earlier and simpler methods have by no means been adequately appreciated and developed.

Legendre's notation, with trifling alterations, will be here retained. The moduli $cc_1c_2c_3\ldots$ are those of the common scale *descending*, which he denotes by $cc^{\circ}c^{\circ\circ}c^{\circ\circ\circ}\ldots$ I propose to employ the notation $\eta\eta^{\circ}\eta_{\circ}$ to imply the relations

$$F(c\eta) + F(c\eta^{\circ}) = F(c, \tfrac{1}{2}\pi),$$
$$F(b\eta) + F(b\eta_{\circ}) = F(b, \tfrac{1}{2}\pi),$$

in which case η° may be called the *conjugate* amplitude to η, and $\eta_{\circ}$ the *lower conjugate*. We then have the well-known relations $\cot\eta . \cot\eta^{\circ} = b$,

$$\sin\eta^{\circ} = \frac{\cos\eta}{\Delta(c\eta)}, \quad \cos\eta^{\circ} = \frac{b\sin\eta}{\Delta(c\eta)}, \quad \Delta(c\eta^{\circ}) = \frac{b}{\Delta(c\eta)} \ldots (1),$$

in which we may change η°, c into $\eta_{\circ}$, b.

Also if from c, η be formed c, η in Lagrange's scale, we get

[redacted]

$$\therefore \int \frac{dQ}{1+TQ^2} = \Pi(p) + \Pi\left(\frac{c^2}{p}\right) - F \ldots\ldots\ldots\ldots (3),$$

where p, c^2p^{-1} are parameters, and c, ω the other elements.

Let $pq = c^2$, then p and q are called *reciprocal* in this theory.

(3) If $(1+p)(1-r) = b^2$, therefore

$$\int_0 \frac{d(\sin\omega \sin\omega^\circ)}{1+pr(\sin\omega\sin\omega^\circ)^2} = \frac{1+p}{p}.\Pi(p) - \frac{1-r}{r}.\Pi(-r) - \frac{c^2}{pr}.F \ldots (4).$$

The parameters p and $-r$ are called *conjugate.*

(4) Various integrals $P(p,\omega) = \int \phi(p,\omega)\,d\omega$ are known, which may be found in some simpler form for *a special* value of ω, $(\omega = \alpha)$ by means of

$$\frac{dP}{dp} = \int \frac{d\phi(p,\omega)}{dp}\,d\omega = \psi(p,\omega).$$

If the function ψ is known, we get

$$\frac{dP(p,\alpha)}{dp} = \psi(p,\alpha), \text{ and } P(p,\alpha) = \int \psi(p,\alpha)\,dp.$$

Legendre applied this method to Π, and deduced not only the value of Π_c in terms of F and E, but certain *commutative* equations, in which the amplitude exchanges places with a certain function of the parameter.

(5) He applied Lagrange's scale to Π, and by it deduced two series, one for descending, the other for ascending, moduli. But both are too complicated for use, especially the latter.

One more property established by Legendre remains to be named; viz. if ζ, ω, η are *amplitudes* which make $F\zeta = F\omega + F\eta$, and p the parameter common to the three Π's which correspond to the F's, then

$$\sqrt{T}.\{\Pi\omega + \Pi\eta - \Pi\zeta\} = \tan^{-1}.\frac{\sqrt{T}.p\sin\omega\sin\eta\sin\zeta}{1+p(1-\cos\omega\cos\eta\cos\zeta)} \ldots\ldots (5).$$

Consequently if $\zeta = \frac{1}{2}\pi$, or $\eta = \omega^\circ$, we get

$$\sqrt{T}\{\Pi\omega + \Pi\omega^\circ - \Pi_c\} = \tan^{-1}.\left\{\frac{p}{1+p}.\sqrt{T}.\sin\omega\sin\omega^\circ\right\} \ldots (6),$$

in which, whenever T is negative, it will be easy to give to the last term the form of a logarithm.

My first business is, to shew that all these integrations of Legendre, when duly simplified, lead to available results.

III. The function $T = (1+p)(1+q)$ may be called the *test product*, since, according as it is negative or positive, Π is of the logarithmic or of the circular class. We may occasionally denote it by $T(p)$, and the definition shews that $T(p) = T(q)$: also

$$T(p).\,T(-r) = b^4 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (7).$$

It is easy to prove that the reciprocals of conjugate parameters are conjugate, and the conjugates of reciprocals are reciprocal. Thus the reciprocal of the conjugate is the conjugate of the reciprocal.

Also two reciprocals, or two conjugates, are either both circular or both logarithmic.

Legendre assigns two forms for logarithmic parameters, viz. $-c^2\sin^2\eta$ and $-\operatorname{cosec}^2\eta$, both negative; the former ranging from 0 to $-c^2$, the latter from -1 to $-\infty$. Evidently if a parameter has the form $-c^2\sin^2\eta$, its reciprocal is $-\operatorname{cosec}^2\eta$. But logarithmic *conjugates* are either both of the form $-c^2\sin^2\eta$ or both of the form $-\operatorname{cosec}^2\eta$. In fact, since $\Delta(c\eta).\Delta(c\eta^\circ) = b$, it follows that $-c^2\sin^2\eta$ and $-c^2\sin^2\eta^\circ$ are logarithmic conjugates. So also, since $\cot\eta.\cot\eta^\circ = b$, therefore $-\operatorname{cosec}^2\eta$ and $-\operatorname{cosec}^2\eta^\circ$ are logarithmic conjugates.

A circular parameter, when positive, may be denoted by $p = \cot^2\theta$; then its reciprocal is $q = c^2\tan^2\theta$, and its conjugate is $-r = -1 + b^2\sin^2\theta$. But we may also write $p = \cot^2\theta$, $q = \cot^2\theta_\circ$, $r = \Delta^2(b, \theta)$. Of two circular conjugates one is necessarily negative, the other positive. Also since

as the P. Thus $\sqrt{T}\Pi(p) + \sqrt{T}\Pi(-r)$ will mean

$$\sqrt{T(p)}.\Pi(p) + \sqrt{T(-r)}.\Pi(-r).$$

When p is infinitesimal, $\Pi = F$; also $T = c^2p^{-1}$; whence

$$\sqrt{T}(F - \Pi) = c\sqrt{p^{-1}}\int_0 p \sin^2\omega . dF = 0.$$

Also when p is infinite, $T = p$, and $\sqrt{T}\Pi$ has no increments while ω is finite; for then $\dfrac{\sqrt{p}}{1 + p\sin^2\omega} = 0$. But while ω is infinitesimal $\sqrt{T}\Pi = \displaystyle\int_0 \frac{\sqrt{p}d\omega}{1 + p\omega^2} = \tan^{-1}(\sqrt{p}.\omega)$, which becomes $\tan^{-1}(\infty)$, as soon as ω rises to a sensible value; hence $\sqrt{T}\Pi = \frac{1}{2}\pi$, when $p = \infty$, whatever the finite value of ω. This reasoning is rather refined; but the conclusion may be equally obtained from the reciprocal equation.

IV. The three integrals F, E, Π have in common the property, that whenever their amplitude $\omega = n.\frac{1}{2}\pi$, the integral $= n$ times the complete integral. It immediately follows, that if for a moment we assume three arcs x, x', x'' such that

$$\frac{F(\omega)}{F_c} = \frac{x}{\frac{1}{2}\pi}, \quad \frac{E(\omega)}{E_c} = \frac{x'}{\frac{1}{2}\pi}, \quad \frac{\Pi(\omega)}{\Pi_c} = \frac{x''}{\frac{1}{2}\pi},$$

each of the three new arcs is equal to ω, as often as ω is a multiple of $\frac{1}{2}\pi$. Hence $(x' - x)$ and $(x'' - x)$, or any functions proportional to them, vanish periodically every time that $\omega = n.\frac{1}{2}\pi$. Such *fluctuating* functions are the appropriate auxiliaries for calculating E and Π, when F is known.

Legendre assumed G as an auxiliary, equivalent to $E - \dfrac{F}{F_c}.E_c$; which is proportional to $x' - x$; and by it he obtained by far the most elegant of the approximations to E; namely, if C now stands for F_c, he found

$$\left.\begin{array}{ll} & CG = C_1G_1 + C_1c_1\sin\omega_1 \text{ [Lagrange's scale]} \\ \text{whence} & CG = C_1c_1\sin\omega_1 + C_2c_2\sin\omega_2 + C_3c_3\sin\omega_3 + \&\text{c}.... \end{array}\right\}...(8).$$

But the properties and uses of G have by no means been fully exhibited, and a digression on that subject, either here or afterwards, is inevitable. If we assume H as a second auxiliary, such that

$$H = E - \left(1 - \frac{E_b}{F_b}\right)F.................(9),$$

we easily get [since by Legendre's equation of complementary moduli, $\frac{1}{2}\pi = F_bE_c + F_cE_b - F_bF_c$,]

therefore $$H = G + \frac{\frac{1}{2}\pi F}{F_b F_c};$$
or, in the other notation,
$$H = G + \tfrac{1}{2}\pi \cdot \frac{F}{BC} \quad\text{.....................(10).}$$

Moreover $$CH = C_1 H_1 + C_1 c_1 \sin\omega_1;$$
or $$BH = \tfrac{1}{2}B_1 H_1 + \tfrac{1}{2}B_1 c_1 \sin\omega_1 \quad\text{............(11),}$$
whence
$$\left.\begin{aligned} BH = B\sin\omega - 2B'(\sin\omega - \sin\omega') - 2^2B''(\sin\omega' - \sin\omega'')\\ - 2^3B'''(\sin\omega'' - \sin\omega''') - \&c.\end{aligned}\right\}\text{...(12),}$$
which is by far the most elegant series for calculating E by *ascending* moduli, (*i.e.* when c is near to 1,) and converges with the usual precipitancy of Lagrange's scale.

Farther, if $\sin\eta = \sqrt{-1}\tan\theta$, we easily obtain
$$\left.\begin{aligned} &G(c\eta) = -\sqrt{-1}\,.\,H(b,\theta) + \sqrt{-1}\,.\,\tan\theta\,.\,\Delta(b,\theta),\\ &H(c\eta) = -\sqrt{-1}\,.\,G(b,\theta) + \sqrt{-1}\,.\,\tan\theta\,.\,\Delta(b,\theta),\\ \text{or}\quad &G(c\eta) - \sqrt{-1}\,.\,G(b,\theta) = H(c\eta) - \sqrt{-1}\,.\,H(b,\theta),\\ \text{also}\quad &G(c\eta) + \sqrt{-1}.G(b,\theta) = \sqrt{-1}\left\{\tan\theta\Delta(b,\theta) - \tfrac{1}{2}\pi.\frac{F(b\theta)}{BC}\right\}\end{aligned}\right\}\text{...(13).}$$

Finally, it is worth observing, that while $G(c\omega)$ vanishes, not only when ω is a multiple of $\frac{1}{2}\pi$, but also when c is evanescent; we have, on the other hand, $BH_c = \frac{1}{2}\pi$, for all values of c; also, for all values of ω, we find $BH = \omega$ when $c = 0$; but $H = \sin\omega$, when $c = 1$. We may add that $G(c, n\pi + \omega) = G(c, \omega)$, but $BH(c, n\pi + \omega) = BH(c, \omega) + n\pi$, or $H(c, n\pi + \omega) = H(c, \omega) + 2nH_c$.

V. Returning to the integral Π, we follow out the analogy of this proceeding, by assuming an auxiliary proportional to $(x'' - x)$.

Let P stand for $\Pi - \dfrac{F}{F_c}\Pi_c$..........................(14),

then the problem of finding Π divides itself into two parts. First, to find the complete integral Π_c: for when this is known, we regard the second term of P to be known. Next, it remains to find the fluctuating portion P, which alone involves three elements; and since it periodically vanishes, we may look on it as a small correction to be applied to the main term; the total value of Π being given by the equation
$$\Pi = \frac{F}{F_c}\Pi_c + P.$$

It is evident that P vanishes with p. But we must first dispose of the case to which this method is essentially inapplicable, viz. that in which Π_c is infinite; namely in which the parameter has the form $p = -\operatorname{cosec}^2\eta$. By the reciprocal equation (3) Legendre reduces this to the indefinite integral $\Pi(-c^2\sin^2\eta)$: nevertheless, it is not amiss to exhibit the equation in a slightly changed form.

When $p = -\operatorname{cosec}^2\eta$, $T = -\cot^2\eta . \Delta^2(c\eta)$, which applies alike to p and q: therefore

$$\sqrt{-T}\{\Pi(p)+\Pi(c^2p^{-1})-F\}$$

$$= \int_0 \frac{\sqrt{-T}.dQ}{1+TQ^2} = \tfrac{1}{2}\log\frac{\tan\eta\Delta\omega+\tan\omega\Delta\eta}{\pm\{\tan\eta\Delta\omega-\tan\omega\Delta\eta\}}.$$

Consequently, if $F\zeta = F\omega + F\eta$ and $F\mathfrak{s} = F\omega - F\eta$, the relations, furnished by Euler's well-known integration, between $\zeta\mathfrak{s}$ and $\omega\eta$ yield

$$\sqrt{-T}\{\Pi(-\operatorname{cosec}^2\eta)+\Pi(-c^2\sin^2\eta)-F\}(c\omega) = \tfrac{1}{2}\log\frac{\sin^2\zeta}{\sin^2\mathfrak{s}} \quad \ldots(15),$$

in which $\Pi(-\operatorname{cosec}^2\eta)$ and the logarithm both become infinite at the crisis $\omega = \eta$, $\mathfrak{s} = 0$. In future, we set aside the case of parameters negative and greater than unity, as sufficiently disposed of by this equation.

Passing to the circular Π, we may doubly modify the reciprocal equation by supposing p positive or negative. But it will suffice to make p positive, and to treat a negative parameter $(-r)$ as its conjugate. Generally, when T is positive, equation (3) may take the form

$$\sqrt{T}.\{\Pi(p)+\Pi(c^2p^{-1})-F\} = \tan^{-1}.(\sqrt{T}Q).$$

But when $p = \cot^2\theta$,

$$\sqrt{T} = \frac{\Delta(b\theta)}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\sin\theta_0} \quad \ldots\ldots\ldots\ldots(16),$$

an expression which is very easy to remember: and the corresponding value of $\sqrt{T}(-r)$, the conjugate, is no additional burden to the memory, if we do but remember the relation $\sqrt{T}(p).\sqrt{T}(-r) = b^2$, from equation (7). Hence

$$\sqrt{T}.Q = \frac{\Delta(b\theta)}{\sin\theta\cos\theta}\cdot\frac{\tan\omega}{\Delta(c\omega)} = \frac{1}{\sin\theta\sin\theta_0}\cdot\frac{\cos\omega^\circ}{b\cos\omega} \quad \ldots(17).$$

Again, when we assume $\omega = \tfrac{1}{2}\pi$, $Q = \infty$, $\tan^{-1}(\sqrt{T}Q) = \tfrac{1}{2}\pi$; whence

$$\sqrt{T}\{\Pi_c(p)+\Pi_c(c^2p^{-1})-F_c\} = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots(17a).$$

Multiply this by $\dfrac{F\omega}{F_c}$ and subtract the product from the general integral; therefore

$$\left.\begin{aligned}\surd T\{P(p)+P(c^2p^{-1})\}(c\omega)=\cot^{-1}&\left\{\sin\theta\sin\theta_\circ.\,\frac{b\cos\omega}{\cos\omega^\circ}\right\}\\ &-\tfrac{1}{2}\pi.\,\frac{F(c\omega)}{F_c}\end{aligned}\right\}\ldots(18),$$

when $p=\cot^2\theta$.

VI. From the reciprocal we proceed to the conjugate equation (4).

When Π is logarithmic, p as well as $-r$ will be negative, and we may write $-r'$ for p, so that we get

$$\left.\begin{aligned}R&=\sin\omega\sin\omega^\circ\\(1-r)(1-r')&=b^2\end{aligned}\right\}\text{ and }\frac{1-r}{r}\Pi(-r)+\frac{1-r'}{r'}\Pi(-r')-\frac{c^2}{rr'}F=\int_0\frac{-dR}{1-rr'.R_2}$$

$$=\frac{-1}{2\surd(rr')}.\log.\frac{1+R\surd(rr')}{1-R\surd(rr')},$$

it being observed that Rrr' are all numerically less than 1.

If $r=c^2\sin^2\eta$, $r'=c^2\sin^2\eta^\circ$; and the logarithmic part is

$$\log\frac{1+c\sin\omega\sin\omega^\circ.\,c\sin\eta\sin\eta^\circ}{1-c\sin\omega\sin\omega^\circ.\,c\sin\eta\sin\eta^\circ}\text{ or }\log\frac{1+c_1\sin\omega_1\sin\eta_1}{1-c_1\sin\omega_1\sin\eta_1},$$

if we form c_1 ω_1 η_1 in Lagrange's scale from c, ω, η.

Multiply the last by $\frac{F\omega}{F_c}$ and subtract the product from (19), therefore

$$\left.\begin{aligned}&\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta)+\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta^\circ)\\ &=-\tfrac{1}{2}\log\frac{1+c_1\sin\omega_1\sin\eta_1}{1-c_1\sin\omega_1\sin\eta_1}=-\tfrac{1}{2}\log\frac{\Delta\varepsilon}{\Delta\zeta}\end{aligned}\right\}\ldots(20).$$

But the more important case is when Π is circular. Let

$$p=\cot^2\theta,\quad r=-1+b^2\sin^2\theta.$$

Observe that

$$\sqrt{(pr)}=\frac{p}{1+p}\sqrt{T(p)}=\frac{r}{1-r}\sqrt{T(-r)},$$

so that

$$\sqrt{T}.\Pi(p)\div\sqrt{T}\Pi(-r)-\frac{c^2}{\sqrt{(pr)}}F=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(21).$$

We now see also that equation (6) admitted of being written

$$\sqrt{T}\{\Pi\omega+\Pi\omega^\circ-\Pi_e\}=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ)\ldots(22),$$

where p is the common parameter: but the advantage of comparing the two last equations is best seen, when we have separated P out of Π. In (21), make $\omega=\frac{1}{2}\pi$, then

$$\sqrt{T}\Pi_c(p)-\sqrt{T}\Pi_c(-r)-\frac{c^2}{\sqrt{(pr)}}F_c=0.$$

Multiply by $\frac{F\omega}{F_c}$ and subtract from (21); then

$$\sqrt{T}\mathrm{P}(p)-\sqrt{T}\mathrm{P}(-r)=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(23).$$

Also subtract from (5) the equation

$$\sqrt{T}\left\{\frac{F\omega}{F_c}+\frac{F\eta}{F_c}-\frac{F\zeta}{F_c}\right\}\ _c=0,$$

and it changes every Π in (5) into P. Observing, then, that $\mathrm{P}_e=0$, we have instead of (22) the simpler result

$$\sqrt{T}\{\mathrm{P}(p\omega)+\mathrm{P}(p\omega^\circ)\}=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(24).$$

Comparing then (23) and (24), we conclude that

$$\sqrt{T}.\mathrm{P}(-r,\omega)=-\sqrt{T}.\mathrm{P}(p,\omega^\circ)\ldots\ldots\ldots\ldots(25).$$

Hence the general enunciation: "In any circular $\sqrt{T}$P we may at pleasure change a parameter into its conjugate, provided that we change the amplitude also into the negative of its conjugate."

It is not difficult to verify (25) directly, by mere differentiation. Moreover, if we adapt the process to the case of a logarithmic Π, then, instead of (24), we get

$$\sqrt{-T}\{\mathrm{P}(-r,\,\omega)+\mathrm{P}(-r,\,\omega^{\circ})\} = -\tfrac{1}{2}\log\frac{\Delta z}{\Delta\zeta},$$

which, compared with (20), gives

$$\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta^{\circ},\,\omega) = \sqrt{-T}\mathrm{P}(-c^2\sin^2\eta,\,\omega^{\circ})\ldots(25a).$$

Thus, "In a logarithmic $\sqrt{-T}\mathrm{P}$ we may change the parameter into its conjugate, provided that we simultaneously change the amplitude into its conjugate. Or, "To get the integral conjugate to $\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta,\,\omega)$, we may at pleasure put η° for η, or ω° for ω."

Generally, even with a circular Π, it suffices to treat of *three* parameters, as p, its reciprocal q, and $-r$ the conjugate to p. But we may reckon *four* in the following method: first, $\mathrm{P}(\cot^2\theta,\,c,\,\omega)$ its reciprocal $\mathrm{P}(c^2\tan^2\theta,\,c,\,\omega)$; conjugate of the first, $-\mathrm{P}(\cot^2\theta,\,c,\,\omega^{\circ})$; conjugate of the second, $-\mathrm{P}(c^2\tan^2\theta,\,c,\,\omega^{\circ})$. And these, though in appearance four, are evidently in form only two. But for the present we shall continue to deal with three.

VII. Let us for conciseness write $\dfrac{F(c\omega)}{F_c} = \dfrac{x}{\frac{1}{2}\pi}$,

$$\Omega = \sqrt{T}P(pc\omega),\quad \overset{2}{\Omega} = \sqrt{T}P(qc\omega),\quad \overset{3}{\Omega} = \sqrt{T}P(-rc\omega),$$

Legendre, under a different notation; and the result is most unexpectedly simple.

For a moment, let

$$h^{-1} = \frac{\sin\theta\cos\theta}{\Delta(b\theta)} \cdot \frac{\Delta(c\omega)}{\tan\omega}, \qquad k = \frac{\sin\omega\cos\omega}{\Delta(c\omega)} \cdot \frac{\Delta(b\theta)}{\tan\theta};$$

or, if $D = \dfrac{\Delta(b\theta)\sin\omega}{\Delta(c\omega)\sin\theta}$, $h = \dfrac{D}{\cos\omega\cos\theta}$, $k = D\cos\omega\cos\theta$.

Also $\qquad x + \Omega + \dot{\Omega} = \tan^{-1}h : \quad \Omega - \dot{\Omega} = \tan^{-1}k;$

whence $\qquad x + \dot{\Omega} + \dot{\Omega} = \tan^{-1}h - \tan^{-1}k,$

or $\qquad \tan(x + \dot{\Omega} + \dot{\Omega}) = \dfrac{h-k}{1+hk}.$

Now $1+hk = 1+D^2 = \dfrac{\Delta^2(c\omega)\sin^2\theta + \Delta^2(b\theta)\sin^2\omega}{\Delta^2(c\omega)\sin^2\theta} = \dfrac{1-\cos^2\omega\cos^2\theta}{\Delta^2(c\omega)\sin^2\theta}$,

and $\qquad h - k = h(1 - \cos^2\omega\cos^2\theta);$

$$\therefore \frac{h-k}{1+hk} = h.\Delta^2(c\omega)\sin^2\theta = \frac{\Delta(b\theta)}{\cot\theta} \cdot \frac{\Delta(c\omega)}{\cot\omega} = \text{also } \frac{\sin\theta}{\sin\theta_0} \cdot \frac{\sin\omega}{\sin\omega^0}.$$

Thus we obtain between $\dot{\Omega}$ and $\dot{\Omega}$ the relation

$$\tan(x + \dot{\Omega} + \dot{\Omega}) = \frac{\sin\theta}{\sin\theta_0} \cdot \frac{\sin\omega}{\sin\omega^0} \quad \text{.........} \quad (28).$$

The three equations (26), (27), (28) are a mere application of the *Reciprocal* and *Conjugates* of Legendre to the case of a circular $\sqrt{TP}$.

VIII. We now turn to the *Commutative* equations.

Making p to vary in Π, we get $\dfrac{d\Pi}{dp} = \displaystyle\int_0 \frac{-\sin^2\omega.dF}{(1+p\sin^2\omega)^2}$. By the general formula of reduction this receives the shape

$$\frac{d\Pi}{dp} = \alpha.\frac{\sin\omega\cos\omega\Delta\omega}{1+p\sin^2\omega} + \beta.F + \gamma.\int_0 \sin^2\omega\, dF + \delta.\Pi,$$

and it is found that $\alpha, \beta, \gamma, \delta$ have a common denominator, which is none other than $T(p)$. Calling the numerators $\alpha', \beta', \gamma', \delta'$, we find

$$\alpha' = \tfrac{1}{2}, \quad \beta' = -\tfrac{1}{2}.\frac{c^2}{p^2}, \quad \gamma' = -\tfrac{1}{2}.\frac{c^2}{p}, \quad \delta' = -\tfrac{1}{2}.\left(1 - \frac{c^2}{p^2}\right),$$

which farther yield

$$\delta' = -\tfrac{1}{2}\cdot\frac{dT}{dp},\quad \beta' = \tfrac{1}{2}\cdot\left(\frac{dT}{dp}-1\right).$$

Observing then that $c^2\int_0 \sin^2\omega . dF = F - E$, we get

$$T.\frac{d\Pi}{dp} = \tfrac{1}{2}.\frac{\sin\omega\cos\omega\Delta\omega}{1+p\sin^2\omega} + \tfrac{1}{2}.\left(\frac{dT}{dp}-1\right)F - \frac{1}{2p}(F-E) - \tfrac{1}{2}.\frac{dT}{dp}.\Pi.$$

Bringing all the T's and Π's to the left, we divide either by $\sqrt{T}$ or by $-\sqrt{-T}$ to make the left-hand an exact integral, and then integrate nearly as Legendre.

It gives

$$\left.\begin{aligned}&\sqrt{T}(\Pi - F)\\ &= \tfrac{1}{2}\sin\omega\cos\omega\Delta\omega.\int\frac{\sqrt{T}^{-1}dp}{1+p\sin^2\omega} - \tfrac{1}{2}F\int\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{T}} + \tfrac{1}{2}E\int\frac{dp}{p\sqrt{T}}\\ &\sqrt{-T}(\Pi - F)\\ &= -\tfrac{1}{2}\sin\omega\cos\omega\Delta\omega.\int\frac{\sqrt{-T}^{-1}dp}{1+p\sin^2\omega} + \tfrac{1}{2}F\int\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{-T}} - \tfrac{1}{2}E\int\frac{dp}{p\sqrt{-T}}\end{aligned}\right\}\text{..(29)},$$

in which the integrals may begin from $p = 0$, since this supposition makes the left-hand member vanish, and also involves no infinite quantities, such as would appear in Legendre's equation.

In the last let $\omega = \tfrac{1}{2}\pi$;

$$\therefore\ \sqrt{-T}(\Pi_c.p - F_c) = \tfrac{1}{2}F_c.\int\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{-T}} - \tfrac{1}{2}E_c\int_0\frac{dp}{p\sqrt{-T}}\ldots(29a).$$

We may deduce $\Pi_c(-1 + b^2 \sin^2\theta)$ by combining the last with (21) and with (17*a*).

From 21,

$$\sqrt{T}\Pi_c(p) - \sqrt{T}\Pi_c(-r) = \frac{c^2}{\sqrt{(pr)}} F_c.$$

Also $\quad \sqrt{T}\{\Pi_c p + \Pi_c q - F_c\} = \frac{1}{2}\pi$ from (17*a*);

therefore $\sqrt{T}\{\Pi_c(q) - F_c\} + \sqrt{T}\Pi_c(-r) = \frac{1}{2}\pi - \frac{c^2}{\sqrt{(pr)}} F_c,$

or $\sqrt{T}\{\Pi_c(-r) - F_c\} - \sqrt{T}\{F_c - \Pi_c(q)\} = \frac{1}{2}\pi - \left\{\sqrt{T(-r)} + \frac{c^2}{\sqrt{(pr)}}\right\} F_c$

$$= \tfrac{1}{2}\pi - \Delta(b\theta)\tan\theta . F_c, \quad \text{if } p = \cot\theta.$$

Add the last to (31), and it gives

$$\sqrt{T}\{\Pi_c(-\Delta^2 . b, \theta) - F_c\} = \tfrac{1}{2}\pi - F_c . H(b\theta) \ldots (32).$$

This completes the equation needed for the integral Π_c, which is entirely reduced to F and E.

We vary the form only, by adding (17*a*) to (31);

therefore $\quad \sqrt{T}\Pi_c(p) = \frac{1}{2}\pi - F_c\{H(b\theta) - \Delta(b\theta)\tan\theta\}$;

subtract it from $\sqrt{T}.F_c = \dfrac{\Delta(b\theta)}{\sin\theta\cos\theta} . F_c$;

then $\sqrt{T}\{F_c - \Pi_c(\cot^2\theta)\} = F_c\{H(b\theta) + \Delta(b\theta)\cot\theta\} - \frac{1}{2}\pi \ldots (33).$

We may also develop the right-hand member of equation (30) by Lagrange's scale; then

$\sqrt{-T}.\{\Pi_c(-c^2\sin^2\eta) - C\} = C_1 c_1 \sin\eta_1 + C_2 c_2 \sin\eta_2 + C_3 c_3 \sin\eta_3 + \&c\ldots$;

and if we then assume $-c_n^2 \sin^2\eta_n = p_n$,

$$\left.\begin{aligned} &\sqrt{-T}.\{\Pi_c - C\} = C_1\sqrt{-p_1} + C_2\sqrt{-p_2} + C_3\sqrt{-p_3} + \&c\ldots \\ \therefore\ &\sqrt{T}.\{C - \Pi_c\} = C_1\sqrt{p_1} + C_2\sqrt{p_2} + C_3\sqrt{p_3} + \&c\ldots \end{aligned}\right\}\ldots(30a),$$

if we suppose p positive in the last. But to this subject we shall recur.

Π_c being now fully known, it remains to investigate P only, and Π will be known.

We must return to equation (29). If we multiply (29*a*) by $\dfrac{F(c\omega)}{F_c}$ and combine it with (29), we get

$$\sqrt{-T}P = -\tfrac{1}{2}\sin\omega\cos\omega\Delta(c\omega)\int_0 \frac{\sqrt{-T^{-1}}dp}{1 + p\sin^2\omega} - G(c\omega)\int_0 \frac{dp}{2p\sqrt{-T}}.$$

We have already found the last integral $= F(c\eta)$, when $p = -c^2 \sin^2\eta$. Also

$$\frac{\sqrt{-T^{-1}}.dp}{1+p\sin^2\omega} = \frac{2c^2\sin\eta\cos\eta\,d\eta}{(1-c^2\sin^2\omega\sin^2\eta)\cot\eta\,\Delta(c\eta)} = \frac{-2c^2.\sin^2\eta.dF(c\eta)}{1-c^2\sin^2\omega\sin^2\eta}$$

$$= \frac{2}{\sin^2\omega}\left\{1 - \frac{1}{1-c^2\sin^2\omega\sin^2\eta}\right\} dF(c\eta);$$

therefore $\sqrt{-TP}\,(-c^2\sin^2\eta, c, \omega)$

$$= \frac{\Delta(c\omega)}{\tan\omega}\{\Pi(-c^2\sin^2\omega, c, \eta) - F(c\eta)\} - G(c\omega).F(c\eta),$$

where we may write simply $\sqrt{-T}$ for the multiplier of the quantity in brackets.

Again, in (30) change η into ω, and multiply by $F(c\eta) \div F_c$; therefore $0 = \sqrt{-T}.\{\Pi_c(-c^2\sin^2\omega) - F_c\}\dfrac{F(c\eta)}{F_c} - G(c\omega)\,F(c\eta)$: subtract this from the preceding; then

$$\sqrt{-TP}\,(-c^2\sin^2\eta, c, \omega) = \sqrt{-TP}\,(-c^2\sin^2\omega, c, \eta)\ldots(34);$$

this is the commutative equation for a logarithmic P.

For a circular P the result is not quite so simple. We have first

$$\sqrt{TP} = \sin\omega\cos\omega\,\Delta(c\omega)\int_0 \frac{\frac{1}{2}\sqrt{T^{-1}}.dp}{1+p\sin^2\omega} + G(c\omega)\int_0 \frac{dp}{2p\sqrt{T}}.$$

Let $$p = \cot^2\theta;$$

Now put $\theta = \frac{1}{2}\pi$, and the left-hand member vanishes;

or $0 = \sin\omega\cos\omega\Delta(c\omega)\left\{\dfrac{F_b}{\cos^2\omega} - \dfrac{\Pi_b(\cot^2\omega)}{\sin^2\omega\cos^2\omega}\right\} - G(c\omega)F_b + (\frac{1}{2}\pi - x)$.

Multiply the last by $\dfrac{F(b\theta)}{F_b} = \dfrac{t}{\frac{1}{2}\pi}$, and subtract from the penultimate; observing that $\dfrac{\Delta(c\omega)}{\sin\omega\cos\omega} = \sqrt{T}(\cot^2\omega)$;

hence

$$\sqrt{TP}(\cot^2\theta, c, \omega) + \sqrt{TP}(\cot^2\omega, b, \theta) = (\tfrac{1}{2}\pi - x)\left(1 - \frac{t}{\frac{1}{2}\pi}\right) \ldots (35),$$

in which the right-hand member could not have been conjectured from the analogy of (34). It has risen from the peculiarity that $\sqrt{T}\Pi(p) = \frac{1}{2}\pi$ when $p = \infty$.

IX. We may now revert to the notation of (26), (27), (28), by writing Θ for that which Ω becomes when $\theta c\omega$ are changed to $\omega b\theta$. Then, in place of (35), we get

$$\tfrac{1}{2}\pi\,\{\Omega + \Theta\} = (\tfrac{1}{2}\pi - x)\,(\tfrac{1}{2}\pi - t) \ldots\ldots\ldots (36).$$

Between the quantities $\Omega, \overset{2}{\dot{\Omega}}, \overset{3}{\dot{\Omega}}, \Theta, \overset{2}{\dot{\Theta}}, \overset{3}{\dot{\Theta}}$ we have obtained three independent equations (26), (27), (36). If we commute θ, c, ω with ω, b, θ, (36) is not changed, but (26) and (27) yield two new equations, so that we have, in all, *five* equations connecting the *six* quantities in pairs. We have, therefore, exhausted all the relations between them; and by mere elimination we can obtain the equation between any pair of them at pleasure.

In a single view the equations are as follows:

$$\left.\begin{aligned} &\cot(x + \Omega + \overset{2}{\dot{\Omega}}) = \tan(\Theta - \overset{3}{\dot{\Theta}}) \\ &= \tan\left\{\frac{t(\frac{1}{2}\pi - x)}{\frac{1}{2}\pi} + \Theta - \overset{2}{\dot{\Omega}}\right\} = \tan\left\{\frac{(\frac{1}{2}\pi - x)(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} - \Omega - \overset{3}{\dot{\Theta}}\right\} \\ &= \sin\theta\sin\theta_\circ \cdot \frac{b\cos\omega}{\cos\omega^\circ} \end{aligned}\right\} \ldots (37),$$

in which we may exchange $\theta, c, \omega, \Omega, x$ with $\omega, b, \theta, \Theta, t$.

$$\left.\begin{aligned} &\text{Again,}\quad \tan(x + \overset{2}{\dot{\Omega}} + \overset{3}{\dot{\Omega}}) = \tan(t + \overset{2}{\dot{\Theta}} + \overset{3}{\dot{\Theta}}) \\ &= \tan\left\{\frac{xt}{\frac{1}{2}\pi} + \overset{2}{\dot{\Omega}} + \overset{2}{\dot{\Theta}}\right\} = \cot\left\{\frac{(\frac{1}{2}\pi - x)(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} - \overset{3}{\dot{\Omega}} - \overset{3}{\dot{\Theta}}\right\} \\ &= \frac{\sin\theta}{\sin\theta_\circ} \cdot \frac{\sin\omega}{\sin\omega^\circ} \end{aligned}\right\} \ldots (38).$$

Lastly,

$$\left.\begin{aligned} \tfrac{1}{2}\pi\,(\Omega+\Theta) &= (\tfrac{1}{2}\pi - x)(\tfrac{1}{2}\pi - t) \\ \tfrac{1}{2}\pi\,(\overset{2}{\dot{\Theta}} - \overset{3}{\dot{\Omega}}) &= x(\tfrac{1}{2}\pi - t) \\ \tfrac{1}{2}\pi\,(\overset{2}{\dot{\Omega}} - \overset{3}{\dot{\Theta}}) &= t(\tfrac{1}{2}\pi - x) \end{aligned}\right\}\text{.........(39).}$$

In (37) there are, with the commutations, 8 equations; in (38) there are 4; so that we have here 15 relations of pairs of the six integrals. Also $15 = \dfrac{5.6}{2}$; so that all the possible pairs are here exhibited.

Thus by a single transformation we can pass from one circular $\sqrt{TP}$ to another, so as to secure at once that p shall be less than q, (and therefore, p less than c), and that the modulus shall be less than its complement. We may also, if we wish it, secure a positive parameter.

X. Now rises the question, of approximating to $\sqrt{TP}$, when we have so transformed it as to obtain a favourable case. We proceed by Lagrange's scale, supposing $c_1\omega_1$ to be deduced as usual from $c\omega$. Let p also be given, but $p_1 r$ be disposable constants. Then if Π_1 stand for $\Pi\,(p_1 c_1 \omega_1)$, and we make the usual substitutions $F(c_1\omega_1) = (1+b)\,F(c\omega)$, $\sin\omega_1 = (1+b)\,\dfrac{\sin\omega\,\cos\omega}{\Delta(c\omega)}$, we get

$$\Pi_1 = \int_0 \frac{dF(c_1\omega_1)}{1+p_1\sin^2\omega_1} = (1+b)\int_0 \frac{\Delta^2(c\omega)\,dF(c\omega)}{\Delta^2(c\omega)+p_1(1+b)^2\sin^2\omega\cos^2\omega}\,.$$

Farther, if we assume

$$\frac{1-c^2v}{(1+pv)(1-rv)} = \frac{M}{1+pv} + \frac{N}{1-rv},$$

which yields

$$\Pi_1 = (1+b)\{M\Pi(p) + N\Pi(-r)\},$$

we get $M + N = 1$, $Mr - Np = c^2$; whence

$$M = \frac{p+c^2}{p+r} = \frac{1+p}{p}.\left(\frac{pr}{p+r}\right), \text{ and } N = \frac{r-c^2}{p+r} = \frac{1-r}{r}.\left(\frac{pr}{p+r}\right),$$

whence $\Pi_1 = (1+b).\dfrac{pr}{p+r}\left\{\dfrac{1+p}{p}\Pi(p) + \dfrac{1-r}{r}\Pi(-r)\right\}.$

Multiply by $\quad \sqrt{T_1} = \dfrac{p+r}{(1+b)\sqrt{(pr)}}$;

therefore $\quad \sqrt{T_1}\,\Pi_1 = \sqrt{T}\,\Pi(p) + \sqrt{T}\,\Pi(-r) \text{.........} (40).$

Make $\omega = \frac{1}{2}\pi$, $\omega_1 = \pi$; $\therefore 2\sqrt{T_1}\Pi_{c1} = \sqrt{T}\Pi_c(p) + \sqrt{T}\Pi_c(-r)$.

Multiply this by $\frac{1}{2}.\dfrac{F_1}{C_1} = \dfrac{F}{C}$, and subtract from (40),

then $\quad \sqrt{\pm T_1}\mathrm{P}_1 = \sqrt{\pm T}\mathrm{P}(p) + \sqrt{\pm T}\mathrm{P}(-r) \text{......} (40a)$,

which for the circular integral is

$$\Omega_1 = \Omega + \overset{s}{\Omega} \text{.....................} (41).$$

But from (27), observing that

$$\sin\omega\sin\omega^\circ.\frac{c\cos\theta}{\cos\theta_0} = \frac{\sin\omega_1}{1+b}.\sqrt{(pr)} = \sqrt{p_1}\sin\omega_1,$$

we have $\quad \Omega - \overset{s}{\Omega} = \tan^{-1}(\sqrt{p_1}\sin\omega_1).$

Add this to (41), and you get

$$\Omega = \tfrac{1}{2}\Omega_1 + \tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\sin\omega_1) \text{............} (42):$$

but in the process there has been nothing up to (40a) to limit p to be circular. When otherwise, it is very obvious how to change $\tan^{-1}$ into $\sqrt{-1}.\log$ in the equation (42) so as to adapt it as in equation (20).

XI. Legendre has investigated the relations of p_1 and p; but we need to add many reflections.

When $\quad p = -c^2\sin^2\eta$, $r = c^2\sin^2\eta^\circ$,

$$\sqrt{(-pr)} = c^2\sin\eta\sin\eta^\circ = \frac{c^2}{1+b}\sin\eta_1;$$

therefore $$\sqrt{-p_1} = \frac{c^2}{(1+b)^2}\sin\eta_1 = c_1 \sin\eta_1.$$

Consequently the series $p, p_1, p_2\ldots$answers the conditions supposed in equation (30*a*), and those developments apply to our newly-derived parameters.

When we assume $\sin\eta = \sqrt{-1}\tan\theta$, or $p = c^2\tan^2\theta$, the general equation of Lagrange's scale, viz. $\tan\eta_1 = \dfrac{(1+b)\tan\eta}{1 - b\tan^2\eta}$, changes into the scale of Gauss, $\sin\theta_1 = \dfrac{(1+b)\sin\theta}{1 + b\sin^2\theta}$, which is that followed by $\theta, \theta_1, \theta_2, \theta_3\ldots$ Consequently, when $\theta = \frac{1}{2}\pi$, every other θ in the scale also $= \frac{1}{2}\pi$. Also $\theta, \theta_1, \theta_2, \theta_3\ldots$ tend to $\frac{1}{2}\pi$ as their limit, since they give

$$\frac{F(b\theta)}{B} = \frac{F(b_1\theta_1)}{B_1} = \frac{F(b_2\theta_2)}{B_2};$$

and since $B_n = \infty$ when $n = \infty$, and $b_n = 1$, therefore

$$F(1\theta_n) = \infty, \text{ or } \theta_n = \tfrac{1}{2}\pi.$$

If we form $\theta, \theta', \theta''\ldots$ in the opposite direction, then, since

$$\frac{t}{\frac{1}{2}\pi} = \frac{F(b\theta)}{B} = \frac{F(b'\theta')}{B'} = \frac{F(b''\theta'')}{B''} = \&\text{c}\ldots$$

and $B^{(n)}$ has $\frac{1}{2}\pi$ for limit, and $b^{(n)}$ is evanescent, therefore $\theta^{(n)} = t$, when $n = \infty$; or $\theta, \theta', \theta'', \theta'''\ldots$ tend to t as their limit.

which last gives inversely

$$b \sin^2\theta = \frac{1 - \Delta(b_1\theta_1)}{1 + \Delta(b_1\theta_1)}.$$

Legendre calculates $p, p_1, p_2 \ldots$ by auxiliary arcs $\lambda, \lambda_1, \lambda_2, \lambda_3 \ldots$ Let $\cos\lambda = \Delta(b\theta)$, $\cos\lambda_1 = \Delta(b_1\theta_1)$, &c.; therefore $\sin\lambda = b\sin\theta$. Consequently $\frac{\sin^2\lambda}{b} = \frac{1 - \cos\lambda_1}{1 + \cos\lambda_1}$, or $\sin\lambda = \sqrt{b} \,.\, \tan\frac{1}{2}\lambda_1$; which is a general relation for $\lambda, \lambda_1, \lambda_2, \lambda_3 \ldots$

The arcs $\lambda, \lambda_1, \lambda_2 \ldots$ are thus suggested by the parameter $\cot^2\theta$, but they equally apply to logarithmic parameters. In fact, if we assume

$$\sin\lambda = \frac{b}{\Delta(c\eta)}, \quad \sin\lambda_1 = \frac{b_1}{\Delta(c_1\eta_1)},$$

and observe that $\Delta^2(c\eta) = b^2 + c^2\cos^2\eta$, we get

$$\cos\lambda = \frac{c\cos\eta}{\Delta(c\eta)}, \quad \cos\lambda_1 = \frac{c_1\cos\eta_1}{\Delta(c_1\eta_1)}.$$

Now, by the known relations in Lagrange's scale,

$$\cos\eta_1 = \frac{1 - (1 + b)\sin^2\eta}{\Delta(c\eta)}, \quad \text{and } \Delta(c_1\eta_1) = \frac{1 - (1 - b)\sin^2\eta}{\Delta(c\eta)},$$

and

$$c_1 = \frac{1 - b}{1 + b};$$

whence

$$\cos\lambda_1 = \frac{1 - b}{1 + b} \cdot \frac{1 - (1 + b)\sin^2\eta}{1 - (1 - b)\sin^2\eta} = \frac{1 - b - c^2\sin^2\eta}{1 + b - c^2\sin^2\eta},$$

and

$$\frac{1 - \cos\lambda_1}{1 + \cos\lambda_1} = \frac{b}{1 - c^2\sin^2\eta}, \quad \text{or } \tan\tfrac{1}{2}\lambda_1 = \frac{\sin\lambda}{\sqrt{b}},$$

as before. Nevertheless, if λ has the same value in both cases, the relation of θ to η is no longer $\sin^2\eta = -\tan^2\theta$, but is $c^2\sin^2\eta = -\cot^2\theta$.

It thus appears that the series $p_1, p_2, p_3 \ldots$ which are derived from p by the law $p_1 = \frac{pr}{(1 + b)^2}$, are calculable for both kinds of integral by the auxiliaries $p = \cot^2\theta = -c^2\sin^2\eta$,

$$\tan\tfrac{1}{2}\lambda_1 = \frac{\sqrt{b}}{\sqrt{(1 + p)}}, \quad \tan\tfrac{1}{2}\lambda_2 = \frac{\sin\lambda_1}{b_1}, \quad \tan\tfrac{1}{2}\lambda_3 = \frac{\sin\lambda_2}{b_2}, \text{ \&c.} \ldots$$

provided only that $1 + p$ be positive.

Put for a moment $\cos\lambda_r = x$, $\cos\lambda_{r+1} = y$; $\therefore \frac{1 - y}{1 + y} = \frac{1 - x^2}{b_r}$; and since $b_1, b_2, b_3 \ldots$ rapidly tend to 1, the last equation tends

to give $y = \dfrac{x^2}{2 - x^2}$, or scarcely more than $y = \frac{1}{2}x^2$. Thus $\cos\lambda_1$, $\cos\lambda_2$, $\cos\lambda_3$... soon tend rapidly to zero.

These arcs enable us to embrace in one expression the two series (30*a*).

Let N for a moment stand for the complete integral $\int_0^{\frac{1}{2}\pi} \dfrac{(1+p)\sin^2\omega}{1+p\sin^2\omega} \,.\, dF(c\omega)$, so that $\dfrac{pN}{1+p} = F_c - \Pi_c$; and when $p = -c^2\sin^2\eta$, $\sqrt{-T} = \cot\eta\,\Delta(c\eta)$, and

$$\sqrt{-T}\{\Pi_c - F_c\} = \frac{-p}{1+p}.\cot\eta\,\Delta(c\eta).N = \frac{c^2\sin\eta\cos\eta}{\Delta(c\eta)}.N = (1-b)\sin\eta_1.N.$$

Consequently, from equation (30) we deduce two forms,

$$(1-b)\sin\eta_1.N = CG(c\eta);\quad \text{and}\quad \frac{c_1^2\sin\eta_1\cos\eta_1}{\Delta(c_1\eta_1)}.N_1 = C_1G(c_1\eta_1):$$

but $CG(c\eta) - C_1G(c_1\eta_1) = C_1c_1\sin\eta_1$. Combine these, and observe that $(1-b)\,C = 2C_1c_1$; and you get

$$\frac{N}{C} - \tfrac{1}{2}\cos\lambda_1.\frac{N_1}{C_1} = \tfrac{1}{2}\ldots\ldots\ldots\ldots\ldots\ldots(43),$$

which now applies alike to both sorts of integrals, and easily gives (when $1+p$ is positive)

$$N = \int_0^{\frac{1}{2}\pi} \frac{(1+p)\sin^2\omega}{1+p\sin^2\omega}\,dF$$

$$= C\{\tfrac{1}{2} + \tfrac{1}{4}\cos\lambda_1 + \tfrac{1}{8}\cos\lambda_1\cos\lambda_2 + \tfrac{1}{16}\cos\lambda_1\cos\lambda_2\cos\lambda_3 + \&c\ldots\}\ldots(43a).$$

Consequently, whether we assume $p = c^2 \tan^2\theta$, $p_1 = c_1^2 \tan^2\theta_1$, &c....; or, on the other hand, assume $p = \cot^2\theta$, $p_1 = \cot^2\theta_1$, &c.... the series $\theta, \theta_1, \theta_2 \ldots$ which determine $p, p_1, p_2 \ldots$ are deduced by the very same law. It amounts to the same thing to remark, that we have

$$p_1 = \frac{p^2}{(1+b)^2} \cdot \frac{1+q}{1+p}; \quad q_1 = \frac{q^2}{(1+b)^2} \cdot \frac{1+p}{1+q};$$

so that q_1 is formed from q, by the same law as p_1 from p.

Thus, "When two original parameters are reciprocals, in reference to the original modulus, so are every derived pair, in reference to the new modulus."

It also follows, that the more rapid the convergence of $p, p_1, p_2, p_3, p_4 \ldots$, the slower is the convergence of $q, q_1, q_2, q_3 \ldots$ if indeed they converge at all. This makes it important whether p or q be selected to approximate from. Legendre's equation is

$$p_1 = \frac{p}{1+p} \cdot \frac{p+c^2}{(1+b)^2};$$

whence, if $p = ec$, and $p_1 = e_1 c_1$, there follows

$$\frac{e_1}{e} = \frac{e+c}{1+ec}, \text{ or } 1 \pm \frac{e_1}{e} = \frac{(1 \pm c)(1 \pm e)}{1+ec}.$$

If then $p^2 < c^2$, $e^2 < 1$, and we deduce that $\left(1 \pm \frac{e_1}{e}\right)$ is positive, or e_1 is numerically less than e. Hence if $p^2 < c^2$, $p, p_1, p_2, p_3 \ldots$ decrease more rapidly than $c, c_1, c_2, c_3 \ldots$

The relation of p_1 to p may also be written

$$\sqrt{p_1} = \frac{p}{1+p} \cdot \frac{\sqrt{T}}{1+b};$$

but our original equation, $\sqrt{p_1} = \dfrac{\sqrt{(pr)}}{1+b}$, with equal clearness shews that p_1 is positive whenever p is circular. Nor only so. It also denotes that if the original integral be $\Pi(-r, c, \omega)$, the value of p_1 is not altered; for to change p to $-r$, does but change r to $-p$, and leaves (pr) unchanged. Nevertheless, a consideration of the process which elicited equation (42), shews that the conjugates $p, -r$ in the first step downwards generate $+\sqrt{p_1}$ and $-\sqrt{p_1}$ differing in sign: but in the second step they coincide, and both produce the same $\sqrt{p_2}$.

When the problem is reversed, and we desire from p_1 to determine p, it is evident that there are two roots, p and $-r$

positive and *negative*, when Π is circular. But we cannot carry the series backward from $-r$, without falling on imaginary parameters. In fact, as p_1 is positive, if it proceed from a real p and r, so $-r$, being negative, cannot proceed from a real p' and r'. It may here deserve remark, that we thus learn of *certain* imaginary parameters, whose integral can be reduced to *one* real Π.

XIII. It remains to develop the actual series.

Let $\Psi = \tan^{-1}(p \sin\omega)$, so that

$$\Omega = \tfrac{1}{2}\Omega_1 + \tfrac{1}{2}\Psi_1.$$

When p is evanescent, $\Pi_c = F_c$, and P becomes identical with $\Pi - F$. We have seen that $\sqrt{T}(\Pi - F)$ vanishes with p; so therefore does $\sqrt{T}\mathrm{P}$. If then $p, p_1, p_2, p_3 \dots$ decrease beyond all limit, so do $\Omega, \Omega_1, \Omega_2, \Omega_3 \dots$; and much more does $2^{-n}\Omega_n$ vanish when $n = \infty$. Now, by repetition of the formula,

$$\Omega - 2^{-n}\Omega_n = \tfrac{1}{2}\Psi_1 + \tfrac{1}{4}\Psi_2 + \tfrac{1}{8}\Psi_3 + \dots + 2^{-n}\Psi_n;$$

whence $$\Omega = \tfrac{1}{2}\Psi_1 + \tfrac{1}{4}\Psi_2 + \tfrac{1}{8}\Psi_3 + \&c. \&c. \dots,$$

which is the final development by *descending* moduli.

In the other notation we have

$$\left.\begin{aligned}\sqrt{T}\mathrm{P}(p, c, \omega) = {} & \tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\sin\omega_1) + \tfrac{1}{4}\tan^{-1}(\sqrt{p_2}\sin\omega_2) \\ & + \tfrac{1}{8}\tan^{-1}(\sqrt{p_3}\sin\omega_3) + \&c.\end{aligned}\right\} \dots (44).$$

One of the equations marked (37) is

$$\frac{t(\frac{1}{2}\pi - x)}{\frac{1}{2}\pi} + \Theta - \overset{2}{\dot{\Omega}} = \tan^{-1}\left\{\sin\theta\sin\theta_0 . \frac{b\cos\omega}{\cos\omega^\circ}\right\}.$$

Change ω, c, θ, x, t to θ, b, ω, t, x; therefore

$$\frac{x(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} + \Omega - \overset{2}{\dot{\Theta}} = \tan^{-1}\{\sqrt{p_1}\sin\omega_1\} = \Psi_1.$$

In the last, change c, ω, θ to c', ω', θ', which changes x to $\frac{1}{2}x$, but leaves t unchanged; therefore

$$\frac{\frac{1}{2}x.(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} + \Omega' - \overset{2}{\dot{\Theta}}{}' = \Psi.$$

Eliminate x and t; then

$$(2\Omega' - \Omega) - (2\overset{2}{\dot{\Theta}}{}' - \overset{2}{\dot{\Theta}}) = 2\Psi - \Psi_1.$$

But from (42) we have $2\Omega' - \Omega = \Psi$.

Hence $$\overset{2}{\dot{\Theta}} - 2\overset{2}{\dot{\Theta}}{}' = (\Psi - \Psi_1) \quad\text{.................}(45),$$

which is our new equation of reduction.

Repeating it n times, we obtain

$$\overset{2}{\dot{\Theta}} - 2^n.\overset{2}{\dot{\Theta}}{}^{(n)} = (\Psi - \Psi_1) + 2(\Psi' - \Psi) + 2^2(\Psi'' - \Psi') + \dots \text{ to } n \text{ terms.}$$

We now need a Lemma, to prove that $2^n.\overset{2}{\dot{\Theta}}{}^{(n)}$ vanishes when $n = \infty$.

First, observe that when c is so small that c^4 and $c^4\tan^4\theta$ are omissible, and $c^2\tan^2\theta$ is the parameter, we have the following values:

$$\Pi(c^2\tan^2\theta, c, \omega) = \int_0 (1 - c^2\tan^2\theta\sin^2\omega)(1 + \tfrac{1}{2}c^2\sin^2\omega)\,d\omega,$$

$$= \omega - \tfrac{1}{2}c^2(\tan^2\theta - \tfrac{1}{2})(\omega - \tfrac{1}{2}\sin 2\omega),$$

$$\Pi_1(c^2\tan^2\theta) = \tfrac{1}{2}\pi\{1 - \tfrac{1}{2}c^2(\tan^2\theta - \tfrac{1}{2})\},$$

$$F(c\omega) = \omega + \tfrac{1}{4}c^2(\omega - \tfrac{1}{2}\sin 2\omega),$$

$$F_c = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}c^2):$$

and combining these, we get

$$\mathrm{P} = \tfrac{1}{4}c^2\tan^2\theta\sin 2\omega = \tfrac{1}{4}p\sin 2\omega.$$

Also $$\sqrt{T} = \frac{\Delta(b\theta)}{\sin\theta\cos\theta},$$

which converges to $\dfrac{1}{\sin\theta}$, therefore

$$\sqrt{T}\mathrm{P} = \frac{1}{4}\frac{c^2\tan^2\theta}{\sin\theta} . \sin 2\omega.$$

Similarly then $\overset{2}{\Theta}$, when b is very small, converges to

$$\tfrac{1}{4}\cdot\frac{b^2\tan^2\omega}{\sin\omega}\cdot\sin 2\theta:$$

and when we change b, θ, ω into $b^{(n)}, \theta^{(n)}, \omega^{(n)}$, we know that $\theta^{(n)}, \omega^{(n)}$ approach to fixed limits t and $\dot{\omega}$ (which are less than $\frac{1}{2}\pi$, if θ, ω are less), so that we have nearly

$$\overset{2}{\Theta}{}^{(n)} = \tfrac{1}{4}b^{(n)2}.\frac{\tan^2\dot{\omega}}{\sin\dot{\omega}}.\sin 2t = \mathrm{k}.b^{(n)2};$$

the quantity k being finite and independent of n. Hence

$$2^n.\overset{2}{\Theta}{}^{(n)} = \mathrm{k}.\{2^n.b^{(n)2}\}.$$

But when $n=\infty$, $2^n.b^{(n)2}$ is evanescent; and indeed the quantity is extremely small when $n=3$ or even $n=2$, if c is very near to 1. Hence we get

$$\overset{2}{\Theta} = (\Psi - \Psi_1) + 2(\Psi' - \Psi) + 2^2(\Psi'' - \Psi') + \&\text{c.}\ldots(46),$$

a series rapidly converging.

When $\overset{2}{\Theta}$ is known, we find Ω or $\overset{2}{\Omega}$ by one of the commutative equations. Thus

$$\Omega = \overset{2}{\Theta} + \Psi_1 - \frac{x}{\frac{1}{2}\pi}(\tfrac{1}{2}\pi - t)$$

It may farther be observed, that since $\Omega - \Omega' = \Omega' - \Psi$, we have also $\Omega^{(n)} - \Omega^{(n+1)} = \Omega^{(n+1)} - \Psi^{(n)}$. Also since $c, c', c'', c'''\ldots$ $\omega, \omega', \omega'', \omega'''\ldots \theta, \theta', \theta''\ldots$ all tend to fixed limits $1, \grave{\omega}, t$, so do $\Omega', \Omega'', \Omega'''\ldots.\ \Psi', \Psi'', \Psi'''\ldots$ tend to fixed limits $`\Omega`\Psi$; and since $\Omega^{(n)} - \Omega^{(n+1)}$ has limit zero, so has $\Omega^{(n+1)} - \Psi^{(n)}$; *i.e.* $`\Omega = `\Psi$;

or $\qquad \sqrt{\mathrm{T}}\mathrm{P}(p`, 1, \grave{\omega}) = \tan^{-1}(\sqrt{p}\ \sin\grave{\omega})$;

or, since $p`\grave{\omega}$ are mutually independent, we have generally

$$\sqrt{\mathrm{T}}\mathrm{P}(p, 1, \omega) = \tan^{-1}(\sqrt{p}\sin\omega)\ldots\ldots\ldots\ldots(47),$$

which may be easily confirmed by direct integration.

Thus in equation (42), as also in the reciprocal, and in the conjugate equation, the function *tan*$^{-1}$ may be replaced by an integral of the form $\sqrt{\mathrm{T}}\mathrm{P}(\cot^2\mu, 1, \psi)$. In equation (8) we might similarly substitute

$$CG(c\omega) - C_1G(c_1\omega_1) = C_1c_1G(1, \omega_1).$$

It is remarkable how the function $\tan^{-1}(\sqrt{p_1}\sin\omega_1)$ derived according to Lagrange's and Gauss's scale seems to intrude itself into more elementary equations, as (21), (24).

Finally, it will here be remarked that equation (46) is only in appearance an equation of *ascending* moduli; for though $c, c', c''\ldots$ascend, yet b is the modulus of $\overset{2}{\Theta}$; and $b, b', b'', b'''\ldots$ *decrease* by the same law as $c, c_1, c_2\ldots$ Nevertheless, the mode in which $\theta, \theta', \theta''\ldots\ \omega, \omega', \omega''\ldots$ are derived, is that which we understand to belong to ascending moduli.

XIV. A similar treatment would manifestly apply to the logarithmic P; but Legendre's adaptation of Jacobi's great discovery here supersedes equation (42), by resolving P into a simpler integral. Indeed, Legendre's reduction of Π, or rather of $\sqrt{\mathrm{T}}(\Pi - F)$, to the integral $\Upsilon = \int_0 EdF$, will be well exchanged into a reduction of $\sqrt{\mathrm{T}}\mathrm{P}$ to the integral $V = \int_0 GdF$. Of course, as $G = E - \dfrac{E_c}{F_c}F$, so $V = \Upsilon - \frac{1}{2}\dfrac{E_c}{F_c}F^2$: and as G is the *fluctuant* to E, and P to Π, so is V to E. The process will then be as follows:

By Euler's integration, if $F\omega + F\eta = F\zeta$,

$$E\omega + E\eta - E\zeta = c^2\sin\omega\sin\eta\sin\zeta,$$

and $$\sin\zeta = \frac{\sin\omega\cos\eta\Delta\eta + \sin\eta\cos\omega\Delta\omega}{1 - c^2\sin^2\omega\sin^2\eta}.$$

When η becomes $-\eta$, let ζ become ε. Observe that we equally have

$$G\omega + G\eta - G\zeta = c^2\sin\omega\sin\eta\sin\zeta,$$

$$\therefore\ G\zeta - G\varepsilon = 2G\eta - c^2\sin\omega\sin\eta(\sin\zeta + \sin\eta).$$

Let η be constant; therefore

$$dF\zeta = dF\omega = dF\varepsilon.$$

Hence

$$G\zeta dF\zeta - G\varepsilon dF\varepsilon = 2G\eta dF\omega - c^2 \sin\omega \sin\eta \,.\, \frac{2\sin\omega\cos\eta\Delta\eta}{1 - c^2\sin^2\eta\sin^2\omega}:$$

or $$\tfrac{1}{2}V\zeta - \tfrac{1}{2}V\varepsilon = G\eta.F\omega - \frac{\Delta\eta}{\tan\eta}.\int_0 \frac{c^2\sin^2\eta\sin^2\omega}{1 - c^2\sin^2\eta\sin^2\omega}\,dF\omega.$$

But the last term

$$= \sqrt{-\mathrm{T}}\,\{\Pi(-c^2\sin^2\eta, c, \omega) - F\},$$

and, by equation (30),

$$G\eta = \sqrt{-\mathrm{T}}\left\{\frac{\Pi_c(-c^2\sin^2\eta)}{F_c} - 1\right\};$$

which indeed might be here at once inferred by making $\omega = \tfrac{1}{2}\pi$. Hence

$$\sqrt{-\mathrm{T}}\mathrm{P}(-c^2\sin^2\eta, c, \omega) = \tfrac{1}{2}V(c\,\varepsilon) - \tfrac{1}{2}V(c\,\zeta) \ldots (48),$$

which throws a new light on equation (34).

The simpler integral $V(c\,\omega)$ now claims a full examination.

XV. Since

$$G(n\pi + \omega) = G\omega, \text{ and } F(n\pi + \omega) = F(n\pi) + F\omega,$$

therefore

$$V(n\pi + \omega) = \int G(n\pi + \omega).dF(n\pi + \omega) = \int G\omega.dF\omega = V(n\pi) + V\omega.$$

Again, since

$$G\omega + G\omega' = c^2 \sin\omega \sin\omega' = \frac{c^2 \sin\omega \cos\omega}{\Delta\omega},$$

and
$$dF\omega = -dF\omega' = \frac{d\omega}{\Delta\omega};$$

multiply these together; therefore

$$dV\omega - dV\omega' = \frac{c^2 \sin\omega \cos\omega d\omega}{1 - c^2 \sin^2\omega};$$

or
$$V\omega - V\omega' = \text{const.} - \tfrac{1}{2}\log(1 - c^2 \sin^2\omega).$$

Let $\omega = 0$, $\omega' = \frac{1}{2}\pi$; therefore const. $= -V_c$, or

$$V\omega' - V\omega = V_c + \log\Delta\omega \ldots\ldots\ldots\ldots (51).$$

Cor. Let $\omega = \frac{1}{2}\pi$, $\omega' = 0$; therefore

$$-2V_c = \log b, \quad V_c = \tfrac{1}{2}\log b^{-1} \ldots\ldots\ldots (51a).$$

Thus, if b be infinitesimal, V_c is infinite. Nevertheless, even for small values of b, V_c is of very moderate amount, since it is only a logarithm.

When c is infinitesimal, $G(c\omega)$ vanishes for all values of ω; hence so also does $V(c\omega)$.

So if, $F\psi = 2F\omega$, or $\sin\psi = \dfrac{2\sin\omega\cos\omega\Delta\omega}{1 - c^2\sin^4\omega}$,

and $G\psi = 2G\omega - c^2\sin^2\omega\sin\psi$, we get

$$V\psi = \int_0 G\psi . dF\psi = \int_0 \{2G\omega - c^2\sin^2\omega\sin\psi\}\, 2dF\omega,$$

or
$$V\psi - 4V\omega = \int_0 - 2c^2\sin^2\omega . \frac{2\sin\omega\cos\omega\Delta\omega}{1 - c^2\sin^4\omega} . \frac{d\omega}{\Delta\omega}$$
$$= \log(1 - c^2\sin^4\omega) \ldots\ldots\ldots\ldots\ldots\ldots (52).$$

Equations (51), (51a), (52) are in close analogy with those established by Legendre concerning the function Υ, the integrations being the very same.

XVI. We proceed to apply Lagrange's scale to V.

Since $CG = C_1G_1 + C_1c_1 \sin\omega_1$,

or
$$CG - C_1G_1 = \tfrac{1}{2}Cc^2\frac{\sin\omega\cos\omega}{\Delta(c\omega)},$$

also
$$\frac{dF}{C} = \tfrac{1}{2}\frac{dF_1}{C_1} = \frac{d\omega}{C\Delta(c\omega)};$$

multiply the two equations; therefore

$$dV - \tfrac{1}{2}dV_1 = \tfrac{1}{2}c^2\frac{\sin\omega\cos\omega d\omega}{1 - c^2\sin^2\omega};$$

or
$$V - \tfrac{1}{2}V_1 = -\tfrac{1}{2}\log\Delta(c\omega) \ldots\ldots\ldots (53),$$

which, for a logarithmic Π, replaces (42) for the circula

As we have $C\{G(c\omega)-G(c\omega^\circ)\}=2C_1G(c_1\omega_1)$
so also $V(c\omega)+V(c\omega^\circ)=V(c_1\omega_1)+V_c$ ......(54).

If we repeat (53) n times, we find

$$V-2^{-n}V_n=-2^{-1}\log\Delta-2^{-2}\log\Delta_1-2^{-3}\log\Delta_2-\text{\&c. to } n \text{ terms.}$$

Also, since $c_n=0$, when $n=\infty$, so is $V_n=0$; therefore

$$V=-\tfrac{1}{2}\log\Delta-\tfrac{1}{4}\log\Delta_1-\tfrac{1}{8}\log\Delta_2-\text{\&c.} \quad\text{......(55)},$$

which is analogous to the development (44). In fact, applying the last to (48), and observing that to form $\eta_1\eta_2\eta_3\ldots$ from η, and $\omega_1, \omega_2, \omega_3\ldots$ from ω, by Lagrange's scale, and then to form $\zeta, \zeta_1, \zeta_2, \zeta_3\ldots$ by coupling ω and η, ω_1 and η_1, &c. amounts to forming $\zeta, \zeta_1, \zeta_2, \zeta_3\ldots$ by Lagrange's scale; and similarly of $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3\ldots$, we get

$$\sqrt{-T}P(-c^2\sin^2\eta, c, \omega)=\tfrac{1}{2}V(c\varepsilon)-\tfrac{1}{2}V(c\zeta)$$

$$=\tfrac{1}{4}\log\frac{\Delta\zeta}{\Delta\varepsilon}+\tfrac{1}{8}\log\frac{\Delta\zeta_1}{\Delta\varepsilon_1}+\tfrac{1}{16}\log\frac{\Delta\zeta_2}{\Delta\varepsilon_2}+\text{\&c.}$$

$$=-\tfrac{1}{4}\log\frac{1+D_1}{1-D_1}-\tfrac{1}{8}\log\frac{1+D_2}{1-D_2}-\tfrac{1}{16}\log\frac{1+D_3}{1-D_3}-\text{\&c.}\ldots(56),$$

if $D=c\sin\eta\sin\omega$. This equation is the transformation of (44) to the case of a logarithmic P, and apparently must be actually used to approximate to $\sqrt{-T}P$, until tables of V are calculated.

Now if $\omega < \frac{1}{2}\pi$, the series $\omega, \omega', \omega'', \omega'''\ldots$ decrease towards a limit $\dot{\omega} < \frac{1}{2}\pi$, so that the last integral is finite and independent of n; while $2^n b^{(n)2}$ is infinitesimal when n is infinite. Hence, for infinite values of n, $2^n E^{(n)} = 2^n.\sin\dot{\omega}$.

It is still easier to see that $\Delta^{(n)} = \cos\dot{\omega}$, when $n = \infty$.

Also $E_e^{(n)} = 1$. Hence

$$2^n.G^{(n)} = 2^n.\sin\dot{\omega} - \frac{F}{C}.$$

But $\dfrac{F^{(n)}}{B^{(n)}} = \dfrac{F}{B}$, and $B^{(n)}$ converges to $\frac{1}{2}\pi$, $F^{(n)}$ to $\dfrac{d\dot{\omega}}{\cos\dot{\omega}}$.

$$\therefore\ 2^n.V^{(n)} \text{ or } 2^n.\int_0 G^{(n)}dF^{(n)} = 2^n.\int_0 \frac{\sin\dot{\omega}}{\cos\dot{\omega}}\,d\dot{\omega} - \tfrac{1}{2}\pi\int_0 \frac{FdF}{BC},$$

$$\left.\begin{aligned} &2^n\{V^{(n)} + \log\cos\dot{\omega}\} = -\tfrac{1}{4}\pi.\frac{F^2}{BC} \\ \text{or}\qquad &2^n\{V^{(n)} + \log\Delta^{(n)}\} = -\tfrac{1}{4}\pi.\frac{F^2}{BC} \end{aligned}\right\} \text{ when } n = \infty \ldots(57).$$

Finally, then,

$$V = -\tfrac{1}{4}\pi.\frac{F^2}{BC} + \log\frac{1}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + 2^2\log\frac{\Delta''}{\Delta'''} + \&c.\ldots(58),$$

which converges excellently when c is near to 1.

XVII. As we had a new integral H so related to G that $H - G = \frac{1}{2}\pi.\dfrac{F}{BC}$, it is well to conceive of W similarly related to V. Namely, as $V = \int_0 GdF$, so let $W = \int_0 HdF$;

$$\therefore\ W - V = \tfrac{1}{4}\pi\,\frac{F^2}{BC}\ldots\ldots\ldots\ldots\ldots\ldots(59).$$

This indicates that the last series is a development of W, analogous to (12),

$$W = \log\frac{1}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + 2^2\log\frac{\Delta''}{\Delta'''} + \&c.\ \ldots\ldots(60).$$

Again, if $\sin\eta = \sqrt{-1}\tan\theta$, we obtain from $F(c\eta) = \sqrt{-1}\,F(b\theta)$, and from (13),

$$\left.\begin{aligned} V(c\eta) &= W(b\theta) + \log\cos\theta \\ W(c\eta) &= V(b\theta) + \log\cos\theta \end{aligned}\right\}\ldots\ldots\ldots\ldots(61),$$

in which it is remarkable that $\sqrt{-1}$ has vanished entirely. In truth, the transformation $\sin\eta = \sqrt{-1}\tan\theta$ is nothing but

a device for enabling the sines and cosines of Trigonometry to do duty for *hyperbolic* sines and cosines; and we might in all cases evade $\sqrt{-1}$ in this transformation by having recourse to hyperbolic functions. If capital letters denote these, thus,

$$\mathrm{Cos}\,x = \tfrac{1}{2}(e^x + e^{-x}),\ \mathrm{Sin}\,x = \tfrac{1}{2}(e^x - e^{-x}),\ \&c.,$$

so that $\quad \mathrm{Cos}^2 x - \mathrm{Sin}^2 x = 1$, and $1 - \mathrm{Tan}^2 x = \mathrm{Sec}^2 x$;

it is manifest that, by assuming $\sin\omega = \mathrm{Tan}\,x$, we get

$$\cos\omega = \mathrm{Sec}\,x,\ \sec\omega = \mathrm{Cos}\,x,\ \tan\omega = \mathrm{Sin}\,x,$$

$$d\omega = \mathrm{Sec}\,x dx,\ dx = \sec\omega d\omega.$$

Hence $F(c\omega) = \displaystyle\int_0 \frac{\sec\omega d\omega}{\sqrt{(1+b^2\tan^2\omega)}} = \int_0 \frac{dx}{\sqrt{(1+b^2\,\mathrm{Sin}^2 x)}} = \phi(b, x).$

Thus, by adopting the double form F and ϕ, we might deal with *real* functions only; and equations (61) seem to indicate that the integrals of the second order, V and W, recover common sines and cosines, which were displaced by hyperbolic sines and cosines in the integrals of the first order.

The equations (61) facilitate many transformations.

We may moreover give another form to (58) by slightly modifying the equation of reduction.

Since $V = \log\Delta' + 2V'$, $V + \log\cos\omega = \log(\Delta'\cos\omega) + 2V'$ $= \log(\Delta'\cos\omega\sec^2\omega') + 2(V' + \log\cos\omega')$. But $\Delta'.\cos\omega\sec^2\omega'$ $= 1 - b'\tan^2\omega'$; therefore

Let $\theta = \frac{1}{2}\pi$, therefore $\theta' = \frac{1}{2}\pi = \theta'' = \theta''' = \&c....$, therefore

$$V_b = \log \frac{c'}{\sqrt{c}} + 2 \log \frac{c''}{\sqrt{c'}} + 2^2 \log \frac{c'''}{\sqrt{c''}} + \&c.... = -\log \sqrt{c},$$

and $V(b\theta) = V_b + \log \dfrac{\sin\theta'}{\sin\theta} + 2\log \dfrac{\sin\theta''}{\sin\theta'} + 2^2 \log \dfrac{\sin\theta'''}{\sin\theta''} + \&c....(63).$

This is adapted to the case of b very small, and b is here the modulus: hence the new series has no real advantage; for it is less convenient than (55). Yet we may step back to $W(c\omega)$, and write

$$1 - b' \tan^2\omega' = \frac{c'}{\sqrt{c}} \cdot \frac{\tan\omega'}{\tan\omega};$$

therefore $W(c\omega) + \log\cos\omega$

$$= V_b + \log \frac{\tan\omega'}{\tan\omega} + 2 \log \frac{\tan\omega''}{\tan\omega'} + 2^2 \log \frac{\tan\omega'''}{\tan\omega''} + \&c....(64),$$

which is adapted to the case of c near to 1.

For $\log\cos\omega - V_b$ we may write $\log(\sqrt{c}\cos\omega)$.

A new development of G by the scale of Gauss, bearing analogy to (63), may deserve notice. Since

$$V(b\theta) = \log(1 + b' \sin^2\theta') + 2V(b'\theta');$$

as the series itself indicates; we get, by differentiating, since

$$\frac{F(b\theta)}{B} = \frac{F(b'\theta')}{B'},$$

$$BG(b\theta) = \frac{2b' \sin\theta' \cos\theta'}{1 + b' \sin^2\theta'} . B'\Delta(b'\theta') + 2B'G(b'\theta')$$

$$= 2B'b' \sin\theta' \cos\theta + 2B'G(b'\theta')..........(63a).$$

It is easy to shew that $2^n . G(b^{(n)}\theta^{(n)})$ vanishes when $n = \infty$; therefore

$$BG(b\theta) = 2B'b' \sin\theta' \cos\theta$$
$$+ 2^2B''b'' \sin\theta'' \cos\theta' + 2^3B'''b''' \sin\theta''' \cos\theta'' + \&c....(63b).$$

But this is less simple than (8).

XVIII. To approximate to $V(c\omega)$ is now as easy as to find $G(c\omega)$ or $E(c\omega)$, except only that we have no tables of it ready calculated. But it is not without interest to consider the result of differentiating V with reference to c as the variable; for which purpose we step back to F, E, and G.

Let $a = \dfrac{B}{C}$, $x = \frac{1}{2}\pi . \dfrac{F(c\omega)}{C}$. Then, in the scale whose

index is n, to pass from $c\omega$ to new elements $c_1\omega_1$ changes ax to na,* nx. Moreover, in the higher theory, if we adopt, with Dr. Gudermann, the notation of *hyperbolic* sines and cosines, the development of G admits of the form

$$CG(c\omega) = \pi\left\{\frac{\sin 2x}{\operatorname{Sin}\pi a} + \frac{\sin 4x}{\operatorname{Sin}2\pi a} + \frac{\sin 6x}{\operatorname{Sin}3\pi a} + \&c.\ldots\right\},$$

which things suggest the advantage of making a rather than c the base of variation.

The same process which demonstrates $\frac{1}{2}\pi = F_cE_b + F_bE_c - F_bF_c$ shews, in passing, that $\frac{1}{2}\pi.\dfrac{dc}{da} = -b^2c.F_c^2$. In the common treatises we have

$$\frac{dF}{dc} = \frac{E}{b^2c} - \frac{F}{c} - \frac{c\sin\omega\cos\omega}{b^2\Delta\omega}, \text{ when } \omega \text{ is constant;}$$

$$\therefore\ \tfrac{1}{2}\pi.\frac{dF}{da} = -F_c^2\{E - b^2F - c^2\sin\omega\sin\omega^\circ\}\ldots..(65).$$

Hence $\quad \frac{1}{2}\pi.\dfrac{dF_c}{da} = -F_c^2\{E_c - b^2F_c\}$, when $\omega = \frac{1}{2}\pi$.

Multiply the last by $\dfrac{F}{F_c}$ and subtract from (65), therefore

$$\tfrac{1}{2}\pi\left\{\frac{dF}{da} - \frac{F}{F_c}.\frac{dF_c}{da}\right\} = -F_c^2\{G - c^2\sin\omega\sin\omega^\circ\}.$$

It is observable, that we also have

$$\tfrac{1}{2}\pi.\frac{d(cF_c)}{da} = -E_cF_c^2c;$$

so that
$$\frac{dA}{da} = \sin\omega\sin\omega^\circ.\frac{d(cF_c)}{da} \;\ldots\ldots\ldots\ldots\; (67a).$$

In all these equations, (65)—(67a), we suppose ω to be constant. But in general, when c and ω both vary,

$$\frac{d(x)}{da} = \frac{dx}{d\omega}\frac{d\omega}{da} + \frac{dx}{da}; \text{ and } \frac{d(A)}{da} = \frac{dA}{d\omega}\frac{d\omega}{da} + \frac{dA}{da};$$

that is,
$$\frac{d(x)}{da} = \frac{\tfrac{1}{2}\pi}{F_c\Delta}\cdot\frac{d\omega}{da} + A(\omega^\circ);$$

$$\tfrac{1}{2}\pi\frac{d(A)}{da} = \tfrac{1}{2}\pi\frac{d\omega}{da}\cdot\left(\frac{dA}{d\omega}\right) - E_cF_c^2c^2\sin\omega\sin\omega^\circ.$$

Now let x be the principal variable instead of ω; and when a varies, let x be constant, or $\frac{d(x)}{da} = 0$; and eliminate $\frac{d\omega}{da}$ from the two last; observing that

$$\frac{dA}{d\omega} = F_c\Delta - \frac{E_c}{\Delta} \text{ and } A(\omega^\circ) = F_cc^2\sin\omega\sin\omega^\circ - A(\omega);$$

$$\therefore\ \tfrac{1}{2}\pi.\frac{dA}{da} = F_c.\Delta.A\frac{dA}{d\omega} - F_c^2c^2\sin\omega\cos\omega\Delta \;\ldots\ldots (68).$$

Multiply by $\frac{dx}{\tfrac{1}{2}\pi} = \frac{d\omega}{F_c\Delta}$; and since x is constant in $\frac{d}{da}$, we may integrate for x under the d; or

$$\int_0\frac{dA}{da}dx = \frac{d}{da}\int_0 Adx = \frac{d}{da}\int_0 GdF.\tfrac{1}{2}\pi;$$

$$\therefore\ \tfrac{1}{2}\pi.\frac{d}{da}V(c\omega) = \tfrac{1}{2}A^2(c\omega) - \tfrac{1}{2}F_c^2c^2\sin^2\omega \;\ldots\ldots (69),$$

when x is constant.

If this have no other interest, it at least shews that A^2 can be expressed in series of the cosines of $2x$ and of its multiples: for the developments of $V(c\omega)$ and $F_c^2c^2\sin^2\omega$ are known.

XIX. The same notation facilitates the management of E, G, and V in the higher scales. To avoid confusion, in the scale whose index is n, let $h\psi$ be the new elements of F

which are called $c_1\omega_1$ in the common scale. Then, since x, a change to nx, na, when $c\omega$ change to $h\psi$, we have, as *total* variations,

$$\left.\begin{aligned}\frac{d.x}{da} &= \frac{\frac{1}{2}\pi}{F_c\Delta(c\omega)}\cdot\frac{d\omega}{da} + F_cG(c\omega^\circ)\\ \frac{d.nx}{nda} &= \frac{\frac{1}{2}\pi}{F_h\Delta(h\psi)}\cdot\frac{d\psi}{nda} + F_hG(h\psi^\circ)\end{aligned}\right\}.$$

The left-hand member is the same in both equations, and we may at pleasure assume any *one* of the variables as constant. If it be ω, $\dfrac{d\omega}{da} = 0$, therefore

$$F_cG(c\omega^\circ) = F_hG(h\psi^\circ) + \frac{\frac{1}{2}\pi}{F_h\Delta(h\psi)}\cdot\frac{d\psi}{nda} \quad \ldots\ldots(70).$$

This is a generalization of $CG = C_1G_1 + C_1c_1 \sin\omega_1$, and supersedes a much more complicated one connecting $E(c\omega)$ with $E(h\psi)$ in Legendre.

Multiply by

$$\frac{dF(c\omega^\circ)}{F_c} = \frac{dF(h\psi^\circ)}{nF_h} = -\frac{1}{nF_h}\cdot\frac{d\psi}{\Delta(h\psi)};$$

$$\therefore\ V(c\omega^\circ) = \frac{1}{n}\,V(h\psi^\circ) - \frac{\frac{1}{2}\pi}{n^2F_h^2}\cdot\int_0\left(\frac{d\psi}{da}\right)\frac{d\psi}{1 - h^2\sin^2\psi} \ldots(71),$$

which also is a generalization of (53).

It is proper also to notice here the relation borne by V and W to Jacobi's new functions. Let q be a small fraction such that $\log q^{-1} = \pi a$, and $\Theta\Lambda$ functions such that

$$\left.\begin{aligned}\Theta &= 1 - 2q\cos 2x + 2q^{2.2}\cos 4x - 2q^{3.3}\cos 6x + \&\text{c}.\ldots\\ \Lambda &= \quad 2q^{\frac{1}{4}}\sin x - 2q^{\frac{3.3}{4}}\sin 3x + 2q^{\frac{5.5}{4}}\sin 5x - \&\text{c}.\ldots\end{aligned}\right\};$$

then, among other equations, Jacobi has proved that

$$\sqrt{(1 - c^2\sin^2\omega)} = \sqrt{b}.\,\frac{\Theta(q,\, \frac{1}{2}\pi + x)}{\Theta(qx)},$$

$$\text{and } \cos\omega = \sqrt{\frac{b}{c}}\,.\,\frac{\Lambda(q,\, x + \frac{1}{2}\pi)}{\Theta(qx)}.$$

Legendre has demonstrated (2nd Supplement, § xi.) that

$$\frac{2F_c}{\pi}\,G(c\omega) = \frac{\Theta'(qx)}{\Theta(qx)}.$$

If we multiply by $\frac{1}{2}\pi.\,\dfrac{dF(c\omega)}{F_c} = dx$, and integrate, we get

$$V = \log(\beta\Theta),$$

where β is a function of c.

When $\omega = \pi$, $V = 0$; therefore $\beta\Theta = 1$, or

$$\beta^{-1} = \Theta(q,\, \pi) = 1 - 2q + 2q^{2.2} - 2q^{3.3} + \&\text{c}.\ldots,$$

a series which is known to be equal to $\sqrt{\dfrac{2bF_c}{\pi}}$.

But it suffices to write $V = \log\dfrac{\Theta(q,\, x)}{\Theta(q,\, \pi)}$ ....(72).

This elegant relation is obscurely expressed by Legendre under the following form

$$\log\Theta(qx) = \Upsilon(c\omega) - \tfrac{1}{2}\,\frac{E_c}{F_c}.F^2(c\omega) + \tfrac{1}{2}\log\frac{2bF_c}{\pi}.$$

Had he used the integral V, he would not have overlooked the following curious inference, which would certainly have had a charm for him.

Since (by the form of Θ when resolved into factors)

$$\Theta(q^n,\, nx) = \text{const.} \times \Theta(q,\, x).\Theta\left(q,\, x + \frac{\pi}{n}\right).\Theta\left(q,\, x + \frac{2\pi}{n}\right)\ldots\ldots$$
$$\Theta\left(q,\, x + \frac{n-1}{n}\,\pi\right):$$

differentiate logarithmically, observing that

$$d\log\Theta = dV = GdF,$$

and putting h the same function of nx as c is of x. Therefore

$$nF_k.G(q^n, nx)$$

$$= F_c\left\{G(q, x) + G\left(q, x+\frac{\pi}{n}\right) + \ldots + G\left(q, x+\frac{n-1}{n}\pi\right)\right\}\ldots(73),$$

in which we write after G the elements qx instead of $c\omega$, but meaning the same quantity.

A process which is laborious in Legendre becomes easy by aid of (72), (59), (61).

When $\sin\omega = \sqrt{-1}\,\tan\theta$, and $F(c\omega) = \sqrt{-1}.F(b\theta)$,

$$x = \tfrac{1}{2}\pi.\frac{F(c\omega)}{C} = \sqrt{-1}.\frac{B}{C}.\frac{F(b\theta)}{B} = \sqrt{-1}.at.$$

Call $T(qt)$ the value assumed by $\Theta(q, x)$; that is,

$$\Theta(qx) = 1 - 2q\,\mathrm{Cos}\,2at + 2q^{2.2}\,\mathrm{Cos}\,4at - \&c.\ldots = T(q, t),$$

$$\therefore \log\beta T(qt) = V(c\omega) = W(b\theta) + \log\cos\theta, \text{ by (61)},$$

$$= V(b\theta) + \tfrac{1}{4}\pi.\frac{F^2(b\theta)}{BC} + \log\cos\theta, \text{ by (59)},$$

$$= \log\gamma\Theta(rt) + \tfrac{1}{4}\pi a t^2 + \log\cos\theta,$$

according to the meaning of $\overset{2}{\Omega}$ in equation (26),

$$\tan\overset{2}{\Omega} = \frac{2q\sin 2x.\mathrm{Sin}\,2at - 2q^{2.2}\sin 4x.\mathrm{Sin}\,4at + 2q^{3.3}\sin 6x\,\mathrm{Sin}\,6at - \&c.}{1 - 2q\cos 2x.\mathrm{Cos}\,2at + 2q^{2.2}\cos 4x.\mathrm{Cos}\,4at - \&c. \ldots}.$$

But it suffices to mention the fact. Dr. Gudermann has several other elegant approximations to $\overset{2}{\Omega}$ by this higher theory.

I have thought that we might call the function $\sqrt{\pm}\,T\{\Pi - F\}$ the *Principal Compound* of the Third Elliptic Species, and $\sqrt{\pm}\,TP$ its *Fluctuant:* also G the Fluctuant of E, H its Companion; Υ the First Integral of the Second Order, V its Fluctuant, and W the Companion of V. To avoid a perpetual appropriation of capital letters, some such notation as flE for G, $fl\Pi$ for P, &c. may sometimes be advisable.

XX. Lastly, it may be worth while to touch on a neglected side of the subject; but, as it involves no difficulty of principle, and possibly is more curious than useful, I may be brief.

In fixing the standard forms of F, E, Π, two arbitrary limitations are introduced,—to use circular sines, and not hyperbolic; and, to make the multiplier a negative proper fraction $(-c^2)$. Imaginary Amplitudes overthrow the former limitation, imaginary Moduli the other. But to change $-c^2$ into $+c^2$ involves nothing imaginary, nor indeed, within certain limits, to change b^2 into b^{-2}, which makes $c^2 > 1$.

For a moment put $z = \tan\omega$, and let $\psi(bz)$, $\chi(bz)$ denote what $F(c\omega)$, $E(c\omega)$ become; or

$$\psi(bz) = \int_0 \frac{dz}{\sqrt{(1+z^2)}\sqrt{(1+b^2z^2)}}, \quad \chi(bz) = \int_0 \sqrt{\left(\frac{1+b^2z^2}{1+z^2}\right)}\cdot\frac{dz}{1+z^2}.$$

Mere inspection of these shews that

$$d\psi(b^{-1}, z^{-1}) = -\,b.\psi(bz) \text{ and } d\chi(b^{-1}, z^{-1}) = -\,b^{-1}.\chi(bz).$$

To change z into z^{-1} changes ω into $\frac{1}{2}\pi - \omega$; and if $c = \sin\gamma$, to change b into b^{-1}, or $\cos\gamma$ into $\sec\gamma$, is equivalent to changing $\sin\gamma$ into $\sqrt{-1}\tan\gamma$. Hence, integrating the two last, we find, if $\omega + \theta = \frac{1}{2}\pi$,

$$\left.\begin{array}{r} b^{-1}F(\sqrt{-1}\tan\gamma, \theta) + F(\sin\gamma, \omega) = F_c \\ bE(\sqrt{-1}\tan\gamma, \theta) + E(\sin\gamma, \omega) = E_c \end{array}\right\} \ldots\ldots(76).$$

It is easy then to transform the developments which express F_cF_b and E_cE_b in terms of c^2, into others in which $-c^2b^{-2}$ stands for $+c^2$. For

$$b^{-1}F(\sqrt{-1}\tan\gamma, \tfrac{1}{2}\pi) = F_c \text{ and } bE(\sqrt{-1}\tan\gamma, \tfrac{1}{2}\pi) = E_c \ldots(76a).$$

It farther follows that

$$bG(\sqrt{-1}\tan\gamma, \theta) = b.E(\sqrt{-1}\ \tan\gamma, \theta) - b^{-1}\frac{E_c}{F_c}F(\sqrt{-1}\tan\gamma, \theta)$$

$$= \{E_c - E(\sin\gamma, \omega)\} - \frac{E_c}{F_c}\{F_c - F(\sin\gamma, \omega)\}$$

$$= - G(\sin\gamma, \omega) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (77).$$

Hence also, if A stands for $F_c E - E_c F$, we obtain

$$A(\sqrt{-1}\tan\gamma, \theta) = - A(\sin\gamma, \omega) \ldots\ldots\ldots (78).$$

Moreover, since $V = \int_0 G dF$,

$$V(\sqrt{-1}\tan\gamma, \theta) = \int - b^{-1}G(c\omega) \times - b dF(c\omega) = \int dV(c\omega)$$

$$= V(c\omega) - V_c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (79).$$

When $\omega = 0$, $\qquad V(\sqrt{-1}\tan\gamma, \tfrac{1}{2}\pi) = - V_c$;

which is evidently correct: for since $V_c = \frac{1}{2}\log b^{-1}$, it does but change the sign, if we commute b with b^{-1}.

For (79) we may thus also write

$$V(\sqrt{-1}\tan\gamma, \theta) = V(\sin\gamma, \omega) + \tfrac{1}{2}\log\cos\gamma \ldots (79^*).$$

It might at first seem possible to reduce the circular Π to integrals V which have imaginary moduli; but these equations shew that such integrals fall back into the common form. Moreover, if in Π the modulus is imaginary, the

Observe that $\dfrac{\Delta(c\eta^\circ)}{\tan\eta^\circ} = \sqrt{-T}(-c^2\sin^2\eta^\circ)$; then we obtain

$$\sqrt{-T}\Pi(\tan^2\gamma\cos^2\eta, \sqrt{-1}\tan\gamma, \theta) + \sqrt{-T}\Pi(-\sin^2\gamma\sin^2\eta^\circ, \sin\gamma, \omega)$$
$$= \sqrt{-T}\Pi(-\sin^2\gamma\sin^2\eta^\circ, \sin\gamma, \tfrac{1}{2}\pi)$$
$$= \sqrt{-T}\Pi_c(-c^2\sin^2\eta^\circ)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (80),$$

where the integrals are all logarithmic, and $\sin\gamma = c$, connecting η and η°.

To transform them into circulars, let

$$\sin\eta^\circ = \sqrt{-1}.\tan\delta_\circ,\ \cos^2\eta = \frac{b^2\sin^2\eta^\circ}{\Delta^2(c\eta^\circ)},$$

$$\text{or}\ \tan^2\gamma\cos^2\eta = \frac{c^2\sin^2\eta^\circ}{1-c^2\sin^2\eta^\circ} = \frac{-c^2\tan^2\delta^\circ}{1+c^2\tan^2\delta_\circ} = \frac{-\cot^2\delta}{1+\cot^2\delta} = -\cos^2\delta.$$

Hence $\sqrt{T}\Pi(-\cos^2\delta, \sqrt{-1}\tan\gamma, \theta) + \sqrt{T}\Pi(\cot^2\delta, \sin\gamma, \omega)$

$$= \sqrt{T}\Pi_c(\cot^2\delta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (81).$$

In all of these, from (76) to (81), we suppose $\omega + \theta = \frac{1}{2}\pi$.

ON TWO NEW METHODS OF DEFINING CURVES OF THE SECOND ORDER, TOGETHER WITH NEW PROPERTIES OF THE SAME DEDUCIBLE THEREFROM.

By PROFESSOR STEINER.*

(*Extract from a paper read before the Berlin Academy of Science, March* 1852.)

Section I.

THE two following methods of generating the Conic Sections are in a measure analogous to, and indeed embrace, the two known methods of generation by means of the two foci, or one focus and the corresponding directrix. The first method consists in making the sum or difference of the lengths of two tangents from the generating point to two given fixed circles, equal to a given constant, instead of, as before, considering the sum or difference of the distances from the said point to the two foci themselves as given and constant. In the second method here employed, the simple directrix is replaced by any number of given

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* Translated by Dr. T. A. Hirst.

right lines, perpendiculars are then let fall from the generating point on each of these lines, and a certain constant relation established between these, the distance of the generating point from the focus, and the perpendiculars from the latter upon the above-named right lines. The two consequent theorems may be thus expressed:

I. "*If in a plane any two circles A^2, B^2* be given, and to them, from a point X_0, tangents α, β be drawn, and it be required, that either the sum* $(\alpha+\beta)$ *or the difference* $(\alpha-\beta$ *or* $\beta-\alpha)$, *of these tangents shall equal a given length l, the locus of the point X_0 will always be a certain conic section C^2, making a double contact (real or imaginary) with the two circles, and of whose axes, one always coincides with the central line AB of the circles.*" And conversely, "*If in a given conic section C^2 any two circles A^2 and B^2 be described, making with it a double contact, aud having their centres A and B situated in the same axis, then the tangents α, β drawn from any point X_0 of the conic section to these circles, will have a sum or difference equal to a constant length l; in fact, both these cases generally present themselves, for the arc of the conic section is divided by its points of contact with the two circles into four parts, for two of these the sum* $(\alpha+\beta=l)$, *for the other two the difference* $(\alpha-\beta=l$ *or* $\beta-\alpha=l)$ *is constant.*"

II. "*If in a plane any n right lines G_1, G_2, G_3 ...... G_n together with any point A be given, and the perpendiculars*

to the focus A, depends only on the n coefficients and the fixed elements, so that when, successively, to the length a all values from 0 *to* ∞ *are given, a group of conic sections is generated, whose constituents have in common the focus A and corresponding directrix G, and in which the ratio of the above radius of curvature to the corresponding length a is constant,* i.e.

$$\frac{r}{a} = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n."$$

If in Theorem I. the given circles A^2, B^2 are reduced to their central points A and B; and if in Theorem II. all right lines except one are disregarded, the two well-known theorems mentioned at the commencement will be obtained.

Section II.

With respect to Theorem II., I will at present merely glance briefly at one circumstance, and then subject Theorem I. to a more complete discussion.

The directrix G, inasmuch as in a certain sense it is an axis of *mean* distance in reference to the n given lines, their corresponding coefficients, and the point A, can be thus determined: If a_0 and x_0 are the perpendiculars let fall from the points A and X upon the directrix, then, for every point X in the plane,

$$\alpha_1 \frac{x_1}{a_1} + \alpha_2 \frac{x_2}{a_2} + \alpha_3 \frac{x_3}{a_3} + \ldots + \alpha_n \frac{x_n}{a_n} = (\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n) \frac{x_0}{a_0}.$$

The directrix G, however, is not thus absolutely determined. For as, in reference to each of the n given lines, the signs + and − have to be attributed to opposite sides, and as these signs can be changed at will, many different directrices and corresponding conic sections, in general 2^{n-1}, ensue from these interchanges in the same given elements (*i.e.* in the same n lines G_1, G_2, G_3 ...... G_n, the same n coefficients $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$, the same point A and the same length a).

For example, when only two lines, G_1 and G_2, are given, there are two different directrices, G and H, possible; both these pass through the intersection point of the given lines and are conjugate harmonical to them, &c. All further development of this subject I here pass over.

Section III.

I. In examining the first theorem (§ *I.* I.) more closely, we will commence with the special case, when the given circles A^2 and B^2 lie without each other, as the circles

$Ua_0U_1a_1$, and $Vb_0V_1b_1$ (see fig. 1, plate 1) described around the centres A and B, and on the segments UU_1 and VV_1 as diameters.

It is necessary to fix the following elements more closely in view, as well as to direct our attention to certain collateral circumstances.

Let a, b represent the length of the radii of the circles A^2, B^2; $2C$ the distance AB between their centres; let this distance AB be bisected in point M, i.e. $MA = MB = C$. We will call the unlimited right line $UABN$, the axis X; U and U_1, V and V_1 are the extremities of the diameters of the given circles in this axis. Further, let v and v_1 be respectively the lengths of the tangents from points V and V_1 to the circle A^2, and similarly u and u_1, the two tangents from points U and U_1 to circle B^2. Let us suppose $a > b$, then, of the four tangents, u is the greatest and u_1 the least, their order being $u > v_1 > v > u_1$. Let the line L be the so-called Line of equal Powers for the two given circles; *i.e.* the locus of the point from which two equal tangents α, β can be drawn to the two circles, so that $\alpha = \beta$ or $\alpha - \beta = 0$. Further, let R and R_1 be the outer common tangents of the two circles, and a_0 and b_0, a_1 and b_1 their points of contact; their intersection point x_0 is the outer point of similitude of the two circles.

Similarly, let S and S_1 be the inner common tangents of the two circles, α_0 and β_0, α_1 and β_1 being their points

Similarly, the four points of contact α_0, β_0, α_1, β_1 of the inner tangents S, S_1 lie in another circle M^2, concentric with last; and in like manner, the four points y_0, y_1, z_0, z_1 of alternate intersection, *i.e.* of an outer with an inner tangent, together with the centres A, B of the given circles, lie all in a third circle M_0^2, having the same centre M, and, as is evident, a radius $MA = C$. The perpendiculars let fall from the point M upon the tangents R, R_1, S, S_1, meet the same in m, m_1, μ, μ_1, respectively; these points therefore are in the line L.

Lastly, let the distance x_0x_1 between the points of similitude x_0 and x_1, be bisected in N. The circle N^2, described around N as centre with radius $Nx_0 = Nx_1 = n$, is called the circle of similitude of the given circles A^2 and B^2.

II. If, successively, to the defining length l all values from 0 to ∞ be given, the complete throng of locus-curves C^2, or $Th(C^2)$,* involved in the above theorem (§ *I.* I.) will be generated. However these curves may cover the plane, it is evident that only two of the same can pass through any point X_0 in the plane, for if α, β be the tangents from X_0 to A^2, B^2; then, for the one curve, we have $l = \alpha + \beta$, and for the other $l = \alpha - \beta$ or $= \beta - \alpha$. As will soon be shewn, it is easy to distinguish for any given length l, whether the corresponding locus-curve C^2 be an ellipse E^2, a hyperbola H^2, or a parabola P^2, as also to determine its more precise relation to the circles A^2 and B^2. The curve C^2 is an H^2, or an E^2, according as $l < AB$, or $l > AB$; from the value $l = AB = 2C$ the only parabola P^2 is obtained. With respect to their relations towards the circles A^2, B^2, the hyperbolas may be divided into three groups, and represented by $Gr(H_1^2)$, $Gr(H_2^2)$, $Gr(H_3^2)$; of these, as well as of the group of ellipses $Gr(E^2)$, the following properties must be noticed.

(1). The first group of hyperbolas $Gr(H_1^2)$ corresponds to values of l between $l = 0$ and $l = \alpha_0\beta_0$ (I.); it commences (when $l = 0$) with the line L,—which is to be considered as double, and as a hyperbola, both branches of which coincide with the second axis,—and concludes with the pair of inner

* In translating I have here, as well as in § *IX.*, had to modify the symbols used by Professor Steiner. For a throng of curves $Th(C^2)$, Prof. S. uses the symbol $S(C^2)$, (*Schaar Curven*); in § *IX.* for a pencil of curves $Pn(C^2)$, he uses the symbol $B(C^2)$ (*Curven-Büschel*). The term *Büschel* as applied to the latter implies merely that the curves constituting the *schaar* have, in this case, more properties in common with each than before.

tangents (SS_1), when $l = \alpha_0\beta_0 = \alpha_1\beta_1$. The principal axis of every H_1^2 coincides with the axis X, and of its two branches, the one encloses the circle A^2, the other the circle B^2; at commencement, however, the contact with both circles is imaginary, until $l = u_1$ (I.), when the greater circle A^2 is touched in point U_1; this is a contact in four successive points, so that A^2 is the circle of curvature in the vertex U_1; from here onwards the H_1^2 touch the circle A^2 in two real points, but the contact with circle B^2 remains still imaginary until $l = v$, when B^2 becomes its circle of curvature in vertex V, from here on the H_1^2 give real contacts with both circles, until they reach the limit (SS_1). Consequently the points of real contact with all H_1^2 are situated along the circular arcs $\alpha_0 U_1 \alpha_1$ and $\beta_0 V \beta_1$.

(2). The second group of hyperbolas $Gr(H_2^2)$ corresponds to values of l between $l = \alpha_0\beta_0$ and $l = a_0 b_0$; it begins with the pair of inner tangents (SS_1), and ends with the outer pair (RR_1). The second axis of every H_2^2 coincides with the axis X, and each of the two branches touches both circles outwardly; all four points of contact are real throughout, and are situated in the two pairs of circular arcs $a_0\alpha_0$ and $a_1\alpha_1$, $b_0\beta_1$ and $b_1\beta_0$.

(3). The third group of hyperbolas $Gr(H_3^2)$ corresponding to values of l between $l = a_0 b_0$ and $l = AB$, commences with the outer tangents (RR_1), and concludes with the

As, by the continuous increase of the length l, the several groups of locus-curves are successively traversed, the centre C of the curve C^2 moves along the axis X in unchanged direction, and in so doing, the centres of the several groups pass over the following segments of the axis X. In the $Gr(H_1^2)$, the centre C moves from m_0 to x_1; in the $Gr(H_2^2)$, from x_1 to x_0; in the $Gr(H_3^2)$, C moves in the same direction as before on to infinity, up to the centre C_∞ of the parabola P^2; and lastly, in the $Gr(E^2)$, C comes from infinity, from C_∞, to U, A... until at length it returns to M, exactly the centre of the last ellipse E_∞, situated entirely in infinity, and corresponding to the value $l=\infty$. Accordingly, it would appear that the centre C traverses the whole axis X, with the exception of segment Mm_0; the centres of imaginary locus-curves are situated in this segment.

Any length l being given, the points of contact of the corresponding curve C^2 with the given circles A^2 and B^2 are easily constructed. For example, to find the point of contact with the circle A^2, set off the length $b_0b_0'=l$ on any tangent R of circle B^2, from the point of contact b_0; then the auxiliary circle B_0^2, described around B as centre, with radius Bb_0', will intersect circle A^2 in the required points of contact, and should it give no real points of intersection with A^2, the contact is imaginary; in this case however the line of equal powers of circles A^2 and B^2, *i.e.* their ideal common secant, is at the same time the ideal chord of contact of C^2 and A^2. In the same manner the points of contact of C^2 with B^2 may be found. By this method, for instance, the points of contact α and α', β and β' of the parabola P^2 may be easily constructed, making $b_0b_0'=AB$. The points of contact of each of the two curves C^2 which pass through any given point X_0, may with facility be obtained in an exactly similar manner, &c.

III. The foci of the $Th(C^2)$ have a noteworthy situation, and are, in their totality, subject to an interesting law.

The circle of similitude N^2 is the locus of the foci of the second group of hyperbolas $Gr(H_2^2)$, so that the extremities of every chord of this circle, perpendicular to the diameter x_0x_1, are at the same time the foci of an H_2^2.

On the other hand, the foci of every other locus-curve are in the axis X, but so situated that the two foci of each C^2 are conjugate harmonical to the points of similitude x_0 and

Accordingly, the centre N of the circle of similitu

be the focus of the parabola P^2, because its conjugate harmonical point in reference to x_0 and x_1 is in infinity. The centres A, B of the given circles are the foci of the above-mentioned special ellipse E^2_∞, for these points are conjugate harmonical to x_0 and x_1 and are equidistant from the centre M of E^2_∞.* Further, this theorem determines also the foci of the first special hyperbola H_1^2, consisting of the double line L(II. 1), for as m_0 is evidently its centre, y and z must be regarded as its foci, for they are two conjugate harmonical points to x_0 and x_1, and are equidistant from m_0, $ym_0 = zm_0$ (I.).

Accordingly the foci of all locus curves fulfil the following common condition.

"*The rectangle under the distances of the two foci* (f *and* f_1) *of each locus-curve* C^2 *from the point* N (*the focus of the parabola* P^2 *or centre of circle of similitude* N^2) *is constant and* $= n^2$, i.e. *equal to the square of the radius of the circle of similitude; hence throughout we have* $fN.f_1N = n^2$."

From this, if the centre C of any locus-curve C^2 be given, its foci f and f_1 are easily determined and constructed. For if C be in the diameter x_0x_1, then, as already stated, the extremities of the chord of circle N^2, erected in C perpendicular to diameter x_0x_1, are the required foci. On the other hand, if C be situated in the production of the diameter on either side, then the tangent, drawn from C to the circle N^2, is equal to the eccentricity of the curve C^2, so

is defined by the proportion

$$l : AB = \alpha : \gamma.$$

IV. Let the points in which any locus-curve C^2 touches the circles A^2 and B^2 be respectively p and p_1, q and q_1.

"*The chords of contact pp_1 and qq_1 are parallel to the line L, are always equidistant from, and like it, perpendicular to the axis X, (this too holds true when the contact is imaginary and the chords are ideal).*"* And conversely, "*Every two lines equidistant from, and parallel to L, are the chords of contact of some locus-curve C^2 with the circles A^2 and B^2.*" Further, "*The tangents β drawn from points p and p_1 to circle B^2 are each equal to the tangents α drawn from q and q_1 to circle A^2, and both lines are equal to the length l corresponding to the curve C^2.*"

"*The four points of contact p, p_1, q, q_1 are always in the circumference of a circle M^2, whose centre is the point M.*" And conversely, "*Every circle M^2, described around M as centre, intersects the given circles A^2 and B^2 in four points, which will be the points of contact of some locus-curve C^2 with them.*"

"*The eight points, in which the given circles are touched by any two locus-curves, lie in a certain third conic section D^2.*" For example, the eight points of contact a_0, a_1, b_0, b_1, and $\alpha_0, \alpha_1, \beta_0, \beta_1$, of the two pairs of common tangents R, R_1 and S, S_1, lie in a certain conic section D^2. And conversely, "*If through the four points of contact p, p_1, q, q_1 of any curve C^2, any conic section D^2 be drawn, it will intersect the circles A^2 and B^2 in four new points p° and p_1°, q° and q_1°, which will likewise be the points of contact of some other locus-curve C_1^2 with the circles.*"

"*The four points in which any two locus-curves C^2 and C_1^2 intersect one another, lie always in a certain circle M^2 around M as centre.*" And conversely, "*Every circle M^2 around M as centre, cuts any locus-curve C^2 in four points, which are at the same time the points wherein some other curve C_1^2 also intersects the same curve C^2.*"

Hence, as a special case, every curve C^2 intersects the tangents R and R_1 in four points, which lie in a certain circle M^2; and further, these tangents determine equal chords in the curve C^2: the same applies also to the inner tangents S and S_1, indeed further:

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* This theorem is also to be found in the memoir above referred to (*Crelle's Journal*, vol. XXXVII. p. 176).

"*The four tangents R, R_1, S, S_1 determine in every locus-curve C^2 four equal chords, and in fact, these chords are each equal to the corresponding length l, and are all bisected by the line L in the points m, m_1, μ, μ_1.*" Consequently, any length l being given, the eight points in which the corresponding locus-curve C^2 intersects the four tangents R, R_1, S, S_1, may be easily found.

If the two pairs of points of contact, p and p_1, q and q_1, of any locus-curve C^2 be joined cross-wise by the right lines pq, pq_1, p_1q, p_1q_1, which may be called *Alternate Chords*, then all alternate chords have the following common property.

Every alternate chord determines, in the given circles, equal chords; i.e. *if, for example, the line pq intersects the circles A^2 and B^2 a second time in the points p° and q°, then chord $pp^\circ = qq^\circ$ for all positions of p and q.* Further,

"*The line L is the locus of the middle m° of every alternate chord, and the perpendicular let fall from M upon any chord meets it in this same bisection point m°. Hence, all alternate chords are tangential to a certain parabola $P_\circ^2$, of which M is the focus, and L the tangent in vertex $m_\circ$ of its axis, and which, in particular, has the four tangents R, R_1, S, S_1 in common with the circles A^2 and B^2* (for these are but special alternate chords)."

Let P and P_1, Q and Q_1 be the common tangents of circles A^2 and B^2 and curve C^2; *i.e.* the tangents which,

certain points t° and t_1°; in another of their positions they will coincide with the inner common tangents of the same two circles, and will then touch N^2 in certain points s° and s_1°; *in each of these cases the chord of contact $t^{\circ}t_1^{\circ}$ or $s^{\circ}s_1^{\circ}$ touches the circle B^2 in points V_1 or V*, for both tangents Q and Q_1 will simultaneously coincide with the corresponding chord, and the curve C^2 will make a contact in four consecutive points with circle B^2 in the respective point V_1 or V. (II.). Conversely, "*If to any two circles A^2 and N^2, situated without each other, the two pairs of common tangents be constructed, and in either circle,* e.g. N^2, *the two chords of contact, $t^{\circ}t_1^{\circ}$ and $s^{\circ}s_1^{\circ}$, of the pair of tangents be drawn; afterwards, on the segment V_1V which these chords of contact determine on the axis* X, *a third circle B^2 be described, then the circle N^2 will be the circle of similitude to the circles A^2 and B^2.*"

Section IV.

If the given circles A^2 and B^2 intersect each other, or the one be situated entirely within the other, the properties established in § *III.* suffer some modification, or new circumstances present themselves, to embrace which, the conditions (§ *I.* I.) for the generating point X_0 must be more generally defined. This more general definition results from a consideration of the powers of the point X_0 in reference to the circles.* As the power of a point X_0 in reference to a circle A^2 can be an outer or an inner one, and as such, represented either by the square of the tangent α from it to the circle, or by the square of half the smallest chord α_1 through it, according as the point is situated *without* or *within* the circle; so also, when two circles A^2 and B^2 are given, the locus of point X_0 may be required, for which the sum $\alpha_1 + \beta_1$, or the difference, $\alpha_1 - \beta_1$ or $\beta_1 - \alpha_1$, of half the smallest chords which can be drawn through it in the given circles, shall equal a given length l. This condition can, however, be combined with the above, with respect to the tangents α and β, in the more embracing problem—

"*To find the locus of the point X_0, for which the* (*square*) *roots of the like powers in reference to the given circles, A^2 and B^2, shall have either a sum or a difference equal to a given line l.*"

If, further, the loci for inner and outer powers, considered separately, correspond to the same value l, it is true that by each method a different conic section will be produced,

* See *Crelle's Journal*, vol. I. p. 163.

yet the two will have a certain connexion, and may be considered as naturally complementary to each other. In a similar manner the locus of point Y_0 may also be required, for which the sum or difference of the *unlike-named* powers (*i.e.* the tangent to one circle, and half the smallest chord in the other,) shall be equal to a given length l. In this case, however, the required locus is in general a curve of the fourth degree.

With reference to the above, the foregoing properties suffer the following modifications, when the mutual position of the given circles undergoes the above-described changes.

Section V.

I. If we allow the two circles A^2 and B^2 (fig. 1, plate I.) to approach each other until, meeting in points U_1 and V, they there touch each other in one point $(U_1 V)$; then the two inner tangents S and S_1 will coincide with the line L, which will then be the common tangent of the circles in the point $(U_1 V)$, with which point, too, the inner point of similitude x_1, together with many others, will now coincide. Consequently, the first group of hyperbolas $Gr(H_1^2)$(§ *III.* II.) vanishes, inasmuch as its last constituent (SS_1) combines with its first L, or in other words, is reduced to the single constituent l, with which at the same time the second group $Gr(H_2^2)$ commences. This group, as before, ends with the pair of outer tangents (RR_1), and it, together with all other groups, remains the same as before.

hand, its two infinitely long segments beyond r and s, as the two branches of an H_2^2, for both correspond to the same value of l, viz. $l = 0$, or respectively $\alpha_1 = \beta_1$ and $\alpha = \beta$; in both, therefore, the two points r and s are to be considered as, at the same time, the foci and principal vertices. Consequently, the loci of the foci of both groups are in intimate connexion with each other, just as the foci of the $Gr(H_2^2)$ are all contained in the arc rx_0s, so also are the foci of the $Gr(E_1^2)$ in the arc rx_1s, of the circle of similitude N^2; hence the extremities of every chord of the arc rx_1s, perpendicular to the segment m_0x_1, are at the same time the foci of an E_1^2. According to this, the centres of the $Gr(E_1^2)$ are all contained in the segment m_0x_1. If the length l be allowed to increase from $l = 0$, the centre E_1 of the corresponding locus-curve E_1^2 moves from m_0 to x_1; here $l (= \alpha_1 + \beta_1)$ attains a certain maximum limit, and the curve is reduced to its centre x_1. In this case, that is, when the locus of point X_0 is restricted to a single point x_1, the said maximum is represented by half the smallest chords passing through x_1, both of which are contained in the line $\alpha_0x_1\beta_0$, perpendicular to the axis x, so that $\alpha_0x_1 + \beta_0x_1 = \alpha_0\beta_0$ is the maximum limit of l. Hence, "*Among all the points within both circles A^2 and B^2, the inner point of similitude x_1 has the property of making the sum of the smallest chords through it a maximum.*"

When the length l is given, the points p and p_1, q and q_1, in which any inner locus-curve E_1^2 touches the circles A^2 and B^2, can be constructed in an analogous manner to the one before mentioned (§ *III.* II.). If, for instance, in the circle B^2, a chord of the length $2l$ be drawn, and bisected in m, then the circle described around B as centre, with radius Bm, will cut the circle A^2 in the required points of contact p and p_1. Further, the limits, where real contacts cease, can be determined in an analogous manner. If v be half the smallest chord in circle A^2 through point V, and u_1 half the smallest chord in circle B^2 through point U_1, then the contacts of curve E_1^2 with circles A^2 and B^2 are only real as long as l remains, respectively, smaller than u_1 and v; if $l = u_1$ or $l = v$, the contact with one of the circles in U_1 or V will be in four consecutive points; hence it will be the circle of curvature to the respective locus-curve. If radius $a > b$, then $v > u_1$, and the imaginary contacts with circle A^2 begin earlier than those with B^2. Conceive a curve E_1^2, which touches the circles A^2 and B^2 in real points p and p_1, q and q_1, it will then be evident, that for all points X_0 in the

elliptic arcs situated between the points of contact of different circles, *i.e.* in the arcs pq and p_1q_1, the sum $\alpha_1 + \beta_1 = l$. On the contrary, for points in the elliptic arcs pp_1 and qq_1, between the points of contact of the same circle, the differences $\beta_1 - \alpha_1$ and $\alpha_1 - \beta_1$ will be respectively equal to the constant length l. When the points p and p_1 are imaginary, *i.e.* if $l > u$, but $< v$, then E_1^2 will be divided by points q and q_1 into two arcs, of which the one situated nearest to point U_1 corresponds to the sum $\alpha_1 + \beta_1$; on the contrary, the one nearest point V corresponds to the difference $\alpha_1 - \beta_1$. If all four contacts are imaginary, then for all points X_0 in E_1^2, the sum $\alpha_1 + \beta_1 = l$. Similar remarks might have been made above (§ *III.* II.), with reference to the $Gr(E^2)$, and the unlike properties of the arcs of the several groups of hyperbolas in this respect may, with facility, be more particularly defined.

III. If the centres of the circles A^2 and B^2 approach each other still more, so that points V_1 and U_1 coincide, and the circles touch each other, but only in one point (V_1U_1), the two outer common tangents will likewise coincide with line L, which will be a tangent to both circles in point (U_1V_1): it is also to be considered as the last remnant of the second group of hyperbolas $Gr(H_2^2)$, now also vanished, and at the same time as the commencing constituent of the third group $Gr(H_3^2)$. The outer point of similitude x_0, together with points of intersection r and s, now coincide with point (U_1V_1), so that the circle of similitude N^2 touches the

bola P^2; at the same time, the latter is the commencement of $Gr(E^2)$ which ends, as before, with E^2_∞(§ 3, II.). On the other hand, with respect to the interior locus-curves, the $Gr(E_1^2)$ begins with the outer point of similitude x_0, and in fact with a value of l corresponding to the minimum of the difference $\alpha_1 - \beta_1$. This follows from the following theorem: "*Among all points* X_0 *within the circle* B^2, *the outer point of similitude* x_0 *has the property of making the difference of the smallest chords,* $2\alpha_1$, $2\beta_1$, *and through it, a minimum.*" The line $\alpha_0\beta_0x_0$ perpendicular to the axis X in point x_0, contains these two particular chords, so that $x_0\alpha^0 - x_0\beta^0 = \alpha^0\beta^0$ is exactly the value of l, for which the first E_1^2 is reduced to the point of similitude x_0. Similarly, the last constituent of $Gr(E_1^2)$ is reduced to the inner point of similitude x_1, and corresponds to that value of l which is the maximum of the sum $\alpha_1 + \beta_1$, and, as before, (II.) is represented by $x_1\alpha_1^{\circ} + x_1\beta_1^{\circ} = \alpha_1^{\circ}\beta_1^{\circ}$, in the line $\alpha_1^{\circ}x_1\beta_1^{\circ}$ perpendicular to axis X in the point x_1. In the $Gr(E_1^2)$, therefore, the length l varies between the limits $l = \alpha^0\beta^0$ and $l = \alpha_1^{\circ}\beta_1^{\circ}$.

In the present situation, the circle A^2 forms real contacts with the exterior locus-curves $Gr(H_3^2)$ and $Gr(E^2)$ alone; on the contrary, the circle B^2 only with the interior $Gr(E_1^2)$. The limits where, in both cases, the real contacts begin and end, may be determined in the above-mentioned manner; and similarly, a length l being given, the points of contact can be easily constructed by the method already explained. A special circumstance with respect to the exterior locus-curves, shall here be more particularly considered.

Whether any of the $Gr(H_3^2)$ attain to real contact with the circle A^2 or not, depends upon whether $u_1 < AB$ or $u_1 > AB$; *i.e.* whether the tangent u_1(§ *III.* I.), from the point U_1 (which, of all points in A^2, is nearest the circle B^2) to the circle B^2, is less or greater than AB. If $u_1 = AB$, the last constituent alone of the $Gr(H_3^2)$, *i.e.* the parabola P^2, makes in U a real contact in four points with the circle A^2. If, on the other hand, $u > AB$, then, after P^2, follow a certain number of ellipses in the $Gr(E^2)$, which form no real contact; these, for distinction, we will represent by $Gr(E^2_-)$. For all points X_0 in such an ellipse E^2_-, the difference only, $\beta - \alpha = l$, (the same remark applies also, in this case, to every H_3^2). The $Gr(E^2_-)$ corresponds to values between $l = AB$ and $l = u_1$. When $l = u_1$ the corresponding ellipse makes a contact in four points with circle A^2 in U_1, and, for its whole circumference, the difference $\beta - \alpha$ yet equals l; but at the

same time it is the commencement of the group of ellipses with real points of contact. From here on, as l increases, the E^2 touch the circle A^2 each in two real points p and p_1, by which their arcs are divided each in two parts; of these, the one which overspans U_1 corresponds to the sum $\alpha+\beta$, and the other over U, to the difference $\beta-\alpha$. The value $l=u$ (tangent from U to B^2) corresponds to the last E^2 whose contact with A^2 is real; in u that contact is one in four consecutive points, and for all points in it, the sum alone $\alpha+\beta=l$. From here on, by the increase of l up to $l=\infty$, a new section of ellipses are generated in the $Gr(E^2)$, which we may represent by $Gr(E^2_+)$; they also form no real contact with circle A^2, but for every point in their peripheries the sum $\alpha+\beta$ alone is made equal to the constant length l. Consequently, under the supposition that $u_1 > AB$, the $Gr(E^2)$ contains two distinct sub-groups, $Gr(E^2_-)$ and $Gr(E^2_+)$, both of which enclose, and give imaginary contacts with circle A^2, but are nevertheless essentially different from each other, inasmuch as the points of first make only the difference $\beta-\alpha$ constant, whilst those of the second make only the sum $\alpha+\beta$ constant. These different properties will be explained by the following nearer relation of the two circles to the respective curves. Let, generally, f and f_1 be the foci of an ellipse, and k and k_1 the centres of curvature of the vertices in its major axis; the two last points are situated between the two first, let k be nearest f, and k_1 nearest f_1.

ellipse is imaginary, both their centres being situated in the same segment fk or f_1k_1; then, for all points X_0 of the ellipse, the difference $\beta - \alpha = l$ is constant, and u_1 is always greater than AB; the constant l, however, greater than AB, but smaller than u_1."

Section VI.

From the foregoing considerations it is easy to infer that if, in a plane, any three circles A^2, B^2, and D^2 be given, whose centres A, B, and D are in the same right line X, in general a certain conic section C^2 exists, which with reference to each two circles, shall be one of their corresponding locus-curves, and which, as a consequence, will form a double contact with each circle. The length l corresponding to each pair of circles can, for example, be thus determined.

Let a, b, and d be the respective radii of the circles, and the distance of their respective centres be thus represented, $AB = 2\gamma$, $AD = 2\gamma_1$, and $BD = 2\gamma_2$; and further, let the length l, corresponding to the pairs of circles A^2 and B^2, A^2 and D^2, B^2 and D^2, be respectively 2λ, $2\lambda_1$, $2\lambda_2$; then, if B lies between A and D, we have the following relations:

$$\lambda^2 = \frac{\gamma}{\gamma_1\gamma_2}(4\gamma\gamma_1\gamma_2 - \gamma_2 a^2 + \gamma_1 b^2 - \gamma d^2),$$

$$\lambda_1^2 = \frac{\gamma_1}{\gamma\gamma_2}(4\gamma\gamma_1\gamma_2 - \gamma_2 a^2 + \gamma_1 b^2 - \gamma d^2),$$

$$\lambda_2^2 = \frac{\gamma_2}{\gamma\gamma_1}(4\gamma\gamma_1\gamma_2 - \gamma_2 a^2 + \gamma_1 b^2 - \gamma d^2).$$

Section VII.

If, further, the locus of the points Y_0 were required, for which, in reference to two given circles A^2 and B^2, the roots of the unlike-named powers shall have a sum ($\alpha+\beta_1$ or $\beta+\alpha_1$), or a difference ($\alpha-\beta_1$, $\beta_1-\alpha$, or $\beta-\alpha_1$, $\alpha_1-\beta$), equal to a given length l (§ *IV.*), (whereby therefore the point Y_0 must necessarily be always within one circle and without the other), it would be found that, in general, this locus is a curve of the fourth degree $= C^4$, which touches each of the two circles in four points (real or imaginary); these are easily constructed by means of the concentric auxiliary circles (B_0^2 and A_0^2), as above explained (§ *III.* II., and § *V.* II.).

When, however, as a special case, $l = 0$,—*i.e.* if merely the locus of a point Y_0 is required, which, in reference to the

two circles, has unlike-named, but equal powers, $\alpha = \beta_1$ or $\beta = \alpha_1$,—then the curve C^4 reduces itself to a doubled circle, inasmuch as the two parts, of which it in general consists, now coincide and form a single circle C_0^2. This circle C_0^2 is also thus defined: its centre is the point M, the middle of AB; and it has, in common with the given circles, the line L as its line of equal powers. If, therefore, the given circles A^2 and B^2 cut each other, as in fig. 2, C_0^2 passes through the points of intersection r and s; if B^2 is completely within A^2, as in fig. 3, then C_0^2 is situated in the space between B^2 and A^2; and lastly, if A^2 and B^2 be without each other, as in fig. 4, but so that M falls within A^2, then the circle C^2 can yet be real, and will be entirely within A^2. From these properties the following theorem may be deduced:

"*The locus of the point having equal but unlike-named powers with respect to two given circles A^2 and B^2, is a certain third circle C_0^2, whose centre M is the middle of the line AB, joining the centres of the given circles; and, in common with these circles, C_0^2 has the line L as its line of equal (and like-named) powers.*"

The same theorem may be somewhat differently expressed thus:

"*The locus of centre Y_0 of the circle Y_0^2, which is intersected by one of the given circles A^2 or B^2 (no matter which),*

Section VIII.

The above considerations revealed the existence of an infinite throng of curves of the second order, $Th(C^2)$, which possess the property of forming two contacts with the given circles A^2 and B^2; nevertheless, all the conic sections possessing this same property are not contained therein; on the contrary, there are in general two more throngs possessing a like property. With respect to these latter, the following particulars are worthy of notice.

The given circles (indeed every two conics in the same plane) have, in common with each other, a triplet of conjugate poles x, y, and z, as well as a triplet of conjugate polars X, Y, and Z; the former are the angles, the latter their respective opposite sides, of one and the same triangle. One of these poles x is in infinity, and in fact, in the direction of line L, of which it is to be considered as the infinitely distant point; this pole is always real, whereas both the others, y and z, are simultaneously *imaginary* or *real*, according as the circles *do* or *do not* intersect each other; for they are at the same time the intersection points of the axis (or polar) X with every circle which intersects both the given circles, A^2 and B^2, perpendicularly; or, provided the circles are without each other, as in fig. 1, the poles y and z are also the respective intersection points of the diagonal $x_0x_1 = X$ with the two remaining diagonals $z_0z_1 = Z$ and $y_0y_1 = Y$, of the quadrilateral RR_1SS_1, formed by the four common tangents. The three throngs of conic sections already mentioned have the following essential relations towards these three poles.

The first locus-curves $Th(C^2)$ have reference to the pole x, and shall therefore be represented by $Th(C_x^2)$, for the chords of contact pp_1 and qq_1 of every curve C_x^2 are parallel to the line L, and hence, with it, directed towards the pole x (§ *III.* IV.). A second throng of conic sections $Th(C_y^2)$ touch each of the given circles twice, and have reference, in a similar manner, to the pole y, inasmuch as the chords of contact, pp_1 and qq_1, of every curve C_y^2 pass through this pole. Similarly, a third throng of conic sections $Th(C_z^2)$ also touch each of the given circles twice; their chords of contact, however, always pass through the pole z. The following are some of the interesting properties possessed by these two last throngs of conic sections.

(1) "*The chords of contact, pp_1 and qq_1, of every curve C_y^2, as well as of every curve C_z^2, are always perpendicular to each*

other; and conversely, *If through the pole y or z, any two secants pp_1 and qq_1 be drawn perpendicular to each other, they will intersect the two circles A^2 and B^2 respectively, in certain points p and p_1, q and q_1, which will be the points of contact of some curve C_y^2 or C_z^2."*

(2) "*Of the two axes of each curve C_y^2 or C_z^2, the one passes through centre A, the other through B. Consequently, the circle M_o^2, on AB as diameter* (§ *III.* I.), *is the locus of the centres of the $Th(C_y^2)$, as well as the $Th(C_z^2)$; so that every point of this circle is at the same time the centre of a curve C_y^2, as well as of a curve C_z^2; and hence, the axes of both these curves coincide."*

(3) "*The individual curves of the $Th(C_y^2)$, as also of the $Th(C_z^2)$, are, among themselves, similar; and, in fact, the squares of the axes of each C_y^2 have to each other the same ratio as the distances of the pole y from the centres A and B; and similarly, the squares of the axes of each C_z^2 are to each other as the segments zA and zB,* viz. *thus, if α, β be the semi-axes of a C_y^2, of which α passes through A, and β through B; then*

$$\alpha^2 : \beta^2 = yB : yA;$$

and, similarly, if α_1, β_1 be the semi-axes of a C_z^2, passing respectively through A, B; then

$$\alpha_1^2 : \beta_1^2 = Bz : Az.$$

well as from the point B, is constant, and equal to the square of the radius a_y *or* b_y *of the corresponding circle* A_y^2 *or* B_y^2; *hence*

$$Af.Af_1 = a_y^2, \text{ and } Bf.Bf_1 = b_y^2.$$

Similarly, the foci of the $Th(C_z^2)$ *are situated in two circles* A_z^2 *and* B_z^2, *which possess analogous properties.*"

(5) "*If between the two pairs of points p and* p_1, *q and* q_1, *in which each curve* C_y^2 *touches the given circles* A^2 *and* B^2, *the four alternate chords* pq, pq_1, p_1q, *and* p_1q_1 *be drawn, all such chords will be tangential to a certain conic section* Y^2, *which has the pole y as focus, and, in common with circles* A^2 *and* B^2, *the four (real or imaginary) tangents R and* R_1, *S and* S_1; *its foci y and (the yet unknown one)* y_1 *are conjugate harmonical points to A and B. Each alternate chord determines in the circles* A^2 *and* B^2 *two chords s and* s_1; *the ratio of these chords is for all alternate chords the same,* i.e. $s : s_1 = k$ *constant. Similarly, the alternate chords of the* $Th(C_z^2)$ *are all tangential to a certain conic section* Z^2, *of which pole z is a focus, and which also has, in common with circles* A^2 *and* B^2, *the same four tangents; its foci z and* z_1 *are conjugate harmonical points to A and B. Here again the alternate chords determine, in the given circles, certain chords s and* s_1, *whose ratio is constant, though different from the preceding,* e.g. $s : s_1 = k_1$ *constant.*"

(6) "*If P and* P_1, *Q and* Q_1 *be the tangents common to circles* A^2 *and* B^2, *and curve* C_y^2 (§ *III.*, IV.) *the intersection points* $PP_1 = r$ *and* $QQ_1 = r_1$ *are always situated in the polar Y, and, in their totality, the pairs r and* r_1 *form a System of Points (Involution). On the other hand, the locus of the four alternate intersections PQ and* P_1Q_1, PQ_1 *and* P_1Q, *or s and* s_1, *t and* t_1 (§ *III.* IV.), *is a certain circle* N_y^2, *which passes through the same pair of opposite corners,* y_0 *and* y_1, *as Y; the line L is common secant to it, and to the circles* A^2 *and* B^2, *so that its centre is also in the axis X. In this respect the* $Th(C_z^2)$ *also has exactly analogous properties.*"

To shew the influence of the several relative situations of the given circles on the above properties, we will examine more closely their most essential positions; that is, when they are entirely external to each other, and when B^2 is completely within A^2. The $Th(C_y^2)$ and $Th(C_z^2)$ for the intermediate position, where the circles intersect each other, are imaginary.

I. "*If the circles are without each other, as in* fig. 1, *both* $Th(C_y^2)$ *and* $Th(C_z^2)$ *consist of hyperbolas* $Th(H_y^2)$ *and* $Th(H_z^2)$; *the constituents of each throng are, among themselves, similar. The circles* A^2 *and* B_y^2, *described around the points A and B as centres, and which contain the foci of* $Th(C_y^2)$, *intersect each other perpendicularly in the opposite corners,* y_0 *and* y_1, *of the quadrilaterals* R, R_1, S, S_1; *and similarly, on the other hand, the circles* A_z^2 *and* B_z^2 *intersect each other perpendicularly in the corners* z_0 *and* z_1. *If the centre of an* H_y^2 *is in the arc* $y_0z_0Az_1y_1$ *of the circle* M_0^2, *the curve itself surrounds the circle* B^2, *and hence its principal axis passes through B and intersects the circle* A_y^2 *in the foci f and* f_1. *On the contrary, if the centre of an* H_y^2 *is in the arc* y_0By_1, *it surrounds the circle* A^2, *its principal axis passes through A, and its foci are in the circle* B_y^2. *The transition from the one section to the other takes place through the pair of tangents* (RS_1) *and* (R_1S), *which are special* H^2, *and have respectively* y_0 *and* y_1 *as centres. The hyperbolas* H_z^2 *have exactly similar properties. The asymptotes of every* H_y^2 *pass through the fixed corner points* z_0 *and* z_1; *and similarly, the asymptotes of every* H_z^2 *pass through* y_0 *and* y_1."

II. "*If circle* B^2 *is entirely within* A^2, *as in* fig. 3, *both throngs* $Th(C_y^2)$ *and* $Th(C_z^2)$ *consist of ellipses* $Th(E_y^2)$ *and* $Th(E_z^2)$. *Every* E_y^2 *encloses the circle* B^2, *and is enclosed by the circle* A^2; *hence its principal axis passes always through the point B, and its foci f and* f_1 *are always in a certain circle* A_y^2, *around A as centre; (here the circle* B_y^2 *around B will be intersected by circle* A_y^2 *in a diameter, but these same intersection points are the only real foci it contains). Similarly, every* E_z^2 *encloses circle* B^2 *and is enclosed by* A^2, *so that its principal axis passes only through B, and its foci are all contained in a certain circle* A_z^2 *around A as centre.*"

Section IX.

NOTE.—In the foregoing, three examples incidentally presented themselves, where the locus of a right line (there called alternate chord, § *III.* IV. and § *VIII.* V.), determining in the given circles A^2 and B^2 certain chords s and s_1, of constant ratio to each other, was found to be a conic section This property is a general one, and furnishes the following theorem:

"*The locus of a line G which intersects two given circles* A^2 *and* B^2, *so that the chords s and* s_1 *thereby formed have to each other a constant ratio k,* i. e. $s : s_1 = k$, *is always a certain conic*

section G^2.* *All conic sections thus generated, provided the value k assumes successively all magnitudes, form a pencil of curves* $Pn\ G^2$ *with four real or imaginary common tangents* (R, R_1, S, S_1, *and, in fact, the given circles* A^2 *and* B^2 *themselves belong to this pencil, for they correspond respectively to the values* $k=0$ *and* $k=\infty$. *As above* (§ *III.* IV.), *the parabola* $P_0^2(=G^2)$, *of which the point M is focus, and the line L tangent in vertex, corresponds to the value* $k=1$ *or* $s=s_1$. *The two points of similitude* x_1 *and* x_2 *together form a special* G^2 *corresponding to the value* $k=a:b$, &c...." And conversely, "*The tangents of every conic section* G^2, *which has four real or imaginary tangents in common with two circles* A^2 *and* B^2, *determine, in these circles, certain chords s and* s_1, *whose ratio to each other is constant*, i.e. *for all tangents this ratio has a certain value k, &c...."*

Instead of a full discussion of this theorem, the following few remarks must here suffice.

The centres of the locus-curves $Pn(G^2)$ are all in the axis X, with which, too, one axis of the curve always coincides. Whether this same axis coinciding with X be the first or the second, depends upon whether its centre is situated without or within the segment AB. Hence the curves may be divided into two groups $Gr(G_1^2)$ and $Gr(G_2^2)$. The foci of these two groups fulfil the following conditions:

"*The foci of the* $Gr(G_1^2)$ *are in the axis* X, *and each pair are conjugate harmonical points to A and B. On the other hand, the foci of the* $Gr(G_2^2)$ *are in the circle* M_0^2, *whose diameter is the segment* $AB = 2c$ (§ *III.* I.), *so that each pair of foci forms, at the same time, the extremities of a chord of the circle* M_0^2, *perpendicular to this diameter AB.*"

From this it follows, as above, (§ *III.* III. and § *VIII.* IV.), that both groups fulfil the common condition; viz.

"*The rectangle under the distances, f and* f_1, *of each curve* G^2 *from the point M, the focus of the parabola* P_0^2, *is constant and equal to* c^2."

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* As early as 1827 I forwarded this, together with several other theorems, to the Editor of the *Annales des Mathématiques*, at Montpellier; afterwards, probably by mistake, he allowed them to be published in another name.

NOTE ON THE MECHANICAL ACTION OF HEAT, AND THE SPECIFIC HEATS OF AIR.

By William Thomson.*

I. *Synthetical Investigation of the Duty of a Perfect Thermo-Dynamic Engine founded on the Expansions and Condensations of a Fluid, for which the gaseous laws hold and the ratio (k) of the specific heat under constant pressure to the specific heat in constant volume is constant; and modification of the result by the assumption of* Mayer's *hypothesis.*†

Let the source from which the heat is supplied be at the temperature S, and let T denote the temperature of the coldest body that can be obtained as a refrigerator. A cycle of the following four operations, *being reversible in every respect*, gives, according to Carnot's principle, first demonstrated for the Dynamical Theory by Clausius, the greatest possible statical mechanical effect that can be obtained in these circumstances from a quantity of heat supplied from the source.

(1) Let a quantity of air contained in a cylinder and piston, at the temperature S, be allowed to expand to any extent, and let heat be supplied to it to keep its temperature constantly S.

(2) Let the air expand farther, without being allowed to take heat from or to part with heat to surrounding matter, until its temperature sinks to T.

(3) Let the air be allowed to part with heat so as to keep

and V_2, P_3 and V_3, P_4 and V_4, denote the pressure and volume respectively, at the ends of the four successive operations, we have by the gaseous laws, and by Poisson's formula and a conclusion from it quoted above,* the following expressions:—

Mechanical effect obtained by the first operation $= PV \log \frac{V_1}{V}$

Mechanical effect obtained by the second operation

$$= P_2 V_2 \cdot \frac{1}{k-1} \cdot \left\{ \left(\frac{V_2}{V_1} \right)^{k-1} - 1 \right\}.$$

Work spent in the third operation

$$= P_3 V_3 \log \frac{V_2}{V_3}.$$

Work spent in the fourth operation

$$= P_3 V_3 \cdot \frac{1}{k-1} \left\{ \left(\frac{V_3}{V_4} \right)^{k-1} - 1 \right\}.$$

Now, according to the gaseous laws, we have

$$P_1 V_1 = PV, \ P_2 V_2 = P_1 V_1 \frac{1 + ET}{1 + ES},$$

and $\qquad P_3 V_3 = P_2 V_2$; and, (since $V_4 = V$,) $P_4 = P$.

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* *From a Letter of the Author's to Mr. Joule.*

"To find the work necessary to compress a given mass of air to a given fraction of its volume, when no heat is permitted to leave the air; let P, V, T be the primitive pressure, volume, and temperature, respectively; let p, v, and t be the pressure, volume, and temperature at any instant during the compression; and let P', V', and T' be what they become when the compression is concluded. Then if k denote the ratio of the specific heat of air at constant pressure to the specific heat of air kept in a space of constant volume, and if, as appears to be nearly, if not rigorously true, k be constant for varying temperatures and pressures, we shall have by the investigation in Miller's 'Hydrostatics' (Edit. 1835, p. 22)—

$$\frac{1 + Et}{1 + ET} = \left(\frac{V}{v} \right)^{k-1}.$$

But $$\frac{pv}{PV} = \frac{1 + Et}{1 + ET},$$

therefore $$pv = PV \left(\frac{V}{v} \right)^{k-1}.$$

Now the work done in compressing the mass from volume v to volume $v - dv$ will be pdv, or by what precedes,

$$PV . V^{k-1} \frac{dv}{v^k}.$$

Hence by the integral calculus we readily find, for the work, W, necessary to compress from V to V',

$$W = PV . \frac{1}{k-1} \left\{ \left(\frac{V}{V'} \right)^{k-1} - 1 \right\}."$$

Also, by Poisson's formula,

$$\left(\frac{V_2}{V_1}\right)^{k-1} = \left(\frac{V_3}{V}\right)^{k-1} = \frac{1+ES}{1+ET}.$$

By means of these we perceive that the work spent in the fourth operation is equal to the mechanical effect gained in the second; and we find, for the whole gain of mechanical effect (denoted by M), the expressions

$$M = (PV - P_3V_3)\log\frac{V_1}{V} = PV\log\frac{V_1}{V}\,.\,\frac{E(S-T)}{1+ES}.$$

All the preceding formulæ are founded on the assumption of the gaseous laws and the constancy of the ratio (k) of the specific heat under constant pressure to the specific heat in constant volume, for the air contained in the cylinder and piston, and involve no other hypothesis.* If now we add the assumption of Mayer's hypothesis, which for the actual circumstances is $PV\log\frac{V_1}{V} = JH$, H denoting the heat abstracted by the air from the surrounding matter in the first operation, and J the mechanical equivalent of a thermal unit, we have

$$M = JH\,.\,\frac{E(S-T)}{1+ES}.$$

[redacted]

which involves no hypothesis, the expression

$$\mu = \frac{J}{\frac{1}{E} + t}$$

for Carnot's function, which Mr. Joule had suggested to me, in a letter dated December 9, 1848, as the expression of Mayer's hypothesis, in terms of the notation of my "Account of Carnot's Theory."* Mr. Rankine† has arrived at a formula agreeing with it (with the exception of a constant term in the denominator, which, as its value is unknown, but probably small, he neglects in the actual use of the formula), as a consequence of the fundamental principles assumed in his Theory of Molecular Vortices, when applied to any fluid whatever, subjected to a cycle of four operations satisfying Carnot's criterion of reversibility (being in fact precisely analogous to those described above, and originally invented by Carnot); and he thus establishes Carnot's law as a consequence of the equations of the mutual conversion of heat and expansive power, which had been given in the first section of his paper on the Mechanical Action of Heat.‡

II. *Note on the Specific Heats of Air.*

Let N be the specific heat of unity of weight of any fluid at the temperature t, kept within constant volume, v; and let kN be the specific heat of the same fluid mass, under constant pressure, p. Without any other assumption than than that of Carnot's principle, the following equation is demonstrated in my paper‖ on the "Dynamical Theory of Heat," § 48,

$$kN - N = \frac{\left(\frac{dp}{dt}\right)^2}{\mu \times -\frac{dp}{dv}},$$

where μ denotes the value of Carnot's function, for the temperature t, and the differentiations indicated are with reference to v and t considered as independent variables, of

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* Royal Society of Edinburgh, Jan. 2, 1849, *Transactions*, vol. xvi. pt. 5.

† On the Economy of Heat in Expansive Engines. Royal Society of Edinburgh, April 21, 1851, *Transactions*, vol. xx. part 2.

‡ Royal Society of Edinburgh, Feb. 4, 1850, *Transactions*, vol. xx. pt. 1.

‖ Royal Society of Edinburgh, Mar. 17, 1851, *Transactions*, vol. xx. pt. 2.

Hence it is probable that the values of the specific heat of air under constant pressure, found by Suermann (·3046), and by De la Roche and Berard (·2669), are both considerably too great; and the true value, to two significant figures, is probably ·24.

Glasgow College, Feb. 19, 1852.

POSTSCRIPT.

In a paper communicated to the Royal Society, along with the above, (March 1852), Mr. Joule described a new experimental determination of the specific heat of air under constant atmospheric pressure, which gave ·23 as a mean result, but he used ·2389 as probably nearer the truth, correcting certain tables, calculated from De la Roche and Berard's result, which he had given in his paper on the Air Engine. M. Regnault has just published (Comtes Rendus, *Ap.* 18, 1853) the results of experimental researches on the specific heat of air, by which he finds that for all temperatures from $-30°$ to $+225°$ centigrade, and for all pressures from one up to ten atmospheres, the specific heat of air is from ·237 to ·2379, and thus both pushes to a minuter degree of accuracy the direct confirmation which the theoretical results published by Mr. Rankine and myself first obtained from Mr. Joule's experiments, and justifies Mr. Joule in the number he actually used in his calculations.

Glasgow College, April 1853.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

By J. J. SYLVESTER.

[*Continued from* Vol. VII., p. 214.]

to that given by Aronhold in the particular case where the three functions are derived from the same cubic, and becoming identical therewith when the coefficients are accommodated to this particular supposition.* I shall confine myself for the present to combinants relating to systems of functions, all of the same degree.

If $\phi_1, \phi_2, \ldots \phi_r$, be homogeneous functions of any number of variables, any invariant or other concomitant of the system which remains unchanged, not only for linear substitutions impressed upon the variables contained within the functions, but also for linear combinations impressed upon the functions themselves, is what I term a Combinant. A Combinant is thus an invariant or other concomitant of a system in its corporate capacity (quâ *system*), being in fact common to the whole family of forms designated by $\lambda_1.\phi_1 + \lambda_2.\phi_2 + \ldots + \lambda_r.\phi_r$, where $\lambda_1, \lambda_2, \ldots \lambda_r$, are arbitrary constants. If the coefficients of $\phi_1, \phi_2, \ldots \phi_r$, be supposed to be written out in (r) lines (the coefficients of corresponding terms occupying the same place in each line), so as to form a rectangular matrix, any combinative invariant will be a function of the determinants corresponding to the several squares of r^2 terms each that can be formed out of such matrix, or, as they may be termed, the *full* determinants belonging to such rectangular matrix. If we call any such combinant K, then, over and above the ordinary partial differential equations which belong to it in its character of an invariant, it will be necessary and sufficient, in order to establish its combinantive character, that K shall be subject to satisfy $(r-1)$ pairs of equations of the form

$$\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc}\ldots\right)K = 0,$$

$$\left(a\frac{d}{da'} + b\frac{d}{db'} + c\frac{d}{dc'}\ldots\right)K = 0,$$

where $a, b, c \ldots;\ a', b', c' \ldots$, are respectively lines in the matrix above referred to.

So any combinantive concomitant will be a function of the full determinants of the matrix formed by the coefficients of the given system of forms and of the variables, and will be

* A similar method will subsequently be applied to the representation of the resultant of two cubic equations as a function of Combinants bearing relations to the quadratic and cubic invariants of a quartic function of x and y, precisely analogous to those which the Combinants that enter into the solution above alluded to bear to the Aronholdian invariants of a cubic function.

subject to satisfy the additional differential equations just above written.

It will readily be understood furthermore, that an invariant or other concomitant may be combinantive in respect to a certain number of forms of a system, and not in respect of other forms therein; or more generally, may be combinantive in respect of each, separately considered, of a series of groups into which a given system may be considered to be subdivided, without being so in respect of the several groups taken collectively.

In the fourth section of my memoir on a Theory of the Conjugate Properties of two rational integral Algebraical Functions, recently presented to the Royal Society of London, the case actually arises of an invariant of a system of three functions, which is combinantive in respect only to two of them.

For greater simplicity, let the attention for the present be kept fixed upon combinants which are such in respect of a single group of functions, all of the same degree in the variables. (It will of course have been perceived that when the system is made up of several groups, there would be nothing gained by limiting the groups to be all of the same degree *inter se;* it is sufficient that all of the same group be of the same degree *per se.*)

All such combinants will admit of an obvious and immediate classification. Let us suppose that a combinant is

of this conjunctive, there results a derivative function or system of functions of the quantities $\lambda_1, \lambda_2, \ldots \lambda_r$, in which every term affecting any power or combination of powers of the (λ) series is necessarily an invariant or concomitant of the given system. If now an invariant or other concomitant be taken of the new system in respect to $\lambda_1, \lambda_2, \ldots \lambda_r$, (the original variables (supposing them to enter) being treated as constants), this secondarily derived invariant will be itself an Invariant, or at all events a Concomitant in respect of the original system, and being unaffected by linear substitutions impressed upon the λ_1 system, is by definition a combinant of such system. A similar method will obviously apply if the original system be made up of various groups; each group will give rise to a conjunctive, and one or more concomitants being taken of this system of conjunctives and treated as in the case first supposed, (the only difference being, that there will on the present supposition be several *unrelated* systems instead of a single system of new variables, *i.e.* several λ systems instead of one only,) the result, when all the λ systems have been *invariantized out* (*i.e.* made to disappear by any process for forming invariants), will be a combinant in respect to each of the groups, severally considered, of the given system of functions.

Here let it be permitted to me to make a momentary digression, in order to be enabled to avoid for the future the inconvenience of using the phrase "invariant or other concomitant," and so to be enabled at one and the same time to simplify the language and to give a more complete unity to the matter of the theory, by shewing how every concomitant may in fact be viewed as a simple invariant, so that the calculus of forms may hereafter admit of being cited, as I propose to cite it, under the name of the Theory of Invariants.

Thus, to begin with the case of *simple* contragredience and congredience, if $\xi, \eta, \zeta \ldots$ are contragredient to $x, y, z \ldots$, any form containing $\xi, \eta, \zeta \ldots$, which is concomitantive to a given form or system of forms S, which contains $x, y, z \ldots$, may be regarded as concomitantive to the system S', made up of S and the superadded *absolute* form $\xi x + \eta y + \zeta z + \ldots$, say ϑ; where $\xi, \eta, \zeta \ldots$ are treated no longer as variables, but as *constants*. In like manner every system of variables contragredient to $x, y, z \ldots$, or to any other system of variables in S, will give rise to a superadded form analogous to ϑ, the totality of which may be termed S_1; and thus the

various systems $\xi, \eta, \zeta \ldots$ will no longer exist as variables in the derived form, but purely as constants. Again, if S contain any system of variables ϕ, ψ, ϑ, &c., contragredient to x, y, z, &c., the system of variables u, v, w, &c., congredient with x, y, z, &c., may be considered as constants belonging to the superadded form $\phi u + \psi v + \vartheta w \ldots$; but if S do not contain any system contragredient to x, y, z, &c., then u, v, w, &c. may be treated as constants belonging to the superadded system of forms $xv - yu$, $yw - zv$, $zu - xw$, &c.; and so in general any concomitant containing any sets of variables in simple relation, whether of cogredience or contragredience, with any of the sets in the given system S, may in all cases be treated (record such sets) as an *invariant* of the system S', made up of S and a certain superadded system S_1, all the forms contained in which are absolute, by which I mean, that they contain no literal coefficient. The same conclusion may be extended to the case of concomitants containing sets of variables in *compound* relation with the sets in the given system of forms S. Thus, suppose $u_1, u_2, \ldots u_n$, to be in compound relation of cogredience with $x^{n-1}, x^{n-2}.y, x^{n-3}.y^2, \ldots y^{n-1}$; $u_1, u_2, \ldots u_n$, may be regarded as constants belonging to the superadded form

$$u_1.y^{n-1} - (n-1)\,u_2.y^{n-2}.x + (n-1).\frac{n-2}{2}\,u_3.y^{n-3}.x^2 \mp \&c. \pm u_n.x^{n-1},$$

say Ω. And thus universally we are now enabled to affirm,

apparent that every resultant of any system of n functions of the same degree of a single set of (n) variables is a combinantive invariant of the system. This is an immediate and simple corollary to the theorem given by me in this Journal, in May, 1851. Accordingly, in proceeding to analyse the composition of the resultant of three quadratic functions, I may, besides impressing linear combinations upon the variables, impress linear combinations upon the functions themselves, in any way most conducive to simplicity and facility of expression and calculation; and whatever relations shall be proved to exist between the resultant and other combinants for such specific representation, must be universal, and hold good for the functions in their most general form.

(1) The system, by means of linear substitutions impressed upon the variables which enter into the functions, may be made to assume the form

$$\begin{gathered} x^2 + y^2 + z^2, \\ ax^2 + by^2 + cz^2, \\ lx^2 + my^2 + nz^2 + 2pyz + 2qzx + 2rxy. \end{gathered}$$

(2) By means of linear combinations of the functions themselves the system may evidently be made to take the form

$$\begin{gathered} (c-a)\,x^2 + (c-b)\,y^2, \\ (a-b)\,y^2 + (a-c)\,z^2, \\ ky^2 + 2pyz + 2qzx + 2rxy; \end{gathered}$$

and finally, by taking suitable multipliers of x, y, z in lieu of x, y, z, it may be made to become

$$\begin{gathered} \rho\,(x^2 - y^2), \\ \sigma\,(y^2 - z^2), \\ y^2 + 2fyz + 2gzx + 2hxy. \end{gathered}$$

We have thus reduced the number of constants in the system from eighteen to five; and as it will readily be seen that in any combinant of the system in its reduced form ρ and σ can only enter as factors of the simple quantity, $(\rho\sigma)^i$, for all purposes of comparison of the combinants of the system of like dimensions with one another, ρ and σ might admit of being treated as being each unity, and accordingly, practically speaking, we have only to deal with three in place of eighteen constants, a marvellous simplification, and which makes it obvious, *à priori*, or at least

affords a presumption almost amounting to and capable of being reduced to certainty, that the number of fundamental combinants of the system, of which all the rest must be explicit rational functions, will be exactly four in number; which, for the canonical form hereinbefore written, on making ρ and σ each unity, will correspond to

$$1,\ f^2+g^2+h^2,\ f^2g^2+g^2h^2+h^2f^2,\ fgh,$$

and will be of the 3rd, 6th, 12th, and 9th degrees respectively. The reason why the squares of f, g, h, instead of the simple terms f, g, h, appear in the 2nd and 3rd of these forms is, because, on changing x into $-x$, y into $-y$, or z into $-z$, two of the quantities f, g, h will change their sign, but the forms representing the invariants of even degrees ought to remain absolutely unaltered for such transformations. I shall in the course of the present section set forth the methods for obtaining these four combinants, which, although of the regularly ascending dimensions 3, 6, 9, 12, belong obviously to two different groups, the one of three dimensions forming a class in itself, and the natural order of the three others being that denoted by the sequence 6, 12, and 9, and not that which would be denoted by the sequence 6, 9, 12, the combinant of the ninth degree being properly to be regarded as in some sort an accidentally rational square root of a combinant of 18 dimensions.

Let now $\rho(x^2-y^2)=U$

Hence the resultant R

$$=\rho^4.\sigma^4.(1+2f+2g+2h)(1-2f-2g+2h)(1+2f-2g-2h)(1-2f+2g-2h)$$

$$=(\rho\sigma)^4\{(1+2h)^2-4(f+g)^2\}\{(1-2h)^2-4(f-g)^2\}$$

$$=(\rho\sigma)^4\{(1+4h^2-4f^2-4g^2)^2-(4h-8fg)^2\}$$

$$=(\rho\sigma)^4[1-8(f^2+g^2+h^2)+16\{(f^4+g^4+h^4)-2(g^2h^2+h^2f^2+f^2g^2)\}$$
$$+64fgh].$$

Let now $$K=\lambda U+\mu V+\nu W,$$

K being what I term a linear conjunctive of U, V, W. The invariant of K, in respect to x, y, z, will be the determinant

$$\begin{vmatrix} \rho\lambda, & h\mu, & g\mu, \\ h\mu, & \mu-\rho\lambda+\sigma\nu, & f\mu, \\ g\mu, & f\mu, & -\sigma\nu, \end{vmatrix}$$

i.e. $$=(2fgh-g^2)\mu^3+\sigma(h^2-g^2)\mu^2\nu-\rho(f^2-g^2)\mu^2\lambda-\rho\sigma\mu\lambda\nu$$
$$+\rho^2\sigma\lambda^2\nu-\rho\sigma^2\lambda\nu^2;$$

or, multiplying by 6, we may write

$$I_{x,y,z}.K=6d\lambda\mu\nu+3b_2\mu^2\nu+3b_1\mu^2\lambda+3a_3\lambda^2\nu+3c_1\lambda\nu^2+b_2\mu^3,$$

where $$d=-\rho\sigma, \qquad b_2=12fgh-6g^2,$$
$$b_1=-2\rho(f^2-g^2), \qquad b_3=2\sigma(h^2-g^2),$$
$$a_3=\rho^2\sigma, \qquad c_1=-2\rho\sigma^2,$$

the notation being accommodated to that employed by Mr. Salmon in *The Higher Plane Curves*, pp. 182, 184, λ, μ, ν in $I.K$ being correspondent to x, y, z in Mr. Salmon's form. If now we employ Mr. Salmon's expression (p. 184) for the S (the biquadratic Aronholdian of $I.K$), observing that

$$a_2=0, \quad c_2=0, \quad a_1=0, \quad c_3=0,$$

we have the complex combinant

$$S_{\lambda,\mu,\nu}.I_{x,y,z}.K=d^4-2d^2(b_1c_1+a_3b_3)+da_3b_2c_1-a_3c_1b_1b_3+b_1^2c_1^2+a_3^2b_3^2$$

$$=\rho^4\sigma^4\begin{pmatrix} 1-8(f^2+h^2-2g^2)+4(12fgh-6g^2) \\ -16(f^2-g^2)(h^2-g^2)+16(f^2-g^2)^2+(h^2-g^2)^2 \end{pmatrix}$$

$$=\rho^4\sigma^4\{1-8(f^2+g^2+h^2)+16(f^4+g^4+h^4-h^2g^2-g^2f^2-f^2h^2)+48fgh\}.$$

Hence, calling the resultant R, we have

$$-3R+4.S_{\lambda,\mu,\nu}.I_{x,y,z}.K=1-8(f^2+g^2+h^2)+16(f^4+g^4+h^4)$$
$$+32(f^2g^2+g^2h^2+h^2f^2)$$
$$=\{1-4(f^2+g^2+h^2)\}^2=P^2.$$

Let Ω be taken the polar reciprocal to the conjunctive

$$-\lambda U+\mu V+\nu W;$$

and for greater simplicity, as we know, *à priori*, from the fundamental definition of a combinant which (save as to a factor, must remain unaltered by any linear modification impressed upon the functions to which it appertains), that ρ and σ can enter factorially only in any combinant, let ρ and σ be each taken equal to unity in performing the intermediary operations.

$$\text{Then}\qquad \Omega=\left\{\begin{matrix} -\lambda, & h\mu, & g\mu, & \xi, \\ h\mu, & \lambda+\mu+\nu, & f\mu, & \eta, \\ g\mu, & f\mu, & -\nu, & \zeta, \\ \xi, & \eta, & \zeta, & 0, \end{matrix}\right\}$$

$$=\left\{\begin{aligned} &\xi^2(\nu^2+\nu\mu+\nu\lambda+f^2\mu^2) \\ &+\eta^2(-\lambda\nu+g^2\mu^2) \\ &+\zeta^2(\lambda^2+\lambda\mu+\lambda\nu+h^2\mu^2) \\ &-2\eta\zeta(f\lambda\mu-hg\mu^2) \\ &-2\zeta\xi\{g(\mu\lambda+\mu\nu)+(g-fh)\mu^2\} \\ &-2\xi\eta(h\mu\nu-fg\mu^2) \end{aligned}\right\}.$$

$$
\begin{matrix}
\frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, \\
\frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, \\
\multicolumn{3}{c}{\left[\frac{d^2}{d\lambda^2}, \frac{d^2}{d\mu^2}, \frac{d^2}{d\nu^2}\right]}
\end{matrix}
$$

$$
-\left\{
\begin{matrix}
\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta} \\
\frac{\dot{d}}{d\xi}, \frac{\dot{d}}{d\eta}, \frac{\dot{d}}{d\zeta} \\
\left[\frac{d^2}{d\lambda^2}, \frac{d}{d\mu}\cdot\frac{d}{d\nu}, \frac{d}{d\mu}\cdot\frac{d}{d\nu}\right]
\end{matrix}
\right\}
$$

$$
-\left\{
\begin{matrix}
\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta} \\
\frac{\dot{d}}{d\xi}, \frac{\dot{d}}{d\eta}, \frac{\dot{d}}{d\zeta} \\
\left[\frac{d}{d\lambda}\cdot\frac{d}{d\nu}, \frac{d^2}{d\mu^2}, \frac{d}{d\lambda}\cdot\frac{d}{d\nu}\right]
\end{matrix}
\right\}
$$

$$
-\left\{
\begin{matrix}
\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta} \\
\frac{\dot{d}}{d\xi}, \frac{\dot{d}}{d\eta}, \frac{\dot{d}}{d\zeta} \\
\left[\frac{d}{d\lambda}\cdot\frac{d}{d\mu}, \frac{d}{d\lambda}\cdot\frac{d}{d\mu}, \frac{d^2}{d\nu^2}\right]
\end{matrix}
\right\}
$$

$$
2\left\{
\begin{matrix}
\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta} \\
\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta} \\
\left[\frac{d}{d\lambda}\cdot\frac{d}{d\mu}, \frac{d}{d\mu}\cdot\frac{d}{d\nu}, \frac{d}{d\nu}\cdot\frac{d}{d\lambda}\right]
\end{matrix}
\right\}.
$$

In this expression the first lines may be considered commutable, the second or dotted lines are subject to the usual process of commutation, which makes three of the six permutations positive and six negative; and the third or bracketed lines are subject to the simple process which

makes all the permutations of the same sign. In the three middle groups two of the terms in the final line are always identical; it will therefore be more convenient to introduce the multiplier 2, and then to consider each such line to represent the three distinct permutations, taken singly.

Let now
$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2};\frac{d^2}{d\eta^2};\frac{d^2}{d\zeta^2}\right\}\Omega=(\Omega),$$

$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2};\frac{d}{d\eta}\cdot\frac{d}{d\zeta};\frac{d}{d\eta}\cdot\frac{d}{d\zeta}\right\}\Omega=(\Omega)',$$

$$\frac{1}{8}\left\{\frac{d}{d\xi}\cdot\frac{d}{d\zeta};\frac{d^2}{d\eta^2};\frac{d}{d\xi}\cdot\frac{d}{d\zeta}\right\}\Omega=(\Omega)'',$$

$$\frac{1}{8}\left\{\frac{d}{d\eta}\cdot\frac{d}{d\xi};\frac{d}{d\eta}\cdot\frac{d}{d\xi};\frac{d^2}{d\zeta^2}\right\}\Omega=(\Omega)''',$$

$$\left\{\frac{d}{d\xi}\cdot\frac{d}{d\eta};\frac{d}{d\eta}\cdot\frac{d}{d\zeta};\frac{d}{d\zeta}\cdot\frac{d}{d\xi}\right\}\Omega=(\Omega)_1.$$

And let
$$\left[\frac{d}{d\lambda^2};\frac{d^2}{d\mu^2};\frac{d}{d\nu^2}\right]=L,$$

$$\left[\frac{d^2}{d\lambda^2};\frac{d}{d\mu}\cdot\frac{d}{d\nu};\frac{d}{d\mu}\cdot\frac{d}{d\nu}\right]=L',$$

$$\left[\frac{d}{\ }\cdot\frac{d}{\ };\frac{d^2}{\ };\frac{d}{\ }\cdot\frac{d}{\ }\right]=L'',$$

and a similar interpretation must be extended to each of the 25 partial products; we have then

$$L(\Omega) = 8g^2, \quad -2L'(\Omega) = 0, \quad -2L'''(\Omega) = 0,$$
$$-2L''(\Omega) = -4g^2, \quad 2L_1(\Omega) = -2,$$
$$-2L(\Omega)' = 0, \quad -2L(\Omega)''' = 0,$$
$$4L'(\Omega)' = 0, \quad 4L''(\Omega)''' = 0,$$
$$4L''(\Omega)' = 0, \quad 4L'''(\Omega)''' = 0,$$
$$4L'''(\Omega)' = 8f^2, \quad 4L'(\Omega''') = 8h^2,$$
$$4L'(\Omega)'' = 0, \quad 4L''(\Omega)'' = 0, \quad 4L'''(\Omega)'' = 0,$$
$$-4L_1(\Omega)' = 0, \quad -4L_1(\Omega)''' = 0,$$
$$-4L_1(\Omega)'' = 4g^2;$$

and, finally, the 5 terms comprised in

$$4\mathrm{E}(\Omega)_1 \text{ each} = 0.$$

All the above equations can be easily verified by direct inspection, it being observed that $8(\Omega)$ represents

$$\nu^2 + \lambda\nu + \lambda^2 + f^2\mu^2; \quad -\lambda\nu + g^2\mu^2; \quad \lambda^2 + \lambda\mu + \lambda\nu + h^2\mu^2;$$

that $8(\Omega)'$ represents

$$\nu^2 + \mu\nu + \lambda\nu + f^2\mu^2; \quad f\lambda\mu - hg\mu^2; \quad f\lambda\mu - hg\mu^2;$$

that $8(\Omega)''$ represents

$$-\lambda\nu + g^2\mu^2; \quad g(\mu\lambda + \mu\nu) + (g - fh)\mu^2; \quad g(\mu\lambda + \mu\nu) + (g - fh)\mu^2;$$

that $8(\Omega)'''$ represents

$$\lambda^2 + \mu\lambda + \nu\lambda + h^2\mu^2; \quad h\mu\nu - fg\mu^2; \quad h\mu\nu - fg\mu^2;$$

and that $(\Omega)_1$ represents

$$f\lambda\mu - hg\mu^2; \quad -(\mu\lambda + \mu\nu) + (g - fh)\mu^2; \quad h\mu\nu - fg\mu^2.$$

We have thus

$$\mathrm{E}(\Omega) = 8g^2 - 4g^2 - 2 + 8f^2 + 8h^2 + 4g^2$$
$$= 2\{4f^2 + 4g^2 + 4h^2 - 1\}.$$

Hence $3R = 4S_{\lambda,\mu,\nu} \cdot I_{x,y,z} \cdot K - \frac{1}{4}\{\mathrm{E}.\Omega\}^2$ .... (A).

If we restore to U, V, W their general values, and make

$$U = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$
$$V = a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy,$$
$$W = a''x^2 + b''y^2 + c''z^2 + 2f''yz + 2g''zx + 2h''xy,$$

$$y = \pi a \cos^2\theta - 2a\theta,$$
$$= \frac{\pi a}{2}(1 + \cos 2\theta) - 2a\theta,$$
$$= \frac{\pi a}{2}\left(A + \sqrt{\left\{1 - \left(\frac{x}{\frac{1}{2}\pi \,.\, a}\right)^2\right\}} + a \sin^{-1}\frac{x}{\frac{1}{2}\pi \,.\, a}\,.\right.$$

If we would have the equation from the vertex, we must write $x - \frac{1}{2}\pi\, a$ in place of x in this equation, and we get, after reduction,

$$y = \sqrt{(2.\tfrac{1}{2}\pi \,.\, ax - x^2)} + a \operatorname{versin}^{-1}\frac{x}{\frac{1}{2}\pi \,.\, a},$$

the equation to a Trochoid.

(Prob. 8, p. 96.)

Let m be the mass of one of the particles, v its velocity, α the angle at which its line of motion is inclined to the diameter through a point at which it strikes the spherical surface. The impulsive force which it exerts in the impact at this point will be $2mv\cos\alpha$. After striking it will move, with an equal velocity, in another straight line equally inclined to the diameter through the point of impact, and it will therefore impinge again at the same angle, with an equal impulsive force; and the same will hold for all the successive impacts. The interval between any two successive impacts, being the time of describing a chord subtending an angle

The second proposition in the enunciation may be demonstrated as a consequence, by simply observing that the pressure at any small part of the surface depends at each instant only on the masses and motions of the particles infinitely near it, and must therefore have the same expression when the vis viva of the motions per unit of volume in its neighbourhood is given, whatever be the dimensions or form of the whole bounding surface.

Second Solution.

The following general demonstration establishes the truth of the proposition at once for a space bounded by a surface of any form.

Let ω denote an infinitely small area of the surface, and let m be the sum of the masses of the infinitely numerous particles in a unit of volume which move in directions towards the same parts contained within definite limits infinitely near a certain straight line inclined at an angle θ to the normal through ω, with velocities between definite limits differing infinitely little from a certain amount, v. The sum of the masses of all these particles which strike the surface within the area ω, in the unit of time, will be

$$mv\omega \cos\theta,$$

since mv is the sum of the masses of all those of them which, in a unit of time, pass a unit plane area perpendicular to their directions of motion. Hence the portion due to the motions of these particles, of the whole pressure experienced by the area ω, is

$$2mv\omega \cos\theta \,.\, v\cos\theta, \quad \text{or} \quad 2\omega \cos^2\theta \,.\, mv^2;$$

and, if Σmv^2 denote the vis viva of all the particles which, whatever be their velocities, move in directions within the prescribed limits, we have

$$2\omega \cos^2\theta \Sigma mv^2$$

for the pressure which they produce on ω. Now if we choose any line parallel to a normal through ω, and any plane through this line, as axis and plane of reference, we may take (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ to express by the method of polar directional coordinates the limits prescribed for the motions we have been considering: so that parallels drawn to all these lines of motion, through any point in the axis of reference, will cut the surface of a sphere of unit radius described from this point as centre in an infinitely small area

$$\sin\theta \, d\theta \, d\phi.$$

We must therefore have

$$\Sigma mv^2 = q\,\frac{\sin\theta\, d\theta\, d\phi}{4\pi},$$

if q be the entire vis viva of the motions of particles in all directions, through a unit of volume. By using this in the preceding expression, we find

$$\frac{2\omega q}{4\pi}\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \cos^2\theta \sin\theta\, d\theta\, d\phi$$

for the whole pressure experienced by ω, (the limits of integration for θ being 0 to $\frac{1}{2}\pi$, instead of 0 to π, so that only the motions *toward* and not the motions *from* ω, may be included); and, performing the indicated integrations, we obtain finally, for the pressure on ω,

$$\tfrac{1}{3}\omega q,$$

from which we conclude that $\frac{1}{3}q$ is the pressure per unit of surface, as was to be proved.

Chamonix, Aug. 18, 1853.

(Prob. 9, p. 96.)

Let X, Y, Z, &c. be the points in which parallel lines

And from theory of volumes, as represented by determinants,

$$V^r(xyzt) = \begin{vmatrix} Xa & Xb & Xc & Xd \\ Ya & Yb & Yc & Yd \\ Za & Zb & Zc & Zd \\ Ta & Tb & Tc & Td \end{vmatrix} = \begin{vmatrix} 0 & \beta & \gamma & \delta\ldots \\ \alpha & 0 & \gamma & \delta\ldots \\ \alpha & \beta & 0 & \delta\ldots \\ \alpha & \beta & \gamma & 0\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \times \frac{(-V)^{r+1}}{\alpha\beta\gamma\delta\ldots}$$

$$= (-V)^{r+1} \begin{vmatrix} 0 & 1 & 1 & 1\ldots \\ 1 & 0 & 1 & 1\ldots \\ 1 & 1 & 0 & 1\ldots \\ 1 & 1 & 1 & 0\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = (-V)^{r+1}(-1)^r.r. = -V^{r+1}.r;$$

therefore $$(xyzt\ldots\ldots) = -rV.$$

In the same way, if the lines passed through the point whose coordinate volumes are Va, Vb, &c., we should find that

$$V_1 = (-1)^r.rV.\frac{VaVbVc\ldots}{(V-Va)(V-Vb)(V-Vc)\ldots}$$

(Prob. 10, p. 96.)

LET x_p, y_p be the coordinates of a point P referred to the axes of the conic; then, if t is its polar, 0_t the perpendicular on it from centre,

$$\frac{x_p}{a^2} = \frac{\sin tx}{0_t}, \quad \frac{y_t}{b^2} = \frac{\cos tx}{0_t}.$$

Hence, for any two points a and b,

$$\frac{x_a y_b - x_b y_a}{a^2b^2} = \frac{\sin 1x \cos 2y - \sin 1y \cos 2x}{0_1 0_2};$$

therefore $$2(0ab) = \frac{\sin(12)}{0_1 0_2}a^2b^2;$$

$$\therefore 2(0ab + 0bc + 0ca) = 2(abc) = \frac{a^2b^2}{0_1 0_2 0_3}\Sigma \sin 12.0_3 = a^2b^2\frac{(123_p)}{0_1 0_2 0_3}.$$

Now if p be the perpendicular from the centre on the tangent, r the radius of curvature, $rp^3 = a^2b^2$. Hence, as the triangle formed from two tangents and the chord of contact is its own polar triangle, in the limit the radii will be as 1, 2, 4; and the areas of the inscribed and circumscribed triangles as 1, 2.

In space of r dimensions, if V be the volume formed from the semiaxes of the quadratic surface, we shall have, using

the notation of the last question,

$$0a.0b.0c...(123...) = V^2(abc...)^r.$$

From these expressions, any pure identities of areas, volumes, &c., considered as formed from points, are dualized. Thus, having the area of a Pascal triangle in terms of the areas derived from the original points, we can write down the corresponding expression for the Brianchon triangle. But there is also another form of this function, which in a different point of view is not of less importance; I mean that in which one of the points is considered as variable. Here, if we have given the point equation to a surface in space r, involving any number of ordinates, by certain operations upon it, we can find its line, plane, &c. equations, the last of which (the $(r-1)$ space equation) involves this function. Thus, in space of 2 dimensions, if the equation involves the ordinates p_1 and p_2 referred to lines 1 and 2, its reciprocal equation will involve $(t12_\rho)$. In cases like this, the most convenient form is $\sin 12.(12_t)$, (12_t) denoting the perpendicular from the point 12 on the line t. Thus, the equation

$$Ap_1^2 + Bp_2^2 + Cp_3^2 = 0$$

gives the reciprocal equation

$$\frac{\{\sin 12.(12_t)\}^2}{A} + \frac{\{\sin 23.(23_t)\}^2}{B} + \frac{\{\sin 31.(31_t)\}^2}{C} = 0.$$

of intersection of the transverse or major axes from each of the conics is one and the same pure imaginary quantity. This property might be deduced from the known properties of focal conics, but it is easier to derive it directly by a method which indeed contains implicitly the demonstration of these properties of focal conics.

Let the centre of the sphere be taken as the origin, the axes of z being perpendicular to the plane of the conics; then, writing for the sphere, and the plane of one of the small circles,

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ lx + my + nz = 1, \end{cases}$$

the equation of the projecting cone is

$$x^2 + y^2 + z^2 - (lx + my + nz)^2 = 0.$$

And if in this equation we suppose z equal to the perpendicular distance of the centre from the plane of the conics ($z = \gamma$ suppose), we have

$$x^2 + y^2 + \gamma^2 - (lx + my + n\gamma)^2 = 0,$$

for the equation of one of the conics: this equation shews that the conic must have a double contact with the imaginary circle
$$x^2 + y^2 + \gamma^2 = 0;$$

i.e. the centre of this circle (viz. the foot of the perpendicular let fall from the centre of the sphere upon the plane of the conics) must lie on an axis of the conic. Moreover the radius of the circle, *i.e.* the normal distance of its centre from the conic, is $i\gamma$, or the perpendicular in question multiplied by the imaginary symbol i; and since the circle $x^2 + y^2 + \gamma^2 = 0$ is the same whatever the small circle on the sphere may be, the other conic must satisfy the same conditions, or the foot of the perpendicular must be the point of intersection of the axes of the two conics, and its normal distance from each of the conics must be one and the same pure imaginary quantity. It only remains to shew that the axes must be the transverse or major (or more correctly 'focal') axes; in fact, the normal distance from a conic of any point upon the non-focal axis is in every case—even when the normal itself is imaginary—a real positive quantity, so that it is only in the case where the point is upon the focal axis that the normal distance can be a pure imaginary quantity.

(Prob. 5, p. 188.)

(*a*) Instead of heating the air directly, we can produce the required effect more economically by means of a perfect thermodynamic engine; and it is easy to shew that this is the most economical way. We will consider the air heated pound by pound, and sent into the building at the end of the heating process. Generally, let T be the temperature of the unheated air, S the temperature to which we wish it heated. T being the temperature of air, water, &c. external to the building, will be the temperature of our refrigerator; the pound of air to be heated will be our source (nominally), and by working the engine backwards instead of taking away, we will give heat to the source.

If a be the specific heat of air, adt units will be required to raise the temperature of the pound of air from t to $t + dt$, and the work which must be spent to supply this will be

$$Jadt \frac{t - T}{t + \frac{1}{E}}.*$$

Let the whole work spent upon the pound of air be denoted by W; then we have

$$W = Ja\int_T^S \frac{t - T}{t + \frac{1}{E}} dt;$$

$$W = 1390 \times \cdot 24 \left\{ 16\cdot 66' - 283\cdot 22404 \log \frac{299\cdot 89071}{283\cdot 22404} \right\}$$

$$= 1390 \times \cdot 24 \left\{ 16\cdot 66' - 16\cdot 194713 \right\}$$

$$= 157\cdot 4438.$$

As one pound of air is heated per second, the H. P. of the engine will be got by dividing this by 550, so that

H. P. of engine = ·28626.

If an engine (probably a steam-engine) be employed to drive the heating machine, and economise only $\frac{1}{10}$th of the fuel, the fuel must have evolved $10\frac{W}{J}$ units. To heat the pound directly $a(S-T)$ units must be supplied, and $\dfrac{10\frac{W}{J}}{a(S-T)} \times 100$ gives the per-centage. In the particular case we have been considering, we find that to heat the air by means of an engine economising $\frac{1}{10}$, would require $\frac{\cdot 47195}{16\cdot 6'} \times 100$, or 28·317 per cent. of the fuel required for direct heating.

(*b*) Conceive two double-stroke cylinders connected by tubes and valves in some convenient way, with a reservoir between them. Conceive the one to be made of perfectly conducting matter, so that there shall be no difference in temperature between internal and external air, (practically this may be approximated to by immersion in running water); the other cylinder must, on the contrary, be perfectly non-conducting. The pressure in the reservoir being kept at an amount depending on the required heating effect, air is admitted (doing work as it enters) by the former, which we shall call the ingress cylinder, and is not allowed to cool below atmospheric temperature. It is pumped out by the latter, called the egress cylinder, and so heated by compression to the required temperature.

After these very general explanations, we proceed to mention more particularly the *details* of this process.

Let p', t' be respectively the atmospheric pressure a temperature, v' the volume of one pound of air

pressure p' and at temperature t', t the temperature to which we wish the air to be raised, p the pressure such, that, if air under it be compressed to pressure p', the temperature will rise from t' to t, v the volume of one pound of air under pressure p and at temperature t', v_1 the volume of air under pressure p' and at temperature t. Now, by Poisson's formula and by the gaseous laws we have

$$\text{(A)} \qquad \frac{\frac{1}{E}+t'}{\frac{1}{E}+t} = \left(\frac{p}{p'}\right)^{\frac{K-1}{K}}, \quad p = p'\left\{\frac{\frac{1}{E}+t'}{\frac{1}{E}+t}\right\}^{\frac{K}{K-1}},$$

from which p may be determined. p must be, moreover, the pressure in the reservoir, as will afterwards appear.

Volume of Cylinder. As the apparatus has to supply one pound of air per second, it will be convenient to suppose the cylinders of such a size, as to contain one pound of air at pressure p and temperature t'.

Operations in Ingress Cylinder. Suppose the piston at the top of its stroke, and the lower part of the cylinder connected with the reservoir, and consequently filled with air at pressure p and temperature t'. Then external air admitted above the piston will push it down ($p' > p$). In

Estimate of total work spent.

(1) In egress cylinder:

mechanical effect obtained during the first part of the stroke

$$= (p' - p)\, v';$$

mechanical effect obtained during the second part

$$= p'v' \log \frac{v}{v'} - p(v - v').$$

(B) Whole gain in ingress cylinder

$$= p'v' \log \frac{v}{v'}.$$

(2) In egress cylinder:

mechanical effect spent during compression

$$= \frac{pv}{K-1}\left\{\left(\frac{v}{v}\right)^{K-1} - 1\right\} - p(v - v_1);$$

work spent during expulsion

$$= p'v_1 - pv_1:$$

but
$$\frac{p'v_1}{pv} = \frac{\frac{1}{E} + t}{\frac{1}{E} + t'} = \left(\frac{v}{v_1}\right)^{K-1}.$$

(C) Whence whole work spent in egress cylinder

$$= \frac{pv}{K-1}\left\{\frac{\frac{1}{E} + t}{\frac{1}{E} + t'} - 1\right\} + pv\left\{\frac{\frac{1}{E} + t}{\frac{1}{E} + t'}\right\} - pv,$$

$$= p'v' \frac{K}{K-1} \frac{t - t'}{\frac{1}{E} + t'}, \text{ since } pv = p'v'.$$

(D) Amount of work spent in both

$$= p'v' \frac{K}{K-1} \frac{t - t'}{\frac{1}{E} + t'} - p'v' \log \frac{v}{v'}.^{*}$$

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* Modifying this by means of the formulas

$$\frac{v}{v'} = \left\{\frac{\frac{1}{E} + t}{\frac{1}{E} + t'}\right\}^{\frac{K}{K-1}} \text{ and } (E),$$

Ratios of expansion :

in the first cylinder, $$\frac{v'}{v} = \frac{p}{p'} = \left\{\frac{\frac{1}{E} + t'}{\frac{1}{E} + t}\right\}^{\frac{K}{K-1}},$$

in the second cylinder, $$\frac{v_1}{v} = \left\{\frac{\frac{1}{E} + t'}{\frac{1}{E} + t}\right\}^{\frac{1}{K-1}}.$$

A slight consideration will shew, that the rates of the cylinders must be the same if we consider them as of the same size, and as each contains one pound of air at pressure p and temperature t', the rate will evidently be 30 double strokes per minute.

Let h be the height of the cylinder, and r the radius of the base ; then volume of cylinder $= v = \pi r^2 h$, and if V_0 be the volume of a pound of air under pressure p' and at 0° Cent.,

(E) $$v' = V_0(1 + Et'),$$

$$v = \frac{p'}{p} V_0(1 + Et').$$

The preceding formulas give us the means of calculating

pounds per square foot. $K = 1{\cdot}41$. Then, by (A),

$$p = 2114 \left(\frac{283{\cdot}22404}{299{\cdot}89071}\right)^{\frac{1{\cdot}41}{{\cdot}41}},$$

$$= 1736{\cdot}6189 = 2114 \times {\cdot}8214848,$$

$$= 2114 \times \frac{1}{1{\cdot}217308}.$$

All these forms are useful.

Volume of cylinder. $v' = V_0(1 + 10E)$,

$$= 12{\cdot}383 \times 1{\cdot}0366,$$

$$= 12{\cdot}836218.$$

$$pv = p'v'. \quad v = \frac{p'}{p} v',$$

$$= 1{\cdot}217308 \times 12{\cdot}836218.$$

Volume of cylinder $= v = 15{\cdot}6256$.

A practically useful height of cylinder might be 4 feet, the corresponding diameter to which is 2·2302 feet.

Again, we have, amount of work spent in cooling in this way one pound of air (D)

$$= \frac{K}{K-1} p'v' \left\{ \frac{\frac{1}{E}+t}{\frac{1}{E}+t'} - 1 \right\} - p'v' \log \frac{v}{v'}$$

$$= \frac{141}{41} \times 2114 \times 12{\cdot}836 \left\{ \frac{299.89071}{283{\cdot}22404} - 1 \right\}$$

$$- 2114 \times 12{\cdot}836 \log 1{\cdot}217308$$

$$= 5491{\cdot}54 - 5336{\cdot}03$$

$$= 155{\cdot}51 \text{ estimated in foot pounds.}$$

As one pound must be supplied per second, H. P. of engine required to drive the apparatus = ·2827. This result ought, inasmuch as this apparatus possesses all the qualifications of a perfect engine, to be identical with the answer found in division (*a*) of this problem; we however find a difference of ·0034 between the two, due to the circumstance that the number ·24 which we employed as the value of the specific heat of air in the previous solution, also 1·41 for K in this, are only approximately true; but the true H. P. to two significant figures is ·28.

Ratios of Expansion. In first cylinder

$$\frac{v'}{v} = \cdot 8214848,$$

so that 3·2859 feet of the stroke passed while air was being admitted at pressure 2114, and ·71406 feet in allowing this to expand to pressure of receiver.

In second cylinder

$$\frac{v_1}{v} = \cdot 869825,$$

or ·5207 feet of the stroke was spent in compressing the air from pressure p to pressure p' or 2114, the remaining 3·4793 feet in expelling it.

(*c*) The first suggestion, we believe, of an apparatus for cooling buildings by compressing air, was to pump in air into a reservoir and allow it to cool to the temperature of the atmosphere, on the supposition that if then allowed to rush out by means of a stopcock, it would, in consequence of the expansion, fall in temperature. Unfortunately however for this scheme, it has been found that there is only an almost imperceptible depression of temperature (after motion ceases in the air) due to a want of perfect rigour in Mayer's hypothesis. The friction of the air in the orifice &c. almost entirely compensates for the cold of expansion.

which under pressure p' would fill the cylinder, there being no change of temperature, v_1 volume under pressure p and temperature t', of a quantity of air which would fill the cylinder under pressure p' and at temperature t.

$$(\text{A}') \qquad \frac{p}{p'} = \left\{ \frac{\frac{1}{E} + t'}{\frac{1}{E} + t} \right\}^{\frac{K}{K-1}}.$$

Operations in Ingress Cylinder. Suppose the piston at the top of its stroke, the cylinder full of air at ordinary pressure. Admitting external air above the piston, push the piston down until the air below is compressed to pressure p, the temperature being kept constant; and then send this compressed air into the reservoir.

Operations in Egress Cylinder. In the first part of the stroke allow so much air to enter the cylinder from the reservoir, that if allowed to expand in the remaining part of the stroke, the pressure and temperature at the end will be p' and t respectively.

In Ingress Cylinder. Work spent in compressing air from p' to p (temperature constant)

$$= p'v' \log \frac{v'}{v} - p'(v' - v).$$

Work spent in sending the air into the receiver

$$= pv - p'v.$$

(B′) Total expenditure per stroke in first cylinder

$$= p'v' \log \frac{v'}{v}.$$

In Egress Cylinder. Mechanical effect gained in partially filling the cylinder from reservoir

$$= pv_1 - p'v_1.$$

Mechanical effect gained during the rest of the stroke

$$= \frac{p'v'}{K-1} \left\{ \left(\frac{v'}{v_1}\right)^{K-1} - 1 \right\} - p'(v' - v_1).$$

(C′) Total gain per single stroke

$$= \frac{p'v'}{K-1} \left\{ \left(\frac{v'}{v_1}\right)^{K-1} - 1 \right\} + pv_1 - p'v'$$

$$= \frac{p'v'}{K-1}\left\{\frac{\frac{1}{E}+t'}{\frac{1}{E}+t} - 1\right\} + p'v'\left\{\frac{\frac{1}{E}+t'}{\frac{1}{E}+t} - 1\right\}$$

$$= p'v'\,\frac{K}{K-1}\,\frac{t'-t}{t+\frac{1}{E}}.$$

Ratios of Expansion. In ingress cylinder,

$$\frac{v}{v'} = \frac{p'}{p} = \left\{\frac{\frac{1}{E}+t}{\frac{1}{E}+t'}\right\}^{\frac{K}{K-1}}.$$

In egress cylinder,

$$\frac{v_1}{v'} = \left\{\frac{\frac{1}{E}+t}{\frac{1}{E}+t'}\right\}^{\frac{1}{K-1}}.$$

It would be a matter of no difficulty to give generally the number of pounds each cylinder would contain, and also the rate of each piston, but it would only tend to confusion of symbols, as we should have to take in the p's, t's, and v's of (b), as well as those of this case; we will give these details in the example, which is

Ratios of Expansion. In ingress cylinder

$$\frac{v}{v'} = \cdot 8214848,$$

and in egress

$$\frac{v_1}{v'} = \cdot 869825,$$

as in the example attached to (*b*).

Let χ be the number of pounds the first cylinder contains at pressure $p'(2114)$, and at temperature t' (26·6′ Cent.).

Volume of χ pounds $= 12\cdot383\ \chi\ (1 + E \times 26\cdot6) =$ volume of cylinder $= 12\cdot383\ (1 + E \times 10) \times 1\cdot217308$, as we found in (*b*); hence

$$\chi = \frac{283\cdot22404}{299\cdot89071}\ 1\cdot217308 = 1\cdot149655.$$

Number of single strokes per second

$$= \frac{1}{\chi} = \cdot 8698348.$$

Number of double strokes per minute

$$= 26\cdot095044.$$

In the second cylinder, let χ' be the number of pounds contained by it at p' (2114) and t(10° Cent.), we easily obtain

$$\chi' = 1\cdot217308.$$

Number of strokes per second

$$= \frac{1}{\chi'} = \cdot 8214848.$$

Number of double strokes per minute

$$= 24\cdot644544.$$

Whole work spent per second in ingress cylinder

$$= \frac{6495\cdot59}{\chi} = 5650\cdot036.$$

Whole mechanical effect gained per second in egress cylinder

$$= \frac{6684\cdot9}{\chi'} = 5491\cdot54.$$

Total motive power required $= 5650\cdot036 - 5491\cdot54$

$= 158\cdot496$ ft. pounds per second.

Hence, H.P. of engine required to drive the apparatus $= \cdot 288$.

(Prob. 7, p. 189.)

Let Δ = area of triangle, $rc = \sin C$,

$$\frac{du}{d\alpha} = u', \quad \frac{du}{d\beta} = v', \quad \frac{du}{d\gamma} = w',$$

and let δ be the distance between two points $(\alpha\beta\gamma)$, $(\alpha_1\beta_1\gamma_1)$; and $(\alpha_2\beta_2\gamma_2)$ the differences of their coordinates. Then it is easy to shew that

$$2\Delta r\delta^2 + a\beta_2\gamma_2 + b\gamma_2\alpha_2 + c\alpha_2\beta_2 = 0 \text{ .... } (1).$$

Let the equation to the inscribed ellipse be

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2ln\gamma\alpha - 2lm\alpha\beta = 0 \ldots (2).$$

Let $(\alpha_1\beta_1\gamma_1)$ be the centre; then δ is a semidiameter, and the equations to the centre are

$$\frac{u_1'}{a} = \frac{v_1'}{b} = \frac{w_1'}{c} = k \text{ .... } (3),$$

or $$\frac{a\alpha_1}{mb^{-1} + nc^{-1}} = \frac{b\beta_1}{nc^{-1} + la^{-1}} = \frac{c\gamma_1}{la^{-1} + mb^{-1}} = -k\frac{abc}{lmn} \ldots (4).$$

But $$a\alpha_1 + b\beta_1 + c\gamma_1 = 2\Delta \text{ identically .... } (5),$$

therefore $$a\alpha_2 + b\beta_2 + c\gamma_2 = 0 \text{ .... } (6),$$

and if $$la^{-1} + mb^{-1} + nc^{-1} = \lambda \text{ .... } (7),$$

by (4), (5), and (7), $$k\lambda + lmn = 0 \text{ .... } (8).$$

By (3) and (5), for all values of α, β, γ,

$$\alpha u_1' + \beta v_1' + \gamma w_1' = 2\Delta k \text{ .... } (9),$$

therefore $$u_1 = \Delta k \text{ .... } (10).$$

By Taylor's theorem,

$$u_2 = u_1 - (\alpha u_1' + \beta v_1' + \gamma w_1') + u \ldots\ldots$$

By (2), (9), (10), $$= -k\Delta \text{ .... } (11).$$

If δ be a semiaxis, δ^2 is a maximum or minimum; therefore if μ_1, μ_2 are arbitrary, we have, from (1), (6), and (11),

$$a\mu_1 + u_2'\mu_2 + b\gamma_2 + c\beta_2 = 0 \text{ .... } (12),$$

$$b\mu_1 + v_2'\mu_2 + c\alpha_2 + a\gamma_2 = 0 \text{ .... } (13),$$

$$c\mu_1 + w_2'\mu_2 + a\beta_2 + b\alpha_2 = 0 \text{ .... } (14);$$

therefore, by (1), (6), and (11),

$$k\mu_2 + 2r\delta^2 = \alpha_2\,(12) + \beta_2(13) + \gamma_2(14) = 0 \ldots (15).$$

The result of substituting in (6) proportionals to $\alpha_2, \beta_2, \gamma_2$,

found from (12), (13), and (14), is a quadratic in μ_2 or δ^2, and the theory of cross-multiplication shews that the absolute term does not contain l, m, or n, and that the coefficient of μ_2

$$= 4a\,(2lmn^2b + 2lm^2nc) + \&c.,$$

and therefore, by (7), $\qquad \propto lmn\lambda$ ....(16).

But $\qquad$ (area of ellipse)$^2 \propto$ product of values of δ^2;

by (15) and (16), $\qquad \propto (k^2lmn\lambda)^{-1}$ .... (17).

This area is to be a maximum; therefore if $\rho_1\rho_2$ are arbitrary, we have from (7), (8), and (17), writing down the coefficients of dl and dm only,

$$\rho_1 l^{-1} + \rho_2 l^{-1} + a^{-1} = 0,$$

$$\rho_1 m^{-1} + \rho_2 m^{-1} + b^{-1} = 0;$$

therefore $\qquad la^{-1} = mb^{-1} = nc^{-1},$

and the equation to the curve becomes

$$(a\alpha)^{\frac{1}{2}} + (b\beta)^{\frac{1}{2}} + (c\gamma)^{\frac{1}{2}} = 0.$$

The equation to the minimum circumscribed ellipse may be established by precisely similar reasoning.

NOTE 1.—If the curve (2) be an inscribed circle,

$$\alpha_1 = \beta_1 = \gamma_1;$$

then, by (4),

$$a^{-1}\left(\frac{m}{b} + \frac{n}{c}\right) = b^{-1}\left(\frac{n}{c} + \frac{l}{a}\right) = c^{-1}\left(\frac{l}{a} + \frac{m}{b}\right),$$

and if $2s = a + b + c$,

$$\frac{l}{a\,(s-a)} = \frac{m}{b\,(s-b)} = \frac{n}{c\,(s-c)},$$

or

$$\frac{l}{\cos^2\frac{1}{2}A} = \frac{m}{\cos^2\frac{1}{2}B} = \frac{n}{\cos^2\frac{1}{2}C},$$

and the equation is

$$\alpha^{\frac{1}{2}}\cos\tfrac{1}{2}A + \beta^{\frac{1}{2}}\cos\tfrac{1}{2}B + \gamma^{\frac{1}{2}}\cos\tfrac{1}{2}C = 0.$$

If the circle touch AB and AC produced, α is negative, and the equation is

$$0 = (-\alpha)^{\frac{1}{2}}\cos\tfrac{1}{2}A + \beta^{\frac{1}{2}}\sin\tfrac{1}{2}B + \gamma^{\frac{1}{2}}\sin\tfrac{1}{2}C.$$

NOTE 2.—If p_1, p_2, p_3 be the perpendiculars from A, B, C on any straight line, its equation will be

$$a\alpha p_1 + b\beta p_2 + c\gamma p_3 = 0.$$

NOTE ON PRECEDING SOLUTION.

[The equations to the maximum and minimum ellipse respectively inscribed in, and circumscribed about, a given triangle, may also be obtained very simply from the consideration, that if the triangle be projected so as to become equilateral, the ellipse must become a circle. Now it is a property of an equilateral triangle inscribed in a circle, that the tangents at its angles are parallel to the sides respectively opposite to them. This property will not be affected by projection. But if the equation to the circumscribed ellipse be

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

the equation to the tangent at $(\beta\gamma)$ will be

$$n\beta + m\gamma = 0;$$

and in order that this may be parallel to $\alpha = 0$, it is necessary that we have

$$\frac{n}{b} = \frac{m}{c},$$

or
$$mb = nc = (\text{by symmetry}) = la.$$

Hence the equation to the minimum circumscribed ellipse becomes

$$(a\alpha)^{-1} + (b\beta)^{-1} + (c\gamma)^{-1} = 0.$$

[redacted]

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EDITED BY

N. M. FERRERS, M.A.,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

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THE

CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL.

NOTE ON THE INTERNAL PRESSURES AT ANY POINT WITHIN A BODY AT REST.

By J. B. Phear, Fellow of Clare Hall.

One of the theorems respecting the internal pressures at any point within a body at rest, to which Mr. Rankine alluded in this *Journal* (No. xxv. p. 54), is so elegant that it seems to deserve a place in our University mathematics.

The following proof of it differs slightly from that given by M. Lamé, in his *Leçons sur la Théorie Mathématique de l'élasticité des corps solides*, and the tensions or internal pressures to which it refers are only restricted to the meaning given by the annexed definition.

Def. Suppose a body to be at rest under the action of any external forces, its constituent particles being maintained in their positions of equilibrium by their mutual actions; then if the portion of the body cut off by any imaginary plane be removed, the equilibrium of the particles in the remaining portion may evidently still be preserved undisturbed by applying to each indefinitely small element ϖ of the plane surface, so formed, a force ϖE having the requisite magnitude and direction: designating the position of ϖ by the letter M, the force E is taken to be the measure per unit of surface of the internal tension of the body at the point M exerted upon the plane area ϖ.

This tension E will generally vary at the same point M for different positions of ϖ, and the theorem in question asserts that if a line be drawn from M to represent it in magnitude and direction, the locus of its extremity will be an ellipsoid; and further, that the plane of ϖ is diametral to those chords of another given surface of the second order, having M for its centre, which are parallel to the direction of E.

Although Mr. Rankine in the before-mentioned paper has enunciated the two fundamental theorems upon which all investigations in this subject are based, and has sketched out the nature of their proof, it will be convenient to re-state them here: they are,

(α) If three elementary planes be taken at right angles to each other through the same point within any body, and if $X_1 Y_1 Z_1$, $X_2 Y_2 Z_2$, $X_3 Y_3 Z_3$ represent the resolved parts of the tensions upon them, estimated according to the preceding definition, in directions parallel to their lines of intersection taken as coordinate axes of x, y, and z respectively, six only of these nine resolved forces are independent; for, the first plane being that perpendicular to the axis of x, the second that perpendicular to the axis of y, and similarly for z; then $Y_3 = Z_2,\ Z_1 = X_3,\ X_2 = Y_1.$

Of these forces the three which are normal to their respective planes may very well be designated by the symbols N_1, N_2, N_3, while the three others, as being tangential to their respective planes, may be distinguished by the letters T_1, T_2, T_3; so that the resolved tensions will be

N_1, T_3, T_2 for the plane perpendicular to axis of x,
T_3, N_2, T_1 . . . . y,
T_2, T_1, N_3 . . . . z.

(β) If E, E' represent the tensions called into action upon

If it be possible that ρ should ever be perpendicular to ϖ, then would $l' = l$, $m' = m$, $n' = n$, and equations (A) would become

$$\left.\begin{aligned} (N_1 - \rho)\, l + T_3 m + T_2 n &= 0 \\ T_3 l + (N_2 - \rho)\, m + T_1 n &= 0 \\ T_2 l + T_1 m + (N_3 - \rho)\, n &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (B),$$

which, together with

$$l^2 + m^2 + n^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

would form four equations for determining ρ and its direction cosines.

Eliminating l, m, and n from (B), we obtain

$$(N_1 - \rho)(N_2 - \rho)(N_3 - \rho) - (N_1 - \rho)\, T_1^2 - (N_2 - \rho)\, T_2^2 \\ - (N_3 - \rho)\, T_3^2 + 2T_1T_2T_3 = 0 \ldots\ldots (2),$$

a well-known cubic which necessarily gives three real roots to ρ; call these ρ_1, ρ_2, ρ_3; the corresponding values of l, m, and n may be easily found from equations (B) and (1).

We see then that there are three, and in general only three, planes passing through any point O of a solid, the tensions upon which are perpendicular to them; these planes, and consequently their tensions ρ_1, ρ_2, ρ_3, are at right angles to each other; for if we solve equations (A) with respect to l, m, and n, we get

$$\left.\begin{aligned} \frac{\lambda}{\rho}\, l &= A_1 l' + B_3 m' + B_2 n' \\ \frac{\lambda}{\rho}\, m &= B_3 l' + A_2 m' + B_1 n' \\ \frac{\lambda}{\rho}\, n &= B_2 l' + B_1 m' + A_3 n' \end{aligned}\right\} \ldots\ldots\ldots\ldots (C),$$

λ, A_1, A_2, A_3, B_1, B_2, B_3 being put for certain combinations of N and T which arise from the operation; and these equations shew that the plane of ϖ is diametral to the system of chords drawn parallel to ρ in the surface of the second order, whose equation is

$$A_1 x^2 + A_2 y^2 + A_3 z^2 + 2B_1 yz + 2B_2 zx + 2B_3 xy + K = 0 \ldots (3),$$

and it is the property of such a surface that the three planes, which bisect their chords at right angles, are perpendicular to each other.

These tensions ρ_1, ρ_2, ρ_3 may be very conveniently termed the principal tensions at O. Take their directions for new axes of coordinates, and let x, y, z be the coordinates of P when referred to them, and l_1, m_1, n_1 the direction-cosines of ϖ, then, by theorem (β), equating the projection of ρ upon the axis of x to that of ρ_1 upon the normal to ϖ,

similarly
$$\left.\begin{aligned} x &= l_1\rho_1 \\ y &= m_1\rho_2 \\ z &= n_1\rho_3 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(D);$$

therefore
$$\frac{x^2}{\rho_1^2} + \frac{y^2}{\rho_2^2} + \frac{z^2}{\rho_3^2} = 1\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

which shews that ρ, the magnitude of the tension upon any elementary plane ϖ at O, is represented by the portion of OP cut off by the ellipsoid whose equation is (4) and whose semiaxes have the same magnitude and direction as $\rho_1\rho_2\rho_3$, which represent the principal tensions at O.

It has been already remarked that OP is always parallel to the systems of chords, which are bisected by the plane of ϖ, in that surface of the second order whose equation referred to the first set of axes was (3); ϖ is therefore parallel to the tangent plane at the point where OP meets this surface.

The direction-cosines of the new axes are given by the

ordinates whose direction cosines are given by (E), produces, as is well known, an equation not involving the products of the coordinates and one where the coefficients of x^2, y^2, z^2 are the roots of (5); therefore (3), by the alteration of co-ordinate axes, becomes

$$\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\frac{z^2}{\rho_3}+\frac{K}{\lambda}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

If $\rho_1\rho_2\rho_3$, the roots of (2), be all of the same sign, *i.e.* if the principal tensions at O be all, either compressive or tensile, (6) must be an ellipsoid, and therefore ρ always, whatever be the position of ϖ, represents a force of the same kind as ρ_1, ρ_2, ρ_3, whether compressive or tensile.

If however the roots of (3) be not all of the same sign, then K in (6) may be either positive or negative, and therefore (6) must be taken to represent at once two hyperboloids, the one of two, the other of one sheet, both having a common asymptotic cone whose equation is

$$\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\frac{z^2}{\rho_3}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

In this case when OP meets the hyperboloid of two sheets, the force represented by it must be of the same kind as those represented by the two *like* axes of the surface, and when it meets the other hyperboloid the force must be the same as that represented by the single *unlike* axis: in intermediate positions to these OP must lie upon the cone (7), and be therefore in the plane of ϖ, which is always parallel to the tangent-plane at the point where OP meets (6).

If one of the roots of (2), as ρ_3, $=0$, then there is one plane through O upon which there is no tension, and consequently, by theorem (β), the directions of the tensions upon all other planes through O must lie in this plane; hence x, y, z being as before the coordinates of P referred to the directions of the principal tension as axes, and l_1, m_1, n_1 the direction-cosines of ϖ, $z=0$: but $\dfrac{z}{\rho_3}=n_1=\cos\gamma$, if γ represent the inclination of ϖ to the plane of xy; therefore

$$\frac{x^2}{\rho_1^2}+\frac{y^2}{\rho_2^2}+\cos^2\gamma=1,$$

or

$$\frac{x^2}{\rho_1^2}+\frac{y^2}{\rho_2^2}=\sin^2\gamma \quad\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

the equation to an ellipse in the plane of xy, representing the locus of the extremity of OP for all such planes ϖ as are inclined at an angle γ to plane of xy.

Also the direction-cosines of the line in which ϖ intersects the plane of xy are $l_1 m_1$, *i.e.* (from (D)) $\frac{x}{\rho_1}$, $\frac{y}{\rho_2}$, and they are therefore proportional to $\frac{x_1}{\rho_1}$ and $\frac{y_1}{\rho_2}$ respectively, where $x_1 y_1$ are the coordinates of the point where OP produced meets the conic section whose equation is

$$\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} = K' \text{ ....(9).}$$

Hence this line of intersection is parallel to the tangent to (9) at point $x_1 y_1$, and the relation between OP and ϖ is completely determined.

If ρ_1 and ρ_2 have the same signs, (9) is an ellipse, but if opposite signs it is an hyperbola and its conjugate, having the same asymptotes, because K may be either positive or negative; the nature of the pressure represented by OP may be obtained by reasoning similar to that just employed in regard to the hyperboloid (6).

If two of the roots of (2), as ρ_3 and ρ_2, each $= 0$, then the theorem (β) shews that the tension upon every small plane

SOLUTIONS OF PROBLEMS.

(Prob. 2, vol. VIII. p. 188).

If from a point in the circumference of a vertical circle two heavy particles be successively projected along the curve, their initial velocities being equal and either in the same or in opposite directions, the subsequent motion will be such that a straight line joining the particles at any instant will touch a given circle.

Note. The particles are supposed not to interfere with each other's motion.

THE direct analytical proof would involve the properties of elliptic integrals, but it may be made to depend upon the following geometrical theorems.

(1) If from a point in one of two circles a right line be drawn cutting the other, the rectangle contained by the segments so formed is double of the rectangle contained by a line drawn from the point perpendicular to the *radical axis* of the two circles, and the line joining their centres.

The radical axis is the line joining the points of intersection of the two circles. It is always a real line, whether the points of intersection of the circles be real or imaginary, and it has the geometrical property—that if from any point on the radical axis, straight lines be drawn cutting the circles, the rectangle contained by the segments formed by one of the circles is equal to the rectangle contained by the segments formed by the other.

The analytical proof of these propositions is very simple, and may be resorted to if a geometrical proof does not suggest itself as soon as the requisite figure is constructed.

If A, B be the centres of the circles, P the given point in the circle whose centre is A, a line drawn from P cuts the first circle in p, the second in Q and q, and the radical axis in R. If PH be drawn perpendicular to the radical axis, then

$$PQ.Pq = 2AB.HP.$$

COR. If the line be drawn from P to *touch* the circle in T, instead of cutting it in Q and q, then the square of the tangent PT is equal to the rectangle $2AB.HP$.

Similarly, if ph be drawn from p perpendicular to the radical axis

$$pT^2 = 2AB.hp.$$

Hence, if a line be drawn touching one circle in T, and cutting the other in P and p, then

$$(PT)^2 : (pT)^2 :: HP : hp.$$

(2) If two straight lines touching one circle and cutting another be made to approach each other indefinitely, the small arcs intercepted by their intersections with the second circle will be ultimately proportional to their distances from the point of contact.

This result may easily be deduced from the properties of the similar triangles $P'PT$ and $p'pT$.

Cor. If particles P, p be constrained to move in the circle A, while the line Pp joining them continually touches the circle B, then the velocity of P at any instant is to that of p as PT to pT; and conversely, if the velocity of P at any instance be to that of p as PT to pT, then the line Pp will continue to be a tangent to the circle B.

Now let the plane of the circles be vertical and the radical axis horizontal, and let gravity act on the particles P, p. The particles were projected from the same point with the same velocity. Let this velocity be that due to the depth of the point of projection below the radical axis. Then the square of the velocity at any other point will be proportional to the perpendicular from that point on the radical axis; or, by the corollary to (1), if P and p be at any time at the extremities of the line PTp, the square of the velocity of P will be to the square of the velocity of p as PH to ph, that is, as $(PT)^2$ to $(pT)^2$. Hence, the velocities of P and p are in the proportion of PT to pT, and therefore, by the

and as this is true at any instant, any number of such polygons may be formed.

Hence, the following geometrical theorem is true:

"If two circles be such that n lines can be drawn touching one of them and having their successive intersections, including that of the last and first, on the circumference of the other, the construction of such a system of lines will be possible, at whatever point of the first circle we draw the first tangent."

(Prob. 3, vol. VIII. p. 188).

A transparent medium is such that the path of a ray of light within it is a given circle, the index of refraction being a function of the distance from a given point in the plane of the circle.

Find the form of this function and shew that for light of the same refrangibility—

(1) The path of *every ray within the medium* is a circle.

(2) All the rays proceeding from any point in the medium will meet accurately in another point.

(3) If rays diverge from a point without the medium and enter it through a spherical surface having that point for its centre, they will be made to converge accurately to a point within the medium.

Lemma I. Let a transparent medium be so constituted, that the refractive index is the same at the same distance from a fixed point, then the path of any ray of light within the medium will be in one plane, and the perpendicular from the fixed point on the tangent to the path of the ray at any point will vary inversely as the refractive index of the medium at that point.

We may easily prove that when a ray of light passes through a spherical surface, separating a medium whose refractive index is μ_1 from another where it is μ_2, the plane of incidence and refraction passes through the centre of the sphere, and the perpendiculars on the direction of the ray before and after refraction are in the ratio of μ_2 to μ_1. Since this is true of any number of spherical shells of different refractive powers, it is also true when the index of refraction varies continuously from one shell to another, and therefore the proposition is true.

Lemma II. If from any fixed point in the plane of a circle, a perpendicular be drawn to a tangent at any point of the circumference, the rectangle contained by this perpendicular and the diameter of the circle is equal to the square of the line joining the point of contact with the fixed

point, together with the rectangle contained by the segments of any chord through the fixed point.

Let APB be the circle, O the fixed point; then

$$OY.PR = OP^2 + AO.OB.$$

Produce PO to Q, and join QR, then the triangles OYP, PQR are similar; therefore

$$OY.PR = OP.PQ$$
$$= OP^2 + OP.OQ;$$

$$\therefore \quad OY.PR = OP^2 + AO.OB.$$

If we put in this expression $AO.OB = a^2$,

$$PO = r, \quad OY = p, \quad PR = 2\rho,$$

it becomes

$$2p\rho = r^2 + a^2,$$
$$\rho p = \frac{r^2 + a^2}{2\rho}.$$

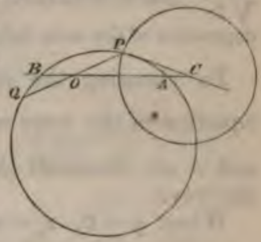
To find the law of the index of refraction of the medium, so that a ray from A may describe the circle APB, μ must be made to vary inversely as p by Lemma I.

Let $AO = r_1$, and let the refractive index at $A = \mu_1$; then generally

$$\mu = \frac{C}{p} = \frac{2C\rho}{r^2 + a^2};$$

a result independent of r_1. This shews that any point A' may be taken as the origin of the ray instead of A, and that the path of the ray will still be circular, and will pass through another point B' on the other side of O, such that $OB' = \dfrac{a^2}{OA'}$.

Next, let CP be a ray from C, a point without the medium, falling at P on a spherical surface whose centre is C.



Let O be the fixed point in the medium as before. Join PO, and produce to Q till $OQ = \dfrac{a^2}{OP}$. Through Q draw a circle touching CP in P, and cutting CO in A and B; then PBQ is the path of the ray within the medium.

Since CP touches the circle, we have

$$CP^2 = CA.CB,$$

$$= (CO - OA)(CO + OB);$$

but $$OA = \frac{a^2}{OB};$$

therefore $$CP^2 = CO^2 + CO\left(OB - \frac{a^2}{OB}\right) - a^2,$$

an equation whence OB may be found, B being the point in the medium through which all rays from C pass.

NOTE. The possibility of the existence of a medium of this kind possessing remarkable optical properties, was suggested by the contemplation of the structure of the crystalline lens in fish; and the method of searching for these properties was deduced by analogy from Newton's *Principia*, lib. I. prop. VII.

It would require a more accurate investigation into the law of the refractive index of the different coats of the lens to test its agreement with the supposed medium, which is an optical instrument theoretically perfect for homogeneous light, and might be made achromatic by proper adaptation of the dispersive power of each coat.

On the other hand, we find that the law of the index of refraction which would give a minimum of aberration for a sphere of this kind placed in water, gives results not discordant with facts, so far as they can be readily ascertained.

(Prob. 4, vol. VIII. p. 188.)

A series of waves, which at sea are twenty feet long from crest to crest, and three feet high from hollow to crest, break on a shore which is parallel to their breadth. How much heat is developed per hour on each foot of the shore, and how much would the temperature of 180 cubic feet of fresh water be raised by receiving an equal quantity? [The form of a wave at sea, of which the height is a small fraction of l, its length, is approximately the curve of sines; its velocity of propagation is $\sqrt{\frac{gl}{2\pi}}$; and its mechanical energy is half that of a double elevation and depression of the same form without velocity.]

By choosing the most convenient origin and axes, the equation of the wave will be of the form $y = k \sin \dfrac{x}{a}$; where k and a are constants depending on the height and length of the wave.

When $y = 0$, $x_0 = am\pi$, and as between two successive crests the wave meets the axis of x twice, we have

$$l = a(m\pi) - a(m-2)\pi = 2a\pi.$$

Also when $x = a(m+\frac{1}{2})\pi$, $y = k$, a maximum;

and when $x = a(m+\frac{3}{2})\pi$, $y = -k$, a minimum;

so that denoting the height of the wave from hollow to crest by h, $k = \frac{1}{2}h$, and the equation of the wave becomes

$$y = \tfrac{1}{2}h \sin \frac{2\pi x}{l}.$$

of the work that can be obtained by this inversion, in other words is the mechanical energy of a standing wave.

Denoting this by $2W$, we have

$$2W = 2Py_1 = 2\int_0^{\frac{1}{2}l} dx \int_0^y py\, dy,$$

and as 62·447 pounds is the weight of a cubic foot of distilled water, and 1·027 is the specific gravity of sea water,

$$p = 62{\cdot}447 \times 1{\cdot}027,$$

$$2W = p\int_0^{\frac{1}{2}l} y^2 dx = ph^2 \int_0^{\frac{1}{2}l}\left(\sin\frac{2\pi x}{l}\right)^2 dx,$$

$$= ph^2\left\{-\frac{l}{2\pi}\sin\frac{2\pi x}{l}\cos\frac{2\pi x}{l} + \int\left(\cos\frac{2\pi x}{l}\right)^2 dx\right\}_0^{\frac{1}{2}l},$$

$$= ph^2\left\{-\frac{l}{4\pi}\sin\frac{2\pi x}{l}\cos\frac{2\pi x}{l} + \frac{x}{2} + C\right\}_0^{\frac{1}{2}l},$$

$$= \tfrac{1}{4}ph^2 l \text{ in foot pounds.}$$

Mechanical energy of one *moving* wave $= W = \frac{1}{8}ph^2l$. Thermal equivalent of the energy of one moving wave $= \dfrac{1}{8}\dfrac{ph^2l}{J}$, J being the number of foot pounds of work equivalent to one unit of heat. Again, velocity of wave $= \sqrt{\dfrac{gl}{2\pi}}$. Period $= \dfrac{l}{v}$. Number of waves per hour $= \dfrac{1}{l}\sqrt{\dfrac{gl}{2\pi}} \times 60^2$. Hence if units of heat developed per hour opposite one foot of the shore $= H$,

$$H = \frac{1}{8}\frac{ph^2l}{J}\cdot\frac{1}{l}\sqrt{\frac{gl}{2\pi}} \times 60^2,$$

$$H = \sqrt{\frac{gl}{2\pi}}\cdot\frac{60^2h^2p}{8J}.$$

If a quantity equal to this be distributed among 180 cubic feet of fresh water, or $180 \times 62{\cdot}447$ pounds, each pound will obtain

$$\sqrt{\frac{gl}{2\pi}}\cdot\frac{60^2h^2p}{8J \times 180 \times 62{\cdot}447} = \sqrt{\frac{gl}{2\pi}}\cdot\frac{60^2h^2 \times 1{\cdot}027}{8 \times J \times 180} \text{ units;}$$

so that if T_0 be the original temperature of the fresh water, and T its temperature at the end of the hour,

$$T = T_0 + \sqrt{\frac{gl}{2\pi}}\cdot\frac{60^2h^2 \times 1{\cdot}027}{8J \times 180}.$$

Ex. If $l = 20$ feet and $h = 3$ feet,

$$H = \sqrt{\left(\frac{32\cdot 2 \times 40}{2 \times \pi}\right)} \cdot \frac{60^2 \times 3^2 \times 62\cdot 447 \times 1\cdot 027}{8 \times J}.$$

From the experiments of Mr. Joule, J is found to be about 772, if we employ the Fahrenheit unit of heat, and consequently about 1390 for the Centigrade. Employing these numbers successively in the formula, we have

$$H = 3406\cdot 22 \text{ units Faht.}$$
$$= 1891\cdot 80 \text{ units Cent.}$$

From the other formula we find

$$T = T_0 + \cdot 3030^\circ \text{ Faht.}$$
$$T = T_0 + \cdot 1683^\circ \text{ Cent.}$$

(Prob. 6, vol. VIII. p. 189.)

Find the amount of "potential energy" (mechanical effect of such a kind as that of weights raised) that can be obtained by equalizing the temperature of two bodies given at different uniform temperatures, and determine the common temperature to which they are reduced.

Ex. 1. Let the bodies be of equal constant thermal capacities, and let their temperatures be 0° and 100° respectively.

Ex. 2. Let the bodies be masses W and W' of water, and let the temperatures at which they are given be 15° and 20° respectively.

considered as a source as we give to it as a refrigerator, we shall obtain exactly as much mechanical effect by employing both engines, as by employing one with source A and refrigerator B. In short if the above relation were not true, then suppose the two engines produced more mechanical effect than the one, the two could then be made to drive the one backwards, and raise weights in addition, and we should be driven to the absurdity of deriving mechanical effect by drawing heat from the refrigerator.

(1) Consider then an engine drawing heat from A at a variable temperature, and giving part of it out to C at temperature θ, and obtaining as much work as possible by the transference.

To diminish the temperature of A from t to $t - dt$, $\phi' t\, dt$ units must be taken away, hence the

$$\text{whole heat given out by } A = \int_\theta^T \phi' t\, dt = (\phi T - \phi\theta).$$

Mechanical effect gained by the fall in temperature of A from t to $t - dt$

$$= J\phi' t\, dt \frac{t - \theta}{t + \frac{1}{E}}.$$

$\phi' t\, dt$ corresponding to H in the formula given in the last number of this *Journal* by Professor Thomson.

Total mechanical effect gained by allowing the temperature of A to fall from T to $\theta = P_1$ suppose, then we have

$$P_1 = J\int_\theta^T \phi' t\, dt \frac{t - \theta}{t + \frac{1}{E}}.$$

Why it should be $\int_\theta^T$ and not $\int_T^\theta$ needs no remark, if attention be paid to the signs.

$$P_1 = J\int_\theta^T \phi' t\, dt - J\int_\theta^T \phi' t\, dt \frac{\theta + \frac{1}{E}}{t + \frac{1}{E}}$$

$$= J\{\phi T - \phi\theta\} - J\left(\theta + \frac{1}{E}\right)(\phi_1 T - \phi_1 \theta),$$

by supposing $\displaystyle\int \frac{\phi' t\, dt}{t + \frac{1}{E}} = \phi_1 t + C;$

but we found before $(\phi T - \phi\theta)$ is the number of units given out by A, and therefore the number of units given out to C must be

$$\left(\theta + \frac{1}{E}\right)(\phi_1 T - \phi_1 \theta).$$

(2) Consider now an engine having C at constant temperature θ as source, and B at variable temperature as refrigerator. Consider the circumstances when the temperature of B is t. To raise the temperature of B from t to $t + dt$, $\psi' t\, dt$ units must be added.

Whole heat received by $B = \int_{T_1}^{\theta} \psi' t\, dt = (\psi\theta - \psi T_1)$.

As in raising B from t to $t + dt$, $\psi' t\, dt$ units are given to it, (the refrigerator) $\psi' t\, dt \dfrac{\theta + \frac{1}{E}}{t + \frac{1}{E}}$ units must have been supplied by the source C. Substituting this for H in the formula, we have

Mechanical effect gained in raising the temperature of B from t to $t + dt$

$$\theta + \frac{1}{E} \quad \theta - t$$

But as the whole heat given out to B is, as we saw,

$$=(\psi\theta-\psi T_1),$$

the whole heat given out by C must have been

$$\left(\theta+\frac{1}{E}\right)(\psi_1\theta-\psi_1 T_1).$$

Now we have supposed the body C of such a nature, that its temperature is constant during the whole operation; but whether we suppose that it receives first all the heat it gets from A, and then gives off to B, or that both engines work together, and that the one takes out as fast as the other gives in, we have at least this condition, that *the amount of heat C gives out is exactly equal to the amount it receives*, for on no other supposition could we substitute two engines for one. This gives us the means of determining θ, for putting the quantity given by the first engine equal to that taken away by the second, we have

$$\left(\theta+\frac{1}{E}\right)(\phi_1 T-\phi_1\theta)=\left(\theta+\frac{1}{E}\right)(\psi_1\theta-\psi_1 T_1),$$

or

$$\phi_1\theta+\psi_1\theta=\phi_1 T+\psi_1 T_1,$$

from which θ may be obtained in any case.

Hence also we conclude that the terms multiplied by $\left(\theta+\frac{1}{E}\right)$ in the expressions for P_1 and P_2, given above, disappear in the addition, and we have the following expression for the required amount of "potential energy,"

$$P=P_1+P_2=J\{(\phi T-\phi\theta)-(\psi\theta-\psi T_1)\}.$$

Ex. 1. Let the thermal capacities be equal and constant.

$$\phi' t=\psi' t=C;$$

then

$$\phi T-\phi\theta=C(T-\theta),$$

and

$$\phi_1 T-\phi_1\theta=C\log\frac{T+\frac{1}{E}}{\theta+\frac{1}{E}},$$

$$\psi\theta-\psi T_1=C(\theta-T_1),$$

$$\psi_1\theta-\psi_1 T_1=C\log\frac{\theta+\frac{1}{E}}{T_1+\frac{1}{E}}.$$

Substituting, we have

$$P = JC(T + T_1 - 2\theta),$$

and
$$\log \frac{T + \frac{1}{E}}{\theta + \frac{1}{E}} = \log \frac{\theta + \frac{1}{E}}{T_1 + \frac{1}{E}};$$

whence
$$\left(T + \frac{1}{E}\right)\left(T_1 + \frac{1}{E}\right) = \left(\theta + \frac{1}{E}\right)^2.$$

which gives $\theta = -\frac{1}{E} \pm \sqrt{\left\{\left(T + \frac{1}{E}\right)\left(T_1 + \frac{1}{E}\right)\right\}}$,

$$P = JC\left[T + T_1 + \frac{2}{E} - 2\sqrt{\left\{\left(T + \frac{1}{E}\right)\left(T_1 + \frac{1}{E}\right)\right\}}\right],$$

$$= JC\left\{\sqrt{\left(T + \frac{1}{E}\right)} - \sqrt{\left(T_1 + \frac{1}{E}\right)}\right\}^2.$$

Def. If we have a body at t Cent., $t + \frac{1}{E}$ is called its absolute temperature.

From the preceding equations we deduce the curious result, that the absolute common temperature is a mean proportional between the absolute original temperatures of A and B. The formula for P is also interesting.

$$P = J\{W(T-\theta) - W'(\theta - T_1)\},$$
$$= J\{WT + W'T_1 - (W+W')\,\theta\}.$$

And $$W \log \frac{T + \frac{1}{E}}{\theta + \frac{1}{E}} = W' \log \frac{\theta + \frac{1}{E}}{T_1 + \frac{1}{E}},$$

$$\left(T + \frac{1}{E}\right)^{W} \left(T_1 + \frac{1}{E}\right)^{W'} = \left(\theta + \frac{1}{E}\right)^{W+W'}.$$

Let $T = 20, \quad T_1 = 15, \quad W = 50, \quad W' = 40.$

$$\theta = -\,273{\cdot}22404 + \{293{\cdot}22404^5 \times 288{\cdot}22404^4\}^{\frac{1}{9}},$$
$$= -\,273{\cdot}22404 + 290{\cdot}9912,$$
$$= 17{\cdot}7672.$$
$$P = 1390 \times {\cdot}952,$$
$$= 1323{\cdot}28.$$

C. A. S.

EXERCISES IN THE HYPERDETERMINANT CALCULUS.

By the Rev. George Salmon.

The readers of this *Journal* have been made acquainted, by several recently-published papers of Mr. Sylvester, with various theorems connecting together different derivatives of homogeneous functions of several variables. A method commonly employed in discovering such relations is the method of "Canonical Forms," in which, when it has been ascertained what is the simplest form to which the general expression is capable of being reduced, the relations found to be true between the derivatives of this simplest form are known to be true in general. Even however when applied to the equations of curves of the fourth degree the simplest expressions become so complicated that it is difficult to work with them, and at all events this method has never appeared to me thoroughly satisfactory, it rather shewing that a relation is true than making it plain why it is true. A recent perusal of Mr. Cayley's remarkable

paper on Linear Transformations (*Journal*, vol. I. p. 104), which I am ashamed to say I never studied carefully before, leads me to think that by the help of the notation there employed, much further progress in this theory may be expected to be made. His notation has the advantage of not only shewing that each derivative he deals with is an invariant or covariant as the case may be, but also of completely identifying each particular derivative and affording the means of comparing it with others. It is true that as the degree of the derivative increases, its symbolical expression becomes more and more complicated; still I should hope that if more attention were directed to this method, simpler means of manipulating these expressions would be discovered. In the following pages however no new principle is made use of; they contain merely selections from different examples to which I applied Mr. Cayley's method when I became acquainted with it, and which I publish in the hope that they may induce others to turn their attention to the same subject.

I commence with the case of homogeneous functions of two variables. In the paper just referred to Mr. Cayley proves that if in a function of equations between several systems of pairs of variables x_1, y_1, x_2, y_2, &c., the analogous variables be transformed in each case by the same linear substitutions; then (as is evident enough) the differentials $\frac{d}{dx_1}, \frac{d}{dy_1}, \frac{d}{dx_2}, \frac{d}{dy_2}$, &c., will also be transformed by linear substitutions; that consequently any symbol of operation of the form $\left(\frac{d}{dx_1}\cdot\frac{d}{dy_2} - \frac{d}{dx_2}\cdot\frac{d}{dy_1}\right)^n$ will be unchanged (to a constant coefficient at least) by linear transformations. For the operative symbol just mentioned he uses the abbreviation $\overline{12}^n$. His method then of obtaining the derivatives of a given function U which are unaltered by linear substitution, is to multiply together any number of times quantities $U_1 U_2 U_3$ &c. (U_1 being the function U with x_1, y_1 substituted for x and y); to operate on the product with the product of any number of the symbols $(12)^\alpha (23)^\beta (14)^\gamma$ &c., and *after the differentiation* to drop all the suffixes and suppose all the variables x_1y_1, x_2y_2 equal to x and y. If we operated on a product of quantities *not* the same, suppose $U_1 V_2 W_3$, &c., we should then obtain covariant derivatives not of a single function, but of a system of different functions. In what follows we only speak of derivatives of a single function,

and therefore drop all mention of the quantity operated on, and write $(12)^n$ instead of at full length $(12)^n U_1 U_2$. The principal equation used for transforming these derivatives is the following, given by Mr. Cayley (*Journal*, vol. VIII. p. 121) and which is easily seen to be identically true,

$$\Xi_1(23) + \Xi_2(31) + \Xi_3(12) = 0 \;\ldots\ldots\ldots\ldots(1),$$

where $$\Xi_1 = x\frac{d}{dx_1} + y\frac{d}{dy_1}, \text{ \&c.}$$

One important result of the above equation is that the expression for any derivative may be so transformed that the highest power of any factor (12) may be an even one. For the derivative is unaltered if we interchange the figures 1 and 2, therefore

$$12^{2m+1}\phi_1 = -12^{2m+1}\phi_2 = \tfrac{1}{2}12^{2m+1}(\phi_1 - \phi_2),$$

and by the help of equation (1) the quantity $\phi_1 - \phi_2$ can be transformed so as to be divisible by (12).

Thus, for example, the derivative $(12)^m(13)$, where m is odd, is transformed as follows,

$$2\Xi_2(12)^m(13) = (12)^m\{\Xi_2(13) - \Xi_1(23)\} = \Xi_3(12)^{m+1};$$

or if n be the degree of U the function operated on,

$$2(n-m)(12)^m(13) = n(12)^{m+1} \;\ldots\ldots\ldots\ldots(2).$$

It is to be noted here and elsewhere that the possibility of making one of the figures disappear from the symbol for any derivative indicates that that derivative is divisible by U.

Having premised this, I proceed, as the first example, to examine a theorem closely connected with one hereafter to be discussed. "The result of substituting in the reciprocal of the function, $\dfrac{dU}{dx}$ for ξ and $\dfrac{dU}{dy}$ for η, is divisible by U," (*Journal*, vol. VII. p. 194). In other words, it is required to calculate $\left(M\dfrac{d}{dx} - L\dfrac{d}{dy}\right)^n U$, where $L = \dfrac{dU}{dx}$ and $M = \dfrac{dU}{dy}$. Writing for shortness the quantity in question Q_n, it is easy to see that

$$Q_2 = (12)(13),\; Q_3 = (12)(13)(14),\; Q_4 = (12)(13)(14)(15), \text{ \&c.},$$

a particular case of the following more general theorem, "If in any contravariant we substitute $\dfrac{dU}{dx}$ for ξ &c., the

result will be a covariant whose symbolical expression is obtained by substituting in every factor of the symbol for the contravariant a new figure for the contravariant letter. This will be better understood from the applications of it made afterwards.

Ex. 1. Let us go on now to calculate Q_2 &c. Squaring equation (1) and assembling the terms which only differ by interchange of figures, we have

$$\Xi_3^2(12)^2 - 2\Xi_2\Xi_3(12)(13) = 0,$$

or

$$2(n-1)^2 Q_2 = n(n-1)B_2 U,$$

where I use Mr. Cayley's abbreviation B_n for $(12)^n U$.

Ex. 2. To calculate Q_3. Squaring the equation

$$\Xi_3(12) - \Xi_2(13) = -\Xi_1(23),$$

we have $2\Xi_2\Xi_3(12)(13) = \Xi_3^2(12)^2 + \Xi_2^2(13)^2 - \Xi_1^2(23)^2$,

$$2(n-1)^2 Q_3 = 2\Xi_3^2(12)^2(14) - \Xi_1^2(23)^2(14);$$

and the last term on the right-hand side vanishing identically, we have

$$(n-1)Q_3 = n(12)^2(13)U,$$

a result obtained by Mr. Cayley and also by M. Burchardt, (*Journal*, vol. VII. p. 194).

$(12)^2(13)$ is a derivative in its simplest form. When U is

$\therefore\ 2(n-2)^2(n-3)^2(12)^2(13)^2 = n(n-1)(n-2)(n-3)UB_4 \ldots (3);$

$\therefore\ 4(n-1)^4(n-2)(n-3)Q_4$
$$= -3n(n-1)(n-2)^2(n-3)^2UB_2^{\,2} + 2n^2(n-1)^2U^3B_4.$$

Ex. 5. To calculate Q_5.

Multiply by (16) the product of the first two equations of the last Example, and

$$4(n-1)^4Q_5 = -4\Xi_1^{\,2}\Xi_5^{\,2}(23)^2(14)^2(16) + 4\Xi_3^{\,2}\Xi_5^{\,2}(12)^2(14)^2(16),$$

or $(n-1)^4Q_5 = -n(n-1)(n-3)(n-4)B_2U\{(12)^2(13)\}$
$$+ n^2(n-1)^2U^2\{(12)^2(13)^2(14)\}.$$

The last term is further reducible by equation (3), which gives

$$2\Xi_2^{\,2}\Xi_3^{\,2}(12)^2(13)^2(14) = \Xi_3^{\,4}(12)^4(14),$$

or $2(n-2)^2(n-3)^2U^2\{(12)^2(13)^2(14)\}$
$$= n(n-1)(n-2)(n-3)U^3(12)^4(13).$$

Ex. 6. To calculate Q_6.

Multiply together

$$-2\Xi_2\Xi_3(12)(13) = \Xi_1^{\,2}(23)^2 - 2\Xi_3^{\,2}(12)^2,$$
$$-2\Xi_4\Xi_5(14)(15) = \Xi_1^{\,2}(45)^2 - 2\Xi_5^{\,2}(14)^2,$$
$$-2\Xi_6\Xi_7(16)(17) = \Xi_1^{\,2}(67)^2 - 2\Xi_7^{\,2}(16)^2,$$

and we have

$$-8(n-1)^6Q_6 = n(n-1)(n-2)(n-3)(n-4)(n-5)(B_2^{\,3}-6B_2^{\,3})U$$
$$+ 12n^2(n-1)^2(n-4)(n-5)U^2B_2(12)^2(13)^2$$
$$- 8n^3(n-1)^3U^3(12)^2(13)^2(14)^2.$$

We have already seen how $(12)^2(13)^2$ is reduced, and it remains to shew how $(12)^2(13)^2(14)^2$ may also be reduced. We have at once, by equation (3),

$$2(n-2)^2(n-3)^2(12)^2(13)^2(14)^2 = U\Xi_4(12)^4(13)^2 \ldots (4).$$

When U is of the sixth degree, this $(12)^4(13)^2$ is the reciprocal of the first evectant of the invariant called by Mr. Cayley $D_{4,2,0}$; but it is more convenient to express it generally by help of $C_2 = (12)^2(23)^2(31)^2$. For multiply together

$$2\Xi_2\Xi_3(12)(13) = \Xi_2^{\,2}(13)^2 + \Xi_3^{\,2}(12)^2 - \Xi_1^{\,2}(23)^2,$$
$$2\Xi_3\Xi_1(21)(23) = \Xi_3^{\,2}(12)^2 + \Xi_1^{\,2}(23)^2 - \Xi_2^{\,2}(13)^2,$$
$$2\Xi_1\Xi_2(31)(32) = \Xi_1^{\,2}(23)^2 + \Xi_2^{\,2}(13)^2 - \Xi_3^{\,2}(12)^2,$$

and we have

$$-8\Xi_1^{\,2}\Xi_2^{\,2}\Xi_3^{\,2}(12)^2(23)^2(31)^2 = -3\Xi_3^{\,6}(12)^6 + 6\Xi_3^{\,4}\Xi_2^{\,2}(12)^4(13)^2$$
$$-2\Xi_1^{\,2}\Xi_2^{\,2}\Xi_3^{\,2}(12)^2(23)^2(31)^2,$$

or $2\Xi_2^{\,2}\Xi_3^{\,2}(12)^4(13)^2 = n(n-1)(n-2)(n-3)B_6U - 2(n-4)^2(n-5)^2C_2$.

Thus Q_6 is expressed in the form

$$U(aB_6U^4 + bC_2U^3 + cB_4B_2U^2 + dB_2^{\,3}),$$

a, b, c, d being constant coefficients.

In general it may be observed that Q_n will be of the form

$$U(aU + bB_2).$$

A series proceeding in powers of U can easily be made for Q_n by multiplying together

$$\Xi_2 13 = \Xi_3 12 + \Xi_1 23,$$
$$\Xi_2 14 = \Xi_4 12 + \Xi_1 24,$$
$$\Xi_2 15 = \Xi_5 12 + \Xi_1 25,$$
&c.;

and observing that the last term in the right hand side is equal with contrary sign to that on the left, we have a series of the form

$$Q_m = A(12)^m + B(12)^{m-1}(13) + C(12)^{m-2}(13)(14) + \text{\&c.}$$
$$+ D(12)^2(13)(14)\ldots$$

But then this admits of further reduction.

and $-4\Xi_1\Xi_2\Xi_3\Xi_4(12)^2(34)^2(13)(24) = \Xi_1^2\Xi_2^2(12)^2(34)^4$
$$+ 2\Xi_3^2\Xi_4^2(12)^2(13)^2(24)^2 - 2\Xi_4^4(12)^2(23)^2(31)^2,$$
or $\quad -4(n-3)^3(12)^2(34)^2(13)(24) = (n-2)^2(n-3)\,B_2B_4$
$$-2n(n-1)(n-2)\,UC_2 + 2(n-2)^2(n-3)\,(12)^2(13)^2(24)^2.$$

Lastly, to calculate the first term, we have

$$(12)^2\{(12)^2(34)^2\} = (12)^2\{(13)(24)-(14)(23)\}^2$$
$$= 2(12)^2(13)^2(24)^2 - 2(12)^2(13)(14)(23)(24),$$
$$\therefore\ 2(12)^2(13)(14)(23)(24) = 2(12)^2(13)^2(24)^2 - B_2B_4.$$

But, multiplying by $(12)^2$ the product of the two equations

$$2\Xi_3\Xi_4(13)(14) = \Xi_4^2(13)^2 + \Xi_3^2(14)^2 - \Xi_1^2(34)^2,$$
$$2\Xi_3\Xi_4(23)(24) = \Xi_4^2(23)^2 + \Xi_3^2(24)^2 - \Xi_2^2(34)^2,$$

we have

$$4\Xi_3^2\Xi_4^2(12)^2(13)(14)(23)(24) = 2\Xi_4^4(12)^2(23)^2(31)^2$$
$$- 2\Xi_3^2\Xi_4^2(12)^2(24)^2(13)^2 + \Xi_1^2\Xi_2^2(12)^2(34)^4,$$

or $4(n-2)(n-3)(12)^2(13)(14)(23)(24) = 2n(n-1)\,C_2U$
$$-2(n-2)(n-3)(12)^2(13)^2(24)^2 + (n-2)(n-3)\,B_2B_4,$$

and comparing this with the value previously obtained,

$$6(n-2)(n-3)(12)^2(13)^2(24)^2$$
$$= 3(n-2)(n-3)\,B_2B_4 + 2n(n-1)\,UC_2\,;$$
$$-12(n-3)^3(12)^2(34)^2(13)(24)$$
$$= 6(n-2)^2(n-3)\,B_2B_4 - 4n(n-1)(n-2)\,UC_2\,;$$
$$6(n-3)^2(12)^2(13)(14)(34)^2$$
$$= 2n(n-1)\,UC_2.$$

Hence the Hessian of the Hessian can be expressed in the form $aB_2B_4 + bUC_2$, a theorem given by Mr. Cayley (*Crelle*, vol. XXXIV. p. 44), and which may also be proved more easily by shewing that it is true when U is of the fourth degree, and therefore, since no differentials higher than the fourth are introduced, must be true in general.

Ex. 8. It is an important question whether the result of elimination between two equations admits of a general symbolical expression. The result of elimination between a simple equation and one of the n^{th} degree is easily seen to be $(1a)(1b)(1c)\ldots(1n)$, where the figures are symbols referring to the equation of the n^{th} degree and the letters to the simple equation. Let us now examine whether a general expression

can be found for the result of elimination between a quadratic and an equation of the n^{th} degree. Resolve the quadratic into its factors and let a and a' refer to the differentials of the separate factors, and the result of elimination is

$$(1a)(1b)(1c)\ldots\ldots(1n)(2a')(2b')(2c')\ldots\ldots(2n');$$

but if A, B, &c. are symbols relating to the quadratic, we have $A = a + a'$, $B = b + b'$, &c.; $(1A)(2A) = (1a\,2a') + (1a')(2a)$ (since $(1a)^2$ vanishes); and $(1A)^2 = 2\,(1a)(1a')$.

We shall write this $(1A)(2A) = [1, 0] + [0, 1]$, where the first figure denotes the number of unaccented letters combined with 1, and the second the number combined with 2. In this notation then

$$(1A)(2A)(1B)(2B)\ldots(1N)(2N)$$
$$= [n, 0] + n[(n-1), 1] + \frac{n(n-1)}{1.2}[(n-2), 2] + \text{\&c.}$$

Similarly $(1A)^2\,(2N)^2(1B)(2B)\ldots(1M)(2M)$
$$= 4\,(\qquad [(n-1), 1] + (n-2)\,[(n-2), 2] + \text{\&c.}$$
$$(1A)^2(1B)^2(1C)(2C)\ldots(1L)(2L)$$
$$= 16\,(\qquad [n-2, 2] + (n-4)\ \text{\&c.}$$

Hence the result of elimination, viz. $[n, 0]$, is

$$(1A)^2(2A)^2(1B)(2B)\ \text{\&c.}$$

where 1, 2 refer to the first, aud 3, 4 to the second. But since we have identically

$$(12)(34) = (13)(24) - (14)(23),$$

$$2(13)(14)(23)(24) = 2(13)^2(24)^2 - (12)^2(34)^2,$$

and the result of elimination is

$$(13)^2(24)^2 - (12)^2(34)^2,$$

or, in terms of the coefficients,

$$(ac' + ca' - 2bb')^2 - 4(ac - b^2)(a'c' - b'^2),$$

a form found by Mr. Boole.

Similarly, the result of elimination between a cubic and quadratic is

$$4\{(1a)(1b)(1c)(2a)(2b)(2c)\} - 3(1a)^2(2b)^2(1c)(2c),$$

and $\quad 2(1a)(2b)(1b)(2a) = 2(1a)^2(2b)^2 - (12)^2(ab)^2;$

therefore the result is

$$(1a)^2(2b)^2(1c)(2c) - 2\{(ab)^2(12)^2(1c)(2c)\},$$

and so generally the series can be transformed in powers of $(ab)^2$, that is of $B^2 - AC$.

I proceed now to the case of homogeneous functions of three variables. We use the abbreviations (123) to denote the determinant

$$\frac{d}{dx_1}\left(\frac{d}{dy_2}\frac{d}{dz_3} - \frac{d}{dy_3}\frac{d}{dz_2}\right) + \frac{d}{dx_2}\left(\frac{d}{dy_3}\frac{d}{dz_1} - \frac{d}{dy_1}\frac{d}{dz_3}\right)$$
$$+ \frac{d}{dx_3}\left(\frac{d}{dy_1}\frac{d}{dz_2} - \frac{d}{dy_2}\frac{d}{dz_1}\right),$$

$(\alpha 12)$ to denote the determinant

$$\alpha\left(\frac{d}{dy_1}\frac{d}{dz_2} - \frac{d}{dy_2}\frac{d}{dz_1}\right) + \beta\left(\frac{d}{dz_1}\frac{d}{dx_2} - \frac{d}{dz_2}\frac{d}{dx_1}\right)$$
$$+ \gamma\left(\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}\right),$$

Ξ_1 denotes $x\dfrac{d}{dx_1} + y\dfrac{d}{dy_1} + z\dfrac{d}{dz_1}$; and $P = \alpha x + \beta y + \gamma z$.

I notice in the first place that $(123)^3$ is six times the Hessian of the function U, and that $(\alpha 12)^2$ is the function called by Mr. Sylvester the bordered Hessian, and is the same as the equation which I have called the polar conic of a line with regard to a curve of the third degree (*Higher Plane Curves*, p. 151). The curve S (*ib.* p. 101) is simply $(\alpha 12)^4$ and is the

evectant of the invariant $(123)^4$, while the curve T (*ib.* p. 102) is $(\alpha 12)^2 (\alpha 23)^2 (\alpha 31)^2$. Any invariant of a function of two variables becomes a contravariant of a function of three by prefixing a contravariant symbol α in each factor of the symbolical expression for the invariant, and Mr. Cayley has noticed that the discriminant of a function of two variables treated in this manner gives the equation of the corresponding reciprocal curve.

These symbols are transformed by the identical equations

$$\Xi_4(123) = \Xi_1(234) - \Xi_2(134) + \Xi_3(124) \ldots\ldots (A),$$
$$P(123) = \Xi_1(\alpha 23) + \Xi_2(\alpha 31) + \Xi_3(\alpha 12) \ldots. (B),$$
$$(123)(145) + (124)(153) + (125)(134) = 0 \ldots (C),$$
$$(\alpha 12)(\alpha 34) + (\alpha 23)(\alpha 14) + (\alpha 31)(\alpha 24) = 0 \ldots (D).$$

Since the terms in the last are obtained by cyclically permuting 1, 2, 3, it follows that if the whole equation be multiplied by any quantity unchanged by such a permutation, every term must be separately zero. In this manner are proved the equations of which use is made afterwards,

$$(123)(\alpha 23)(\alpha 14) = 0; \quad (123)(\alpha 23)(\alpha 14)(\alpha 45) = 0, \text{ \&c.}$$

I proceed now to investigate by this method the problem of finding the curve which passes through the points of contact of the double tangents to a given curve. It is proved (see Mr. Cayley's paper, *Crelle*, vol. XXXIV., and *Higher Plane Curves*, p. 81) that if we denote by R_n the result of substituting

$$x_1 = \gamma \frac{du}{dy} - \beta \frac{du}{dz}, \quad y_1 = \alpha \frac{du}{dz} - \gamma \frac{du}{dx}, \quad z_1 = \beta \frac{du}{dx} - \alpha \frac{du}{dy},$$

in $\left(x_1 \dfrac{d}{dx} + y_1 \dfrac{d}{dy} + z_1 \dfrac{d}{dz}\right)^n$, then R_n will always be of the form

$$R_n = P_n U + Q_n(\alpha x + \beta y + \gamma z)^2,$$

and that if we form the discriminant of the equation

$$\lambda^{n-2} Q_2 + \lambda^{n-3}\mu \frac{Q_3}{3} + \lambda^{n-4}\mu^2 \frac{Q_4}{3.4} + \text{\&c.},$$

the result will be divisible by $(\alpha x + \beta y + \gamma z)^{(n-2)(n-3)}$, and will give the equation of the double tangent curve required. The object of what follows is to perform the actual calculation of Q_2, Q_3, &c. And first it is easy to see that R_n may be symbolically expressed as follows:

$$R_2 = (\alpha 12)(\alpha 13), \quad R_3 = (\alpha 12)(\alpha 13)(\alpha 14),$$
$$R_4 = (\alpha 12)(\alpha 13)(\alpha 14)(\alpha 15), \text{ \&c.}$$

Ex. 10. To calculate R_2. Square the equation B, and we have

$$P^2(123)^2 = 3\Xi_3^{\,2}(\alpha 12)^2 - 6\Xi_2\Xi_3(\alpha 12)(\alpha 13).$$

We shall denote $(\alpha 12)^n$ by G_n and $(123)^n$ by H_n,

then
$$6(n-1)^2 R_2 = 3n(n-1)\,G_2 U - P^2 H_2,$$
$$6(n-1)^2 Q_2 = -H_2.$$

Ex. 11. To calculate R_3. From equation B,

$$-2\Xi_2\Xi_3(\alpha 12)(\alpha 13) = P^2(123)^2 - 2P\Xi_1(123)(\alpha 23)$$
$$+\Xi_1^{\,2}(\alpha 23)^2 - \Xi_2(\alpha 31)^2 - \Xi_3^{\,2}(\alpha 12)^2.$$

Multiply by $(\alpha 14)$, two of the terms vanish identically and

$$-2(n-1)^2 R_3 = P^2(123)^2(\alpha 14) - 2n(n-1)\,U(\alpha 12)^2(\alpha 13)$$
$$-2(n-1)^2 Q_3 = (123)^2(\alpha 14).$$

Ex. 12. To calculate Q_4. In what follows we omit the terms multiplied by U indicated by one or more figures disappearing from the symbolical expression. Multiply together

$$-2\Xi_2\Xi_3(\alpha 12)(\alpha 13) = P^2(123)^2 - 2P\Xi_1(123)(\alpha 23)$$
$$+\Xi_1^{\,2}(\alpha 23)^2 - \Xi_2^{\,2}(\alpha 31)^2 - \Xi_3^{\,2}(\alpha 12)^2$$
$$-2\Xi_4\Xi_5(\alpha 14)(\alpha 15) = P^2(145)^2 - 2P\Xi_1(145)(\alpha 45)$$
$$+\Xi_1^{\,2}(\alpha 45)^2 - \Xi_4^{\,2}(\alpha 51)^2 + \Xi_5^{\,2}(\alpha 14)^2,$$

and
$$4(n-1)^4 R_4 = P^4(123)^2(145)^2 - 4P^3\Xi_1(123)^2(145)(\alpha 45)$$
$$+\,2P^2\Xi_1^{\,2}(145)^2(\alpha 23)^2$$
$$+\,4P^2\Xi_1^{\,2}(123)(\alpha 23)(145)(\alpha 45) - 4P\Xi_1^{\,3}(123)(\alpha 23)(\alpha 45)^2.$$

But the last two terms destroy each other, since, on substituting in the first of them for $P(145)$,

$$\Xi_1(\alpha 45) + \Xi_4(\alpha 51) + \Xi_5(\alpha 14),$$

the difference vanishes identically. Hence

$$4(n-1)^4 Q_4 = P^2(123)^2(145)^2 - 4(n-3)\,P(123)^2(145)(\alpha 45)$$
$$+\,2(n-2)(n-3)\,H_2 G_2.$$

Ex. 13. To calculate the double tangent curve when U is of the fourth degree.

Its equation is (see *Higher Plane Curves*)

$$Q_3^{\,2} = 3Q_2 Q_4.$$

Now
$$4(n-1)^4 Q_3^{\,2} = (123)^2(\alpha 17)(456)^2(\alpha 48),$$

but $\quad \Xi_8(\alpha 17) - \Xi_7(\alpha 18) = P(178) - \Xi_1(\alpha 78),$

$\quad \Xi_8(\alpha 47) - \Xi_7(\alpha 48) = P(478) - \Xi_4(\alpha 78).$

Multiplying, we have

$-2\Xi_7\Xi_8(\alpha 17)(\alpha 48) = P^2(178)(478) - 2P\Xi_1(178)(\alpha 78) + \Xi_1\Xi_4(\alpha 78)^2,$

or $\; -8(n-1)^6 Q_3^2 = P^2(123)^2(456)^2(178)(478)$

$$-2(n-2)\,PH_2(123)^2(178)(\alpha 78) + (n-2)^2 H_2^2 G_2.$$

Hence, $2(n-3)\,Q_3^2 - 3(n-2)\,Q_2Q_4$ is proportional to

$2(n-3)(123)^2(456)^2(178)(478) = (n-2)(123)^2(456)^2(178)^2 \ldots (E),$

which when $n = 4$ reduces by equation A to the single term

$$(123)^2(456)^2(178)(148) = 0.^*$$

If it be required to find whether this equation be or be not identical with the form which M. Hesse has given for the same curve (Crelle, XLI. p. 285, and *Higher Plane Curves*, p. 89); first I answer that $(123)^2(456)^2(178)(478)$ *is* identical with the left-hand side of his equation. The function given by M. Hesse is obtained by substituting in the bordered Hessian $\dfrac{dH}{dx}$ for α, &c., and therefore may be written $(\alpha 78)$ $(\beta 78)(123)^2(456)^2$, where $\alpha = 1+2+3$, $\beta = 4+5+6$, and this expanded gives only terms of the form $(123)^2(456)^2(178)(478)$. But the right-hand side of my equation is *not* identical with that of M. Hesse. His right-hand side is found by operating on H with $\dfrac{d}{dx}$ for α, &c., substituted in the bordered Hessian, that is to say $(\alpha 45)^2(123)^2$, where $\alpha = 1+2+3$, or

$$(123)^2(145)^2 + 2\{(145)(245)(123)^2\}.$$

There is then apparent disagreement between our results. But I shall prove as follows that the last terms altogether vanish on the supposition $U = 0$. First let us denote by S $(123)(124)(234)(314)$, (it is in fact Aronhold's invariant); then from equation A

$$US = 3\{(125)(124)(234)(314)\},$$

* At first I imagined that this equation might be represented as a symmetrical function of the curve and the Hessian

$$(123)(234)\ UUHH,$$

the first two figures referring to the curve and the second two to the Hessian. It would be so if the sign had been + in equation E. It is worth mentioning, however, that the points of contact of common tangents to two conics lie on a conic, whose equation is

$$(123)(234)\ UUVV = 0.$$

and $(125)(124)(234)(134) = (125)^2(134)(234)$
$$+ 2(245)(125)(234)(134).$$

But multiply together the two identical equations

$$(245)(134) - (124)(345) - (145)(234) = 0,$$
$$(245)(123) - (124)(235) + (125)(234) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and $- 3(125)^2(134)(234) + 6\{(125)(245)(134)(234)\} = 0.$

Therefore the quantity $(123)^2(145)(245)$ reduces to US and therefore vanishes on the supposition $U = 0$. It is to be noted that these are the only terms which exist when U is of the third degree.

The omission then of these superfluous terms renders the actual calculation of the quantity $(123)^2(145)^2$ much more easy in the form I have given it, than in that given by M. Hesse. In fact we have only to square the bordered Hessian $(\alpha 12)^2$ and in it write $\dfrac{d}{dx}$ for α, &c.

As a step to the consideration of the corresponding problems for the fifth and sixth degrees, I add the values of Q_5 and Q_6, which however are probably capable of further reduction:

$$4(n-1)^4 Q_5 = P^2(123)^2(145)^2(\alpha 16) - 4(n-4)P(123)^2(145)(\alpha 45)(\alpha 16)$$
$$+ 2(n-3)(n-4)G_2(123)^2(\alpha 14)$$
$$+ 4(n-3)(n-4)(123)(\alpha 23)(145)(\alpha 45)(\alpha 16)$$

$$- 8(n-1)^6 Q_6 = P^4(123)^2(145)^2(167)^2$$
$$- 6P^3(n-5)(123)^2(145)^2(167)(\alpha 67)$$
$$+ 3P^2(n-4)(n-5)G_2(123)^2(145)$$
$$+ 12P^2(n-4)(n-5)(123)^2(145)(\alpha 45)(167)(\alpha 67)$$
$$- 8P(n-3)(n-4)(n-5)(123)(\alpha 23)(145)(\alpha 45)(167)(\alpha 67)$$
$$- 12P(n-3)(n-4)(n-5)G_2(123)^2(145)(\alpha 45)$$
$$+ 5(n-2)(n-3)(n-4)(n-5)G_2^2H_2.$$

I proceed to give an illustration of the application of this method to a different problem. Mr. Sylvester observed that a curve of the fourth degree has got a covariant conic section of the fifth degree in the coefficient of the original equation. I noticed myself that it has got another covariant conic of the eighth degree in the coefficients which I ascer-

tained to be distinct from the former. The rules for the calculation of each leading to very laborious processes, it remains to be seen whether any simplification is gained by the use of this method. The first conic is obtained by operating on H with $\frac{d}{dx}$ and substituted for α, &c. in G_4, that is $\{(145)+(245)+(345)\}^4(123)^2$; and since the curve is only of the fourth degree this reduces to the two terms $(145)^2(245)^2(123)^2$ and $(123)^2(145)^2(245)(345)$. Here then it appears we have *two* covariant conics whose coefficients contain those of the original equation in the fifth degree. It remains to be seen whether they are the same or different, but on this question I cannot pronounce at present. The calculation of the former is comparatively easy. It is effected by substituting $\frac{d}{dx}$ for α, &c. in the contravariant conic

$$(\alpha 12)^2(134)^2(234)^2,$$

which involves the coefficients only in the fourth degree.

Sept. 23, 1853.

POSTSCRIPT.

Ex. 14. To find the discriminant of a function of two variables of the fourth degree.

The following is a general process by which invariants may be formed of the same degree in the coefficients of the equation, as the discriminant. Take two sets of $(n-1)$ symbols, 1, 3, 5, &c., 2, 4, 6, &c., and form a product of $(n-1)$ factors, each factor containing one of each set. Thus, in the present instance, we form $(12)(34)(56)=\mathfrak{A}$. Then, by cyclically permuting one of the sets of the symbols, we form $(n-1)$ products in all. Thus, in the present instance, we have

$$(14)(36)(52)=\mathfrak{B},\quad (16)(32)(54)=\mathfrak{C}.$$

Then any function of the n^{th} degree of these quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. will be an invariant of the given equation, and of the same degree in the letters as the discriminant, and will vanish identically if the function have more than $\frac{1}{2}n$ equal factors. The two difficulties in the general theory are, first, to ascertain which of these functions *is* the discriminant, and secondly, to discover the relations which exist between the different invariants which may be so formed, and to find the simplest form to which they may be reduced. Thus, for instance, when n is even it is easy to see that

$\mathfrak{A}^n = 2^k\mathfrak{A}^{n-1}\mathfrak{B}$; $k+1$ being the number of factors in the product $\mathfrak{A}$, &c.

In the case then of an equation of the fourth degree, the different invariants that can be formed are $\mathfrak{A}^4$, $\mathfrak{A}^2\mathfrak{B}^2$, $\mathfrak{A}^2\mathfrak{B}\mathfrak{C}$, omitting $\mathfrak{A}^3\mathfrak{B}$, which we know to be only $\frac{1}{4}\mathfrak{A}^4$. The first problem is to form a combination of these invariants, which will vanish identically if the given function be of the form u^2v, where u is of linear dimensions. To do this, suppose the function to be of this form; substitute for 1, $a+\alpha$ (a referring to v and α to u^2) for 2, $b+\beta$, &c., and examine the effect of this substitution on each of the invariants in question. It is obvious that the terms in which either a or α enter above the second degree vanish; and denoting by M the condition $(a\alpha)^2$, that u should be a factor in v, we can find without much trouble that on this substitution we should have

$$\mathfrak{A}^4 = 216M^6, \quad \mathfrak{A}^2\mathfrak{B}^2 = 66M^6, \quad \mathfrak{A}^2\mathfrak{B}\mathfrak{C} = 42M^6;$$

and hence that the discriminant may be expressed by any of the three equations

$$\frac{\mathfrak{A}^4}{36} = \frac{\mathfrak{A}^2\mathfrak{B}^2}{11} = \frac{\mathfrak{A}^2\mathfrak{B}\mathfrak{C}}{7}.$$

Next, for the reduction of these quantities, $\mathfrak{A}^4$ obviously $= B_4^3$. Multiply together the three identical equations,

$$\begin{aligned}
&2\{(12)(34)(14)(32)\} = (12)^2(34)^2 + (14)^2(32)^2 - (13)^2(24)^2,\\
&2\{(34)(56)(36)(54)\} = (34)^2(56)^2 + (36)^2(54)^2 - (35)^2(46)^2,\\
&\underline{2\{(56)(12)(52)(16)\} = (56)^2(12)^2 + (52)^2(16)^2 - (51)^2(26)^2,}
\end{aligned}$$

and $$8\mathfrak{A}^2\mathfrak{B}\mathfrak{C} = B_4^3 + 4\mathfrak{A}^2\mathfrak{B}^2 - 4C_2^2.$$

Again,

$$\begin{aligned}
C_2^2 = (15)^2(35)^2(46)^2(26)^2\{(12)(34) + (14)(23)\}^2 = 2\mathfrak{A}^2\mathfrak{B}^2\\
+ 2[(15)^2(35)^2\{(14)(26)(12)(46)\}\{(34)(26)(23)(46)\}];
\end{aligned}$$

but $$\begin{aligned}
2\{(14)(26)(12)(46)\} &= (12)^2(46)^2 + (14)^2(26)^2 - (16)^2(24)^2,\\
-2\{(34)(26)(23)(46)\} &= (23)^2(46)^2 + (34)^2(26)^2 - (36)^2(24)^2;
\end{aligned}$$

hence $$2C_2^2 = 6\mathfrak{A}^2\mathfrak{B}^2 - \tfrac{3}{2}B_4^3.$$

Hence the expression for the discriminant becomes

$$B_4^3 = 6C_2^2,$$

and we have $$\mathfrak{A}^2\mathfrak{B}^2 = \tfrac{1}{4}B_4^3 + \tfrac{1}{3}C_2^2,$$

$$\mathfrak{A}^2\mathfrak{B}\mathfrak{C} = \tfrac{1}{4}B_4^3 - \tfrac{1}{3}C_2^2.$$

It is to be observed, that we have here made use of one equation more than was necessary for the solution of the problem.

GENERAL CONSIDERATIONS ON CONIC SECTIONS IN DOUBLE CONTACT WITH EACH OTHER.

By Prof. Steiner.

Section X.

In the former memoir,* the properties of all conic sections touching each of two given fixed circles twice were considered. I propose here to append more general considerations, where, instead of two circles, any two conic sections in fixed position will be given, (which may again be represented by A^2 and B^2), and where, in a similar manner, the properties of all conic sections touching each of the given ones twice will be considered.

I. In order to have a definite case (figure) in view, let us imagine, or construct, two ellipses A^2 and B^2, intersecting each other in four points r, s, t, u, and consequently, having four real tangents R, S, T, U in common; let R be the tangent from whose points of contact two elliptical arcs lead immediately to the intersection r, and similarly for the rest. The four intersections form a complete quadrangle $rstu$, and the four tangents a complete quadrilateral $RSTU$. Let us distinguish the three pairs of opposite sides of the first, and their intersections, as well the three pairs of opposite angles of the latter, and its three diagonals, by the following letters:

Side	$rs = \Xi$	and	$tu = \Xi_1$;	Intersection	$\Xi\Xi_1 = x$.
"	$rt = \mathrm{H}$	and	$su = \mathrm{H}_1$;	"	$\mathrm{HH}_1 = y$.
"	$ru = \mathrm{Z}$	and	$st = \mathrm{Z}_1$;	"	$\mathrm{ZZ}_1 = z$.
Angle	$RS = \xi$	and	$TU = \xi_1$;	Diagonal	$\xi\xi_1 = X$.
"	$RT = \eta$	and	$SU = \eta_1$;	"	$\eta\eta_1 = Y$.
"	$RU = \zeta$	and	$ST = \zeta_1$;	"	$\zeta\zeta_1 = Z$.

The intersections x, y, z of the three pairs of opposite sides of the quadrangle are the common triplet of conjugate poles of the two ellipses, and the three diagonals X, Y, Z of the quadrilateral are the common triplet of conjugate polars of the same: hence it follows that

Intersection $XY = z$, $XZ = y$; $YZ = x$, and

Line $xy = Z$, $xz = Y$; $yz = X$.

Further x, ζ, y and ζ_1; x, η, z and η_1; y, ξ, z and ξ_1 are four

* *Math. Jour.*, vol. VIII. p. 227.

harmonical points, as well as X, Z, Y and Z_1; X, H, Z and H_1; Y, Ξ, Z and Ξ_1 four harmonical lines.

Employing this, together with the notation and nomenclature already introduced in the former memoir, the properties under consideration may be thus expressed.

II. All conic sections C^2 in double contact with each of the given ellipses A^2 and B^2, in consequence of their relation to the three poles x, y, z, may be divided into three distinct throngs $Th(C_x^2)$, $Th(C_y^2)$, and $Th(C_z^2)$ (§ *VIII.*) which in general, however, have like properties, so that, for brevity's sake, we need here only consider the $Th(C_x^2)$.

(1) "*If the curve C_x^2 touches the ellipse A^2 in the points p and p_1, and the ellipse B^2 in the points q and q_1, the chords of contact pp_1 and qq_1 pass through the pole x, and are always conjugate harmonical to the opposite sides Ξ and Ξ_1.*" And conversely: "*If through the pole x any two lines G and H be drawn conjugate harmonical to the sides Ξ and Ξ_1, they will intersect the ellipses A^2 and B^2 respectively in certain points p, p_1 and q, q_1, so that in these points the ellipses will be touched by some curve C_x^2; and alternately, H will intersect A^2, and G will intersect B^2 in certain points p°, p_1° and q°, q_1°, so that in these same points A^2 and B^2 will be touched by some other curve C_x^2.*"

(2) "*If between each two pairs of corresponding points of contact p and p_1, q and q_1, the four alternate chords pq, pq_1, p_1q and p_1q_1 be drawn* (§ *VIII.* 5), *they will all be tangents to a certain conic section X^2, inscribed in the quadrilateral $RSTU$, and touching the two opposite sides Ξ and Ξ_1: (inasmuch as the latter as well as the tangents R, S, T, U are themselves special alternate chords).*" And further: "*The two pairs of tangents, drawn from the two intersections of the polar X with the ellipse B^2 (or A^2) to the conic section X^2, touch the latter in the same points in which it is intersected by the other ellipse A^2 (or B^2).*"

(3) "*If through the four points of contact p and p_1, q and q_1, any arbitrary conic section D^2 be drawn, it will intersect the given curves A^2 and B^2 in four new points p° and p_1°, q° and q_1°, so that in these same points A^2 and B^2 will be touched by some other curve C_x^2.*" And conversely: "*The eight points of contact of any two curves C_x^2 with the ellipses A^2 and B^2 lie always in some conic section D^2.*"

"*All curves C_x^2 have, in common, x and Ξ as pole and polar. Of the common secants to any two curves C_x^2, one pair, G and H, always passes through the pole x, and they are always conjugate harmonical to Ξ and Ξ_1.*"

(4) "*Every four points of contact* p, p_1, q, q_1 *lie on the one hand with the angles* r *and* s *in a conic section* M^2, *and on the other hand with the angles* t *and* u *in a conic section* M_1^2. *All the conic sections* M^2, *hereby defined, touch each other in the points* r *and* s, *so that they there have common tangents* Ρ *and* Σ, *and the common chord of contact* $rs = \Xi$, *thus forming a special pencil of curves* $Pl(M^2)$. *Let* m *be the intersection of the tangents* Ρ *and* Σ; *it is situated in the polar* Ξ, *and* m *and* Ξ *are pole and polar in reference to all* M^2 *or* $Pl(M^2)$. *Let* a *and* b *be the poles of the side* Ξ *in reference to* A^2 *and* B^2, *these are also situated in* X, *and let* c *be the intersection of* X *and* Ξ: *then the four points* a, m, b, c *are harmonical, so that the pole* m *is defined by the three points* a, b, c, *which are to be considered as given, and by means of* m, *the tangents* Ρ *and* Σ (= mr *and* ms) *are also determined. In an exactly similar manner, all conic sections* M_1^2 *touch each other in the points* t *and* u, *so that they there have common tangents of contact* Τ *and* Υ, *and the common chord of contact* $tu = \Xi_1$; *hence they form a special pencil of curves* $Pl(M_1^2)$; *and further, the intersection* m_1 *of* Τ *and* Υ, *together with the poles* a_1 *and* b_1 *of the side* Ξ_1 *in reference to the ellipses* A^2 *and* B^2, *lie all in the same polar* X; *and if* c_1 *be the intersection of* X *with* Ξ_1, *then the four points* a_1, m_1, b_1, c_1 *are harmonical; therefore from* a_1, b_1, c_1 *the pole* m_1 *is determined, and from the latter the tangents* Τ *and* Υ." "*The two pairs of tangents* Ρ *and* Σ, Τ *and* Υ, *thus determined, also touch the above conic section* X^2, *the locus of all alternate chords* (2); *and indeed* Ρ *and* Σ *touch it in its intersections with the side* Ξ_1, *and similarly* Τ *and* Υ *touch it in its intersections with the side* Ξ, *so that alternately, with respect to* X^2, m *is the pole of* Ξ_1, *and* m_1 *the pole of* Ξ." If these two pairs of tangents are assumed as given, we can conversely assert that "*Every conic section* M^2, *which touches the lines* Ρ *and* Σ *in the points* r *and* s, *intersects the given curves* A^2 *and* B^2 *in four points* p *and* p_1, q *and* q_1, (*as well as in* r *and* s), *which are the points of contact of some curve* C_x^2 *with them.*" "*The four mutual intersections of every two* C_x^2 *are always situated in a conic section* M^2 (*which touches* Ρ *and* Σ *in* r *and* s); *and conversely, every conic section* M^2 *intersects any curve* C_x^2 *in four points, through which some other curve* C_x^2 *also passes. The same applies to the conic sections* M_1^2."

III. (1) "*Let* P *and* P_1, Q *and* Q_1 *be the tangents of contact of a curve* C_x^2 *with the given curves* A^2 *and* B^2, *then the intersections* $PP_1 = p$ *and* $QQ_1 = q$ *lie always in the polar* X,

and are always conjugate harmonical to the angles ξ *and* ξ_1." And conversely: "*If in the polar* X, *any two points* g *and* h *are taken, conjugate harmonical to the opposite angles* ξ *and* ξ_1, *then the tangents* P *and* P_1, Q *and* Q_1 *drawn therefrom to the curves* A^2 *and* B^2 *are at the same time tangents of contact of these curves with some curve* C_x^2; *and similarly, the tangents* P° *and* P_1°, Q° *and* Q_1° *drawn (alternately) from* h *to* A^2, *and from* g *to* B^2 *are at the same time the tangents of contact of some other curve* C_x^2 *with* A^2 *and* B^2."

(2) "*Every two pairs of corresponding tangents of contact* P *and* P_1, Q *and* Q_1 *have four alternate intersections* PQ *and* PQ_1, P_1Q *and* P_1Q_1; *the locus of all these alternate intersections is a certain conic section* Ξ^2, *which circumscribes the quadrangle* $rstu$, *and also passes through the opposite angles* ξ *and* ξ_1." And further: "*The tangents drawn from the pole* x *to the ellipse* B^2 (*or* A^2) *intersect the conic section* Ξ^2 *in the same points in which it is touched by the four tangents which it has in common with the other ellipse* A^2 (*or* B^2)."

(3) "*If any arbitrary conic section* D^2 *be drawn, touching the four tangents of contact* P *and* P_1, Q *and* Q_1, *it will have, in common with the given curves* A^2 *and* B^2, *two more pairs of tangents* P° *and* P_1°, Q° *and* Q_1°, *which will always be the tangents of contact of some other curve* C_x^2 *with* A^2 *and* B^2." And conversely: "*The eight tangents of contact of any two curves* C_x^2 *with the curves* A^2 *and* B^2 *are always tangential to some conic section* D^2." "*Of the mutual intersections of the tangents common to any two curves* C_x^2, *one pair,* g *and* h, *are always situated in the polar* X, *and are always conjugate harmonical to the opposite angles* ξ *and* ξ_1."

(4) "*Every four tangents of contact* P, P_1, Q, Q_1 *are on the one hand, together with the tangents* R *and* S, *touched by a certain conic section* M^2, *and on the other hand, together with the tangents* T *and* U, *are touched by a conic section* M_1^2. *All the conic sections* M^2, *hereby defined, touch the tangents* R *and* S *in the same points* ρ *and* σ, *and hence they there touch each other also, so as to have* $\rho\sigma = \mathrm{M}$ *as there common chord of contact, and* ξ *and* M *as their common pole and polar; they form consequently a special pencil of curves* $Pl(\mathrm{M}^2)$. *The chord of contact* M *passes through the pole* x, *as also do the polars of* ξ *in reference to* A^2 *and* B^2, *which we may call* A *and* B; *and if further, the line* $x\xi$ *be* Γ, *then the four lines* A, M, B, Γ *are harmonical, consequently* M *is defined by the three others, which may be considered as given, and* M *determines the points*

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I. In order to have a definite case (figure) in view, let us imagine, or construct, two ellipses A^2 and B^2, intersecting each other in four points r, s, t, u, and consequently, having four real tangents R, S, T, U in common; let R be the tangent from whose points of contact two elliptical arcs lead immediately to the intersection r, and similarly for the rest. The four intersections form a complete quadrangle $rstu$, and the four tangents a complete quadrilateral $RSTU$. Let us distinguish the three pairs of opposite sides of the first, and their intersections, as well the three pairs of opposite angles of the latter, and its three diagonals, by the following letters:

Side	$rs = \Xi$	and	$tu = \Xi_1$;	Intersection	$\Xi\Xi_1 = x$.
"	$rt = \mathrm{H}$	and	$su = \mathrm{H}_1$;	"	$\mathrm{HH}_1 = y$.
"	$ru = \mathrm{Z}$	and	$st = \mathrm{Z}_1$;	"	$\mathrm{ZZ}_1 = z$.
Angle	$RS = \xi$	and	$TU = \xi_1$;	Diagonal	$\xi\xi_1 = X$.
"	$RT = \eta$	and	$SU = \eta_1$;	"	$\eta\eta_1 = Y$.
"	$RU = \zeta$	and	$ST = \zeta_1$;	"	$\zeta\zeta_1 = Z$.

The intersections x, y, z of the three pairs of opposite sides of the quadrangle are the common triplet of conjugate poles of the two ellipses, and the three diagonals X, Y, Z of the quadrilateral are the common triplet of conjugate polars of the same: hence it follows that

Intersection $XY = z, \quad XZ = y; \quad YZ = x$, and

Line $\qquad xy = Z, \quad xz = Y; \quad yz = X$.

Further x, ζ, y and ζ_1; x, η, z and η_1; y, ξ, z and ξ_1 are four

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* *Math. Jour.*, vol. VIII. p. 227.

harmonical points, as well as X, Z, Y and Z_1; X, H, Z and H_1; Y, Ξ, Z and Ξ_1 four harmonical lines.

Employing this, together with the notation and nomenclature already introduced in the former memoir, the properties under consideration may be thus expressed.

II. All conic sections C^2 in double contact with each of the given ellipses A^2 and B^2, in consequence of their relation to the three poles x, y, z, may be divided into three distinct throngs $Th(C_x^2)$, $Th(C_y^2)$, and $Th(C_z^2)$ (§ *VIII.*) which in general, however, have like properties, so that, for brevity's sake, we need here only consider the $Th(C_x^2)$.

(1) "*If the curve C_x^2 touches the ellipse A^2 in the points p and p_1, and the ellipse B^2 in the points q and q_1, the chords of contact pp_1 and qq_1 pass through the pole x, and are always conjugate harmonical to the opposite sides Ξ and Ξ_1.*" And conversely: "*If through the pole x any two lines G and H be drawn conjugate harmonical to the sides Ξ and Ξ_1, they will intersect the ellipses A^2 and B^2 respectively in certain points p, p_1 and q, q_1, so that in these points the ellipses will be touched by some curve C_x^2; and alternately, H will intersect A^2, and G will intersect B^2 in certain points p°, p_1° and q°, q_1°, so that in these same points A^2 and B^2 will be touched by some other curve C_x^2.*"

(2) "*If between each two pairs of corresponding points of contact p and p_1, q and q_1, the four alternate chords pq, pq_1, p_1q and p_1q_1 be drawn* (§ *VIII.* 5), *they will all be tangents to a certain conic section X^2, inscribed in the quadrilateral $RSTU$, and touching the two opposite sides Ξ and Ξ_1: (inasmuch as the latter as well as the tangents R, S, T, U are themselves special alternate chords).*" And further: "*The two pairs of tangents, drawn from the two intersections of the polar X with the ellipse B^2 (or A^2) to the conic section X^2, touch the latter in the same points in which it is intersected by the other ellipse A^2 (or B^2).*"

(3) "*If through the four points of contact p and p_1, q and q_1, any arbitrary conic section D^2 be drawn, it will intersect the given curves A^2 and B^2 in four new points p° and p_1°, q° and q_1°, so that in these same points A^2 and B^2 will be touched by some other curve C_x^2.*" And conversely: "*The eight points of contact of any two curves C_x^2 with the ellipses A^2 and B^2 lie always in some conic section D^2.*"

"*All curves C_x^2 have, in common, x and Ξ as pole and polar. Of the common secants to any two curves C^2. one pair, G and H, always passes through the po*
are always conjugate harmonical to Ξ and Ξ_1

ρ and σ, inasmuch as they are its intersections with R and S. In an exactly similar manner, the conic sections M_1^2, *or* $Pl(M_1^2)$, *touch the tangents T and U in two certain points τ and υ, &c., &c.*" "*The two pairs of points of contact ρ and σ, τ and υ thus defined are situated in the above conic section* Ξ^2, *the locus of alternate intersections* (2), *and in fact, the tangents in ρ and σ to* Ξ^2 *pass through the angle* ξ_1, *and the tangents to the same conic section in τ and υ both pass through the angle* ξ, *so that in reference to* Ξ, *inversely,* M *is the polar of* ξ_1, *and* M_1 *the polar of* ξ." Assuming the four points ρ and σ, τ and υ as known, we may conversely assert that, "*Every conic section* M^2 (*or* M_1^2) *which touches the tangents R and S* (*or T and U*) *in the points ρ and σ* (*or τ and υ*) *has, in common with the given curves A^2 and B^2, two other pairs of tangents P and P_1, Q and Q_1 which at the same time are tangents of contact of some curve C_x^2 with A^2 and B^2.*" And further: "*The common tangents of any two curves C_x^2 always touch one of the conic sections* M^2, *as well as one of the* M_1^2; *and conversely: Every conic section* M^2, *or* M_1^2, *has four tangents in common with every curve C_x^2, which are also touched by some other curve C_x^2.*"

IV. "*If in the two points where the side* H (I.) *is intersected by any curve C_x^2, the tangents to the latter be drawn, one of them will always pass through the angle ζ, the other through ζ_1; and similarly, of the two tangents to the same curve in its two points of intersection with side* H_1, *the one always passes through ζ, the other through ζ_1.*" Or conversely: "*If from the angle ζ, or ζ_1, the two tangents to a curve C_x^2 be drawn, the point of contact of the one tangent will always be in the side* H, *and that of the other in the side* H_1." "*And in like manner, of the two tangents to any curve C_x^2 in its points of intersection with the side* Z, *or* Z_1, *the one always passes through the angle η, the other through η_1; or conversely, Of the points of contact made by two tangents drawn from the angle η, or η_1, to any curve curve C_x^2, one is situated in the side* Z, *the other in the side* Z_1."

As before remarked, all the foregoing theorems, referring to the $Th(C_x^2)$, are, in analogous manner, applicable to the throngs $Th(C_y^2)$ and $Th(C_z^2)$, so that each theorem is virtually thrice present. The elements which are each time involved may be easily recognised. For instance, in theorem (IV.) where unlike-named elements are i[illegible] bination is,

$$Th(C_x^2) \text{ with } \begin{cases} \text{H, H}_1 \text{ and } \zeta, \\ \text{Z, Z}_1 \text{ and } \cdot \end{cases}$$

and hence, the combinations for the two other cases are

$$Th(C_y^2) \text{ with } \begin{Bmatrix} \Xi,\ \Xi_1 \text{ and } \zeta,\ \zeta_1; \\ Z,\ Z_1 \text{ and } \xi,\ \xi_1; \end{Bmatrix};$$

and $$Th(C_z^2) \text{ with } \begin{Bmatrix} \Xi,\ \Xi_1 \text{ and } \eta,\ \eta_1; \\ H,\ H_1 \text{ and } \xi,\ \xi_1; \end{Bmatrix}.$$

There are theorems, however, which at the same time refer to two throngs; for example, the following:

V. (1) "*All poles of the side* Ξ, *in reference to the* $Th(C_y^2)$ *as well as in reference to the* $Th(C_z^2)$, *together with its two poles* a *and* b, *in reference to* A^2 *and* B^2, *lie entirely in one and the same conic section* M_x^2, *which is one of the above pencil* $Pl(M^2)$ (II. 4), *therefore touches the lines* P *and* Σ *in the angular points* r *and* s; *it passes also through the two pairs of opposite angles* η *and* η_1, ζ *and* ζ_1 *of the quadrilateral* *RSTU*." "*In like manner, all poles of the side* Ξ_1 *in reference to the* $Th(C_y^2)$ *and* $Th(C_z^2)$, *together with its poles* a_1 *and* b_1 *in reference to* A^2 *and* B^2, *are situated in a certain conic section* M_{1x}^2, *which belongs to the above pencil* $Pl(M_1^2)$ (II. 4), *therefore touches the lines* T *and* Υ *in the angular points* t *and* u, *and passes also through the same opposite angles* η *and* η_1, ζ *and* ζ_1." This theorem also is thrice present, *i.e.* with respect to each of the three pairs of opposite sides of the quadrangle *rstu*, and to the two throngs of unlike name with the side each time involved.

(2) "*All polars of the angle* ξ *in reference to the* $Th(C_y^2)$ *and* $Th(C_z^2)$, *together with its polars* A *and* B *in reference to* A^2 *and* B^2, *are tangents to a certain conic section* M_x^2, *which latter belongs to the above pencil* $Pl(M^2)$ (III. 4), *therefore touches the tangents* *R* *and* *S* *in the points* ρ *and* σ, *and further, is also tangential to the two pairs of opposite sides* H *and* H_1, Z *and* Z_1 *of the quadrangle* *rstu*. *And in similar manner, all polars of the angle* ξ_1 *in reference to the* $Th(C_y^2)$ *and* $Th(C_z^2)$, *together with its polars* A_1 *and* B_1 *in reference to* A^2 *and* B^2, *touch a certain conic section* M_{1x}^2, *which latter belongs to the above pencil* $Pl(M_1^2)$, *and therefore touches the tangents* *T* *and* *U* *in the points* τ *and* υ, *as well as the sides* H *and* H_1, Z *and* Z_1."

VI. *Note.* The above properties are true for the case supposed, that is to say, when the four mutual intersections,
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ρ and σ, inasmuch as they are its intersections with R and S. In an exactly similar manner, the conic sections M_1^2, *or* $Pl(M_1^2)$, *touch the tangents T and U in two certain points τ and υ, &c., &c."* "*The two pairs of points of contact ρ and σ, τ and υ thus defined are situated in the above conic section Ξ^2, the locus of alternate intersections* (2), *and in fact, the tangents in ρ and σ to Ξ^2 pass through the angle ξ_1, and the tangents to the same conic section in τ and υ both pass through the angle ξ, so that in reference to Ξ, inversely,* M *is the polar of ξ_1, and* M_1 *the polar of ξ.*" Assuming the four points ρ and σ, τ and υ as known, we may conversely assert that, "*Every conic section* M^2 (*or* M_1^2) *which touches the tangents R and S* (*or T and U*) *in the points ρ and σ* (*or τ and υ*) *has, in common with the given curves A^2 and B^2, two other pairs of tangents P and P_1, Q and Q_1 which at the same time are tangents of contact of some curve C_x^2 with A^2 and B^2.*" And further: "*The common tangents of any two curves C_x^2 always touch one of the conic sections* M^2, *as well as one of the* M_1^2; *and conversely: Every conic section* M^2, *or* M_1^2, *has four tangents in common with every curve C_x^2, which are also touched by some other curve C_x^2.*"

IV. "*If in the two points where the side* H (I.) *is intersected by any curve C_x^2, the tangents to the latter be drawn, one of them will always pass through the angle ζ, the other through ζ_1; and similarly, of the two tangents to the same curve in its two points of intersection with side* H_1, *the one always passes through ζ, the other through ζ_1.*" Or conversely: "*If from the angle ζ, or ζ_1, the two tangents to a curve C_x^2 be drawn, the point of contact of the one tangent will always be in the side* H, *and that of the other in the side* H_1." "*And in like manner, of the two tangents to any curve C_x^2 in its points of intersection with the side* Z, *or* Z_1, *the one always passes through the angle η, the other through η_1; or conversely, Of the points of contact made by two tangents drawn from the angle η, or η_1, to any curve curve C_x^2, one is situated in the side* Z, *the other in the side* Z_1."

As before remarked, all the foregoing theorems, referring to the $Th(C_x^2)$, are, in analogous manner, applicable to the throngs $Th(C_y^2)$ and $Th(C_z^2)$, so that each theorem is virtually thrice present. The elements which are each time involved may be easily recognised. For instance, in theorem (IV.) where unlike-named elements are involved, the combination is,

$$Th(C_x^2) \text{ with } \begin{Bmatrix} \text{H, H}_1 \text{ and } \zeta,\ \zeta_1; \\ \text{Z, Z}_1 \text{ and } \eta,\ \eta_1; \end{Bmatrix}$$

and hence, the combinations for the two other cases are

$$Th(C_y^2) \text{ with } \begin{Bmatrix} \Xi,\ \Xi_1 \text{ and } \zeta,\ \zeta_1; \\ \mathrm{Z},\ \mathrm{Z}_1 \text{ and } \xi,\ \xi_1; \end{Bmatrix};$$

and $$Th(C_z^2) \text{ with } \begin{Bmatrix} \Xi,\ \Xi_1 \text{ and } \eta,\ \eta_1; \\ \mathrm{H},\ \mathrm{H}_1 \text{ and } \xi,\ \xi_1; \end{Bmatrix}.$$

There are theorems, however, which at the same time refer to two throngs; for example, the following:

V. (1) "*All poles of the side* Ξ, *in reference to the* $Th(C_y^2)$ *as well as in reference to the* $Th(C_z^2)$, *together with its two poles a and b, in reference to* A^2 *and* B^2, *lie entirely in one and the same conic section* M_x^2, *which is one of the above pencil* $Pl(M^2)$ (II. 4), *therefore touches the lines* P *and* Σ *in the angular points r and s; it passes also through the two pairs of opposite angles* η *and* η_1, ζ *and* ζ_1 *of the quadrilateral RSTU.*" "*In like manner, all poles of the side* Ξ_1 *in reference to the* $Th(C_y^2)$ *and* $Th(C_z^2)$, *together with its poles* a_1 *and* b_1 *in reference to* A^2 *and* B^2, *are situated in a certain conic section* M_{1x}^2, *which belongs to the above pencil* $Pl(M_1^2)$ (II. 4), *therefore touches the lines* T *and* Υ *in the angular points t and u, and passes also through the same opposite angles* η *and* η_1, ζ *and* ζ_1." This theorem also is thrice present, *i.e.* with respect to each of the three pairs of opposite sides of the quadrangle *rstu*, and to the two throngs of unlike name with the side each time involved.

(2) "*All polars of the angle* ξ *in reference to the* $Th(C_y^2)$ *and* $Th(C_z^2)$, *together with its polars* A *and* B *in reference to* A^2 *and* B^2, *are tangents to a certain conic section* M_x^2, *which latter belongs to the above pencil* $Pl(\mathrm{M}^2)$ (III. 4), *therefore touches the tangents R and S in the points* ρ *and* σ, *and further, is also tangential to the two pairs of opposite sides* H *and* H_1, Z *and* Z_1 *of the quadrangle rstu. And in similar manner, all polars of the angle* ξ_1 *in reference to the* $Th(C_y^2)$ *and* $Th(C_z^2)$, *together with its polars* A_1 *and* B_1 *in reference to* A^2 *and* B^2, *touch a certain conic section* M_{1x}^2, *which latter belongs to the above pencil* $Pl(\mathrm{M}_1^2)$, *and therefore touches the tangents T and U in the points* τ *and* υ, *as well as the sides* H *and* H_1, Z *and* Z_1."

VI. *Note.* The above properties are true for the case supposed, that is to say, when the four mutual intersections, as well as the four common tangents of the given conic sections A^2 and B^2 are real, whereby, however, the latter need not be ellipses, but may be any kind of conic sections.

From the case supposed we can pass to other cases, where any portion of the above elements is imaginary. The most essential of these cases are the following three. If the relative situation of the given arbitrary conic sections A^2 and B^2 be such, that either (1) the four intersections r, s, t, and u only are real, the four common tangents being, on the contrary, imaginary; or (2) the four common tangents are real, the four intersections being imaginary; or (3) only two intersections, and two common tangents are real. In each of these three cases, a portion of the elements described above (I.) will always be imaginary; hence in the above properties and theorems, corresponding, more or less considerable changes must be introduced in a similar manner as was before done in § *VIII.* For example, if the opposite sides Ξ and Ξ_1 become imaginary, but their intersection, the pole x, remains real, the theorem (II. 1) will be so changed, that all the pairs of chords of contact pp_1 and qq_1 will form an *elliptic* system of rays, instead of a *hyperbolic* one, as before, &c.

Section XI.

I. The following special theorems, included in the above general theorems (§ *X.*), may be here particularized.

(1) "*If in a complete quadrilateral RSTU, any two conic sections A^2 and B^2 be inscribed, the eight points of contact are always situated in some third conic section D^2.*" And, "*If through the four points ρ, σ, τ, and υ, in which any conic section A^2 touches the sides R, S, T, and U of the quadrilateral, any conic section D^2 be drawn, the same will intersect the sides in four other points ρ_1, σ_1, τ_1, and υ_1, so that in these latter points the said sides are also touched by some other conic section B^2.*"* Further: "*The four mutual intersections r, s, t, u of any two conic sections A^2 and B^2, inscribed in the same quadrilateral RSTU, together with any one of the three pairs of opposite angles ξ and ξ_1, η and η_1, or ζ and ζ_1 of the same, are situated in a conic section Ξ^2, H^2, or Z^2.*" And further: "*Of the 8 points of contact (ρ, σ, τ, υ; ρ_1, σ_1, τ_1, υ_1) of any two conic sections A^2 and B^2, inscribed in the quadrilateral, 12 times 4 lie, together with two of the four intersections r, s, t, u of these conic sections, in a certain conic section M^2 (or M_1^2); and these 12 conic sections arrange themselves*

* The analogous theorems to the above two, but in reference to trilateral I have already proved in *Gergonne's Annales des Mat^s* XIX. or XX. 1828.

in 6 *pairs which touch each other twice, for through each two of the four intersections* r, s, t, u *pass two conic sections* M^2, *which there touch each other. The several* 6 *points situated together in some conic section* M^2, *are*

$$\begin{vmatrix} \rho, \sigma, \rho_1, \sigma_1 \\ \tau, \upsilon, \tau_1, \upsilon_1 \end{vmatrix} \text{ with } \begin{vmatrix} r, s \\ t, u \end{vmatrix}; \quad \begin{vmatrix} \rho, \tau, \rho_1, \tau_1 \\ \sigma, \upsilon, \sigma_1, \upsilon_1 \end{vmatrix} \text{ with } \begin{vmatrix} r, t \\ s, u \end{vmatrix};$$

$$\begin{vmatrix} \rho, \upsilon, \rho_1, \upsilon_1 \\ \sigma, \tau, \sigma_1, \tau_1 \end{vmatrix} \text{ with } \begin{vmatrix} r, u \\ s, t \end{vmatrix},$$

that is, both the four points in the first parenthesis, together with each of the two pairs in the second parenthesis, are situated in some conic section M^2.

(2) "*If a quadrangle* $rstu$ *is circumscribed by any two conic sections* A^2 *and* B^2, *their* 8 *tangents in the four angles* r, s, t, *and* u *always touch some third conic section* D^2." And, "*If the quadrangle is circumscribed by any conic section* A^2, *and its four tangents* P, Σ, T, *and* Υ, *in the angles* r, s, t, *and* u, *are touched by any other conic section* D^2, *then there are four tangents* P_1, Σ_1, T_1, *and* Υ_1 *to the latter, passing through the same angles of the quadrangle, which are always touched in those angles by some third conic section* B^2." Further: "*The four common tangents* R, S, T, U *of any two conic sections* A^2 *and* B^2 *circumscribing the same quadrangle* $rstu$, *together with any one of the three pairs of opposite sides of the quadrangle* Ξ *and* Ξ_1, H *and* H_1, *or* Z *and* Z_1 *are touched by some conic section* X^2, Y^2, *or* Z^2." And further: "*Of the* 8 *tangents* (P, Σ, T, Υ; P_1, Σ_1, T_1, Υ_1) *in the angles of a quadrangle* $rstu$ *to any two conic sections* A^2 *and* B^2 *circumscribing the same,* 12 *times* 4, *together with two of the four common tangents* R, S, T, U, *are touched by some conic section* M^2; *and further, these* 12 *conic sections* M^2 *arrange themselves in* 6 *pairs, touching each other twice, which have the four common tangents* R, S, T, *and* U, *taken in pairs, as tangents of contact.*"

II. From the above considerations (§ X.), we can also pass to the special cases where the given conic sections A^2 and B^2, separately considered, consist of a pair of points or lines. In this respect the following five cases may be noticed.

(1) Let B^2 consist of two lines Z and Z_1; then these are a pair of opposite sides of the quadrangle *rstu*, and their mutual intersection is the pole z. The four common tangents R, S, T, and U coincide in pairs, (RU) and (ST), and are the two tangents to the curve A^2, passing through z. Hence,

two pairs ξ and ξ_1, η and η_1 of the six angles of the former quadrilateral $RSTU$ coincide with each other in the point z, and the remaining two ζ and ζ_1 are the points of contact of the tangents (RU) and (ST); they are situated in the polar Z, and are conjugate harmonical to the poles x and y. Hereby the $Th(C_z^2)$ resolves itself into a pencil of rays around the centre z, *i.e.* each ray (right line) passing through z, conceived as double, is to be regarded as a C_z^2; its intersections with A^2 are at the same time its points of contact with the same; whereas its points of contact with $B^2 = (ZZ_1)$ are all combined in z. The throngs $Th(C_x^2)$ and $Th(C_y^2)$ remain actual curves and retain, with a few suitable modifications, all their former properties.

(2) Let B^2 consist of two points ζ and ζ_1; then these are a pair of opposite angles of the quadrilateral $RSTU$, and are situated in the polar Z. The four intersections r, s, t, and u coincide in pairs, (ru) and (st), in the intersections of $\zeta\zeta_1 = Z$ with the curve A^2; so that two pairs of opposite sides of the quadrangle $rstu$, viz. Ξ and Ξ_1, H and H_1, coincide with Z, and the third pair, Z and Z_1, become the tangents to A^2 in points (ru) and (st); these latter intersect each other in z, and are conjugate harmonical to X and Y. Here the throng $Th(C_z^2)$ consists of all pairs of tangents to the curve A^2 which intersect each other on the line Z. On the other hand, the $Th(C_x^2)$ and $Th(C_y^2)$ contain actual conic sections C^2, which pass through the given points ζ and ζ_1, and touch the given curve A^2 twice.

(3) Let A^2 and B^2 consist each of two lines H and H_1, Z and Z_1; these must be considered as two pairs of opposite sides of the quadrangle $rstu$, their own intersections being the poles y and z, and their alternate intersections the points r, s, t, and u. The four common tangents R, S, T, and U all coincide with the line $yz = \mathrm{X}$. The $Th(C_y^2)$ and $Th(C_z^2)$ resolve themselves into pencils of rays around the poles y and z in the same manner as above (1); and the $Th(C_x^2)$ alone contains actual curves, whose chords of contact pass through the pole x; their alternate chords, however, pass in pairs through the poles y and z (§ X. II.).

(4) Let A^2 and B^2 consist each of a pair of points η and η_1, ζ and ζ_1; these must then be considered as opposite angles of the quadrilateral $RSTU$, the lines $\eta\eta_1$ and $\zeta\zeta_1$ joining them, as the polars Y and Z, and the two pairs of lines joining them alternately, as the common tangents R, S, T, and U. The four intersections r, s, t, and u all coincide

in the point $x = YZ$. The $Th(C_y^2)$ resolves itself into all pairs of lines which pass through the points ζ and ζ_1, and intersect on Y; and similarly, the $Th(C_z^2)$ consists of all pairs of lines passing through the points η and η_1, and intersecting each other on Z. The $Th(C_x^2)$ continue actual curves, circumscribing the quadrangle $\eta\eta_1\zeta\zeta_1$.

(5) Lastly, let A^2 consist of two points ζ and ζ_1, and B^2 of two lines Z and Z_1; these must be considered as the elements which these letters represent, so that the line $\zeta\zeta_1 = Z$, and the intersection $ZZ_1 = z$. The four common tangents coincide in pairs with the lines $z\zeta$ and $z\zeta_1$, *i.e.* $(RU) = z\zeta$ and $(ST) = z\zeta_1$, and hence the two pairs of opposite angles ξ and ξ_1, η and η_1 coincide in z. Similarly, the four intersections r, s, t, u coincide in pairs (r and u, s and t) with the intersections of Z with Z and Z_1, so that $(ru) = ZZ$, and $(st) = ZZ_1$, and therefore both pairs of opposite sides Ξ and Ξ_1, H and H_1 coincide with Z; their intersections however, the poles x and y, remain still defined, inasmuch as they are conjugate harmonical to ζ and ζ_1, as well as to the intersections ZZ and ZZ_1. The $Th(C_z^2)$ here resolves itself into a pencil of rays around z. On the contrary, the $Th(C_x^2)$ and $Th(C_y^2)$ contain actual curves, which pass through the points ζ and ζ_1, and touch the lines Z and Z_1.

III. Knowing these special cases (II.) as well as the above general one (§ X.), the following problems can be easily treated, and the number of solutions of which they are capable can be at once determined.

(1) *To find a curve C^2, which touches each of two given conic sections A^2 and B^2 twice, and also, either*

α. touches a given line G; or

β. passes through a given point p.

Each of these problems admits of six solutions, and the curves solving them consist of $2C_x^2$, $2C_y^2$ and $2C_z^2$. The four points p, p_1, p_2, and p_3, in which the pair of curves $2C_x^2$ intersect each other, are situated in one of the conic sections M^2 (§ X. II. 4), which is defined by the point p; the intersections p_1, p_2, and p_3 are also determined, for one, p_2, lies in the line xp; again, the line p_1p_3 also passes through x, and further, both lines pp_2 and p_1p_3 are conjugate harmonical to the sides Ξ and Ξ_1.

(2) *To find a curve C^2, which touches a given curve A^2 twice, and also, either*

α. *touches three given lines* Z, Z_1 *and* G*; or*

β. *touches two given lines* Z, Z_1, *and passes through a given point* p*; or*

γ. *touches a given line* G, *and passes through two given points* ζ *and* ζ_1*; or lastly*

δ. *passes through three given points* ζ, ζ_1, *and* p.

Each of these four problems admits, in general, of six solutions as before (1).

(3) *To find a curve* C^2, *which either*

α. *touches three given lines* Z, Z_1, *and* G, *and passes through two given points* ζ *and* ζ_1*; or*

β. *touches two given lines* Z *and* Z_1, *and passes through three given points* ζ, ζ_1, *and* p.

In each case there are four solutions.

(4) *To find a curve* C^2, *which either*

α. *touches four given lines* (II. 3), *and passes through a given point* p; *or*

β. *passes through four given points* (II. 4), *and touches a given line* G.

In each case there are two solutions. And lastly:

(5) *To find a curve* C^2, *which either*

α. *touches five given lines; or*

β. *passes through five given points.*

In each case there is but one solution, hence C^2 is perfectly defined.

Section XII.

Note. The solution of the problem, "*To find a curve* C^2, *which makes a double contact with each of the three given curves* A^2, B^2 *and* D^2," is in general impossible, as is evident from the above (§ X.); it is only possible when the given curves have a certain more intimate relation to each other; as has already been shewn in the memoir so often referred to,* and which may be easily demonstrated as follows:

For, let us suppose the curve C^2 touches each of the given circles A^2, B^2, and D^2 twice; let A, B, and D be the respective chords of contact, and let further x, y, z; x', y', z'; x'', y'', z'' be the common triplets of conjugate poles, and X, Y, Z; X', Y', Z'; X'', Y'', Z'' the common triplets of conjugate polars of the pairs of curves A^2 and B^2, A^2 and D^2, B^2 and D^2; then the chord of contact A must pass through a pole of the first triplet, *e.g.* x, as well as through a pole

* *Crelle's Journal*, Bd. 37, S. 187.

of the second triplet, *e.g.* x', (because A^2 belongs to the first as well as to the second pair of curves) (§ X. II. 1); and then the chords of contact B and D must pass respectively through the same poles, x and x', as well as both, through one and the same pole of the third triplet, *e.g.* x''. Hence the three chords of contact A, B, D must be the sides of a certain triangle whose angles are any three poles, provided there be one from each triplet, as for example the triangle $xx'x''$; by combination, there are 27 such triangles. Further, as x and X, x' and X', and x'' and X'' are each pole and polar, in reference to the curve C^2 (§ X. II. 3), the triangle $xx'x''$ must be perspectively situated with respect to the trilateral $\Xi\Xi'\Xi''$; *i.e.* the three lines which join their angles, taken by pairs in a certain order, must meet in some point p; and the three intersections of the corresponding pairs of sides must be situated in some line P; in other words, if a, b, d be the angles of the trilateral, respectively opposite to the sides X, X', X'', (they are at the same time the poles of the sides A, B, D of the triangle $xx'x''$ in reference to the curve C^2, and respectively so in reference to the given curves A^2, B^2, D^2,) then the three lines ax, bx', dx'' must meet in a certain point p; and the three intersections AX, BX', DX'' must be in a line P (p and P are pole and polar in reference to C^2). These properties are accompanied further by the following. If the four points, in which the three pairs of curves each mutually intersect each other, be respectively r, s, t, u; r', s', t', u'; r'', s'', t'', u'', and their four common tangents respectively R, S, T, U; R', S', T', U'; R'', S'', T'', U'', and the remaining elements be represented in a like manner; then, the pairs of sides Ξ and Ξ_1, Ξ' and Ξ_1', Ξ'' and Ξ_1'' pass respectively through the poles x, x', x'', and the pairs of angles ξ and ξ_1, ξ' and ξ_1', ξ'' and ξ_1'' are situated respectively in the polars X, X', X''; of the first, four times three must intersect in a point (*e.g.* $\Xi\Xi'\Xi''$, $\Xi\Xi_1'\Xi_1''$, $\Xi'\Xi_1\Xi_1''$, $\Xi''\Xi_1\Xi_1'$), and of the last, four times three must be in a line (*e.g.* $\xi\xi'\xi''$, $\xi\xi_1'\xi_1''$, $\xi'\xi_1\xi_1''$, $\xi''\xi_1\xi_1'$).

If, therefore, a curve C^2 touch each of the three given curves A^2, B^2, D^2 twice, the elements of the latter must, among others, have the above properties. As these properties, however, condition each other, or are themselves dependent upon each other, the condition under which C^2 is possible, limits itself solely to any one portion of the same, viz.

"*The curve C^2, which shall touch each of the three given curves A^2, B^2, D^2 twice, is possible if*

(1) *of the 27 triangles, whose corners are any three poles, selected one out of each triplet, any one (as $xx'x''$) is perspectively situated with respect to its corresponding trilateral ($XX'X''$); or if*

(2) *of the sides of the three quadrangles $rstu$, $r's't'u'$, $r''s''t''u''$ any three, one being selected from each quadrangle (as $\Xi\Xi'\Xi''$), meet in one point p; or lastly, if*

(3) *of the angles of the three quadrilaterals $RSTU$, $R'S'T'U'$, $R''S''T''U''$ any three, one being selected from each quadrilateral (as $\xi\xi'\xi''$), are situated in a right line P."*

ON CLAIRAUT'S THEOREM, AND SOME MATTERS CONNECTED WITH IT.

By Matthew Collins, B.A.,

Senior Moderator in Mathematics and Physics, Trinity College, Dublin, and Professor of Mathematics in the Mechanics Institute, Liverpool.

To prove that a body has three *principal* axes relative to any point, *i.e.* three right lines meeting at this point, at right angles to each other, and such that

$$\int xy\,dm = 0, \quad \int xz\,dm = 0, \quad \text{and } \int yz\,dm = 0,$$

these integrals extending to the entire mass (μ) of the body.

Draw any three lines from this point perpendicular to each other, and relative to these as axes of xyz, let

$$A = \int(y^2+z^2)\,dm, \quad B = \int(x^2+z^2)\,dm, \quad C = \int(x^2+y^2)\,dm;$$

the moments of inertia of the body about the assumed axes of xyz, put also

$$P = \int xy\,dm, \quad Q = \int xz\,dm, \quad R = \int yz\,dm,$$

and draw any right line from the point making angles α, β, γ with these axes, and let r' = radius of gyration about this line, and take on the line a portion $r \propto \frac{1}{r'}$ say $= \frac{k^2}{r'}$, one end of r being the origin (or given point); I say, in the first place, that the locus of the other end of r will be an ellipsoid: for, by Poisson's *Mécanique* (vol. II. p. 55), the moment of inertia about the line, mass $\times\, r'^2$ or $\mu\,\frac{k^4}{r^2}$ will be

$$= \sin^2\alpha \int x^2 dm + \sin^2\beta \int y^2 dm + \sin^2\gamma \int z^2 dm - 2\cos\alpha\cos\beta \int xy\,dm$$
$$- 2\cos\alpha\cos\gamma \int xz\,dm - 2\cos\beta\cos\gamma \int yz\,dm,$$

but $$\sin^2\alpha = \cos^2\beta + \cos^2\gamma,$$
since $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\,;$$
therefore $$\mu\frac{k^4}{r^2} = \cos^2\alpha \int(y^2+z^2)\,dm + \cos^2\beta \int(x^2+z^2)\,dm$$
$$+ \cos^2\gamma \int(x^2+y^2)\,dm - 2\cos\alpha\cos\beta \int xy\,dm - \&c.\,;$$
therefore $$= A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2P\cos\alpha\cos\beta$$
$$- 2Q\cos\alpha\cos\gamma - 2R\cos\beta\cos\gamma.$$

Now multiply off by r^2, and remembering that $r\cos\alpha$, $r\cos\beta$, $r\cos\gamma$ are the coordinates x, y, z of the end of r, we find

$$\mu k^4 = Ax^2 + By^2 + Cz^2 - 2Pxy - 2Qxz - 2Ryz\,;$$

therefore the locus of the end of r or point xyz is an ellipsoid, having the given point as centre. Now this ellipsoid remains unaltered, whatever three lines perpendicular to each other be taken as axes of x, y, z, since its radius r is $=\dfrac{k^2}{r'}$, and r' depends only on the position of the drawn line, as is evident: and since this ellipsoid has three *principal* axes, so that its equation referred to them is of the form

$$\mu k^4 = A'x^2 + B'y^2 + C'z^2,$$

therefore relatively to these *principal* axes P, Q, and R, vanish, *i.e.*

$$\int xy\,dm = 0, \quad \int xz\,dm = 0, \quad \text{and } \int yz\,dm = 0. \quad \text{Q.E.D.}$$

The ellipsoid just determined is evidently Poinsot's Central Ellipsoid, since the moment of inertia (by construction) about any diameter of it varies inversely as the square of the said diameter, and its three *principal* axes are in the directions of the three *principal* axes of the body relatively to this point, as was just proved.

If, on each line drawn from the fixed point, a portion be taken $= r'$ the radius of gyration about this line, then, since $rr' = k^2$, therefore $r' =$ a perpendicular from centre, on a tangent plane to another ellipsoid, called by its discoverer (M'Cullagh) the Ellipsoid of Inertia, which is reciprocal to Poinsot's, and also more intimately connected with the body than Poinsot's, since the radius of gyration itself (and not its reciprocal) about any diameter of this ellipsoid of inertia is given, being equal to the portion of that diameter etween the centre and a tangent plane perpendicular to it.

The equation of an ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equation of a tangent plane to it, at the point xyz, is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1.$$

Now, at the point where this plane cuts the axis of x, $y' = 0$ and $z' = 0$, therefore there $xx' = a^2$; let r = the perpendicular let fall from the centre on this tangent plane, and α, β, γ the angles r makes with the three semiaxes, then

$$r = x' \cos\alpha = y' \cos\beta = z' \cos\gamma,$$

and so $rx = a^2 \cos\alpha$; in like manner

$$ry = b^2 \cos\beta \text{ and } rz = c^2 \cos\gamma,$$

and therefore

$$a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma = r^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right), \therefore = r^2,$$

since the equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Now, if $rr' = k^2$, r' being measured from the centre along r, then

$$\frac{k^4}{r'^2} = r^2, \therefore = a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma;$$

multiply off by r'^2, and remembering that

$$r' \cos\alpha, \quad r' \cos\beta, \quad r' \cos\gamma,$$

are the coordinates x, y, z of the end of r', we get

$$a^2x^2 + b^2y^2 + c^2z^2 = k^4,$$

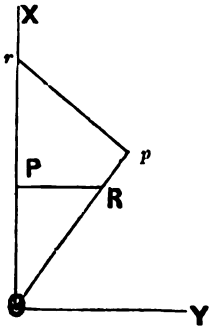
and if $aa' = bb' = cc' = k^2$, this equation gives

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1,$$

and so the locus of the end of r' is another ellipsoid; the two are called reciprocal ellipsoids, since the semidiameter of *either* is proportional to the reciprocal of the distance of the centre from the tangent plane to the other, which is perpendicular to the said semidiameter.

[redacted]

The equation of Poinsot's ellipsoid referred to *any* three rectangular axes drawn from the given point O in the body was found to be



$$Ax^2 + By^2 + Cz^2 - 2Pxy - 2Qxz - 2Ryz = \mu k^4,$$

where A equals the moment of inertia about axis of x, &c., also $P = \int xy\,dm$, &c.; now let the axis of x be OP, perpendicular to tangent plane at R to M'Cullagh's ellipsoid of inertia having its centre at O; let also the axis of y be parallel to PR, so that the point of contact R may be within the angle XOY, the axis of z is, of course, perpendicular to those of x and y; let OP meet Poinsot's ellipsoid in r, and by the property of the reciprocal ellipsoids ORp will be perpendicular to the tangent plane, at r, to Poinsot's ellipsoid, and

$$Or \times OP = OR \times Op = k^2.$$

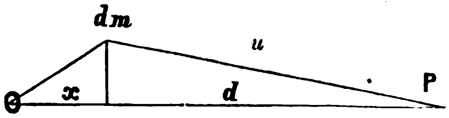
Now, the coordinates of the point r are $y' = 0$, $z' = 0$, and $x' = Or$; and as the tangent plane at r is perpendicular to ORp, and therefore parallel to axis of z, therefore $Q = 0$, else this tangent plane would meet the axis of z, and this being so, the equation of the said tangent plane at r is

$$Axx' - Px'y = \mu k^4, \text{ or } Ax - Py = \frac{\mu k^4}{x'},$$

which shows that tangent of rOp or $\dfrac{PR}{PO}$ is $= -\dfrac{P}{A}$, and as $A = \mu \times OP^2$, therefore

$$P \text{ or } \int xy\,dm = -\mu \times OP \times PR, \therefore = -\mu \times 2\Delta OPR.$$

Now, let O be the centre of gravity of a body of any



shape, attracting a very remote point P, take $OP = d$ as axis of x, let dm be a particle of the body, and u its distance from P, then

$$X = \Sigma \frac{dm}{u^2} \times \frac{d - x}{u}, \quad Y = \Sigma \frac{dm}{u^2} \times \frac{y}{u}, \quad \text{and } Z = \Sigma \frac{dm}{u^2} \times \frac{z}{u},$$

and as $$u^2 = d^2 - 2dx + x^2 + y^2 + z^2;$$

therefore $u^{-3} = d^{-3}\left(1 + \dfrac{3x}{d} + \dfrac{12x^2 - 3y^2 - 3z^2}{2d^2}\right)$ nearly,

and this multiplied by $d - x = d\left(1 - \dfrac{x}{d}\right)$ gives

$$d^{-2}\left(1 + \frac{2x}{d} + \frac{6x^2 - 3y^2 - 3z^2}{2d^2}\right),$$

and as O is centre of gravity, therefore

$$\Sigma x\,dm = 0 = \Sigma y\,dm = \Sigma z\,dm;$$

therefore $X = d^{-2}\,\Sigma dm\left(1 + \dfrac{2x}{d} + \dfrac{6x^2 - 3y^2 - 3z^2}{2d^2}\right),$

therefore $\qquad = \dfrac{\mu}{d^2} + \dfrac{3}{2d^4}\,\Sigma(2x^2 - y^2 - z^2)\,dm.$

Now $\Sigma(2x^2 - y^2 - z^2)dm = \Sigma(2x^2 + 2y^2 + 2z^2)dm - \Sigma(3y^2 + 3z^2)dm$, and as $x^2 + y^2 + z^2$ equals the square of the distance of dm from O, therefore $= x'^2 + y'^2 + z'^2$, x', y', z' being the coordinates of dm referred to the *principal* axes at O; and as

$$\Sigma(y^2 + z^2)\,dm = M,$$

the moment of inertia about OP and

$A = \Sigma(y'^2 + z'^2)\,dm, \quad B = \Sigma(x'^2 + z'^2)\,dm, \quad C = \Sigma(x'^2 + y'^2)\,dm;$

and $Y = \Sigma \frac{ydm}{u^3} = d^{-3}\Sigma dm\left(y + \frac{3xy}{d}\right), \therefore = \frac{3}{d^4}\Sigma xy\,dm$;

therefore (by what is already proved)

$$= -\frac{3\mu}{d^4} \times OP' \times P'R,$$

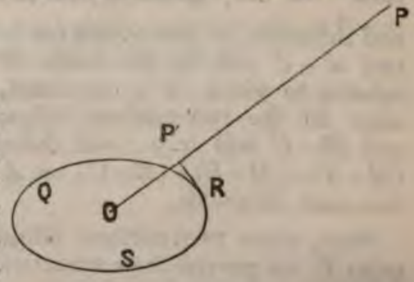
the negative sign indicating that the attraction of the body on P, in a direction perpendicular to OP, is parallel to RP' and not to $P'R$.

If two confocal ellipsoids attract an external point, their two resultants are coincident in direction, and proportional to their masses.

A being the attracting ellipsoid, whose foci are given, and P the external attracted point, let B be an ellipsoid, confocal to A, and passing through P, and let P' be the point on A *corresponding* to P on B; let also abc, $a'b'c'$ be the semiaxes of A and B, $a'b'c'$ being fixed, let $x'y'z'$ be the coordinates of P and therefore given, and let xyz be the coordinates of P' on the variable ellipsoid A, whose foci are given. Now the attraction of B on P' parallel to $a \propto x$,* therefore $\propto a$ since $\frac{x}{a} = \frac{x'}{a'}$, therefore equal a constant; but by Ivory's theorem, attraction of A on P, parallel to a, is to foregoing attraction :: $bc : b'c'$, and so attraction of A on P, parallel to $a \propto abc$, therefore varies as mass of A, and in like manner the components of A's attraction on P, parallel to b and c, are varies as mass of A; thus, when A is changed into another ellipsoid, confocal to it, the three components of its attraction on P, parallel to the semiaxes a, b, c, are all changed in the ratio of the mass of A, therefore the direction of the resultant remains invariable, but its quantity is altered in proportion to the mass of A.

We found

$$\frac{3\mu}{d^4} \times OP' \times P'R$$



above for the attraction of a body of any shape on a very distant point P, in a direction perpendicular to OP, μ being the mass of the attracting body, O its centre of gravity,

* I shall give an elementary geometrical proof of this hereaf

$OP = d$ and RSQ the ellipsoid of inertia (M'Cullagh's, and not Poinsot's) relative to the centre of gravity O, and R the point of contact of a tangent plane to this ellipsoid which cuts OP at right angles in P'. The said attraction perpendicular to OP was also proved parallel to RP' (not to $P'R$): let the semiaxes of the ellipsoid be abc, and let $OP' = r$ make angles $\alpha\beta\gamma$ with abc, and let the coordinates of R be xyz; then the said attraction of force acting on P parallel to RP', being resolved into three components parallel to a, b, c respectively, the component parallel to a will obviously be

$$\frac{3\mu}{d^4} r(r\cos\alpha - x),$$

since $r\cos\alpha - x$ is the projection of RP' on a. But we proved before that $rx = a^2\cos\alpha$, therefore the component parallel to a

$$= \frac{3\mu}{d^4}\cos\alpha\,(r^2 - a^2), \quad \therefore = \frac{3\cos\alpha}{d^4}(M - A),$$

since by the property of the ellipsoid of inertia $\mu \times OP'^2 = M$, the moment of inertia about OP, and $A = \mu a^2$; similarly, of course, the components parallel to b and c are

$$\frac{3\cos\beta}{d^4}(M - B), \text{ and } \frac{3\cos\gamma}{d^4}(M - C).$$

to be confocal to it, and very small, so that P may be a remote point relatively to it; then the components parallel to a, b, c of the small one's attraction on P, perpendicularly to the straight line PO, joining P to the centre O, being

$$\frac{3\cos\alpha}{d^4}(M' - A'), \text{ \&c },$$

therefore the analogous components of the large one's attraction will be $\frac{3\cos\alpha}{d^4}(M - A)$, for the $M - A$ of such ellipsoids was just proved to vary as their masses: and since the small ellipsoid's attraction on P (along $PO = d$), which may be considered a remote point relatively to this small one, is

$$\frac{\mu'}{d^2} + \frac{3}{2d^4}(A' + B' + C' - 3M'),$$

and since the attractions of the two ellipsoids are in the same direction, and as their masses, therefore the attraction of the large given ellipsoid on the external point P, along PO, is

$$\frac{\mu}{d^2} + \frac{3}{2d^4}(A + B + C - 3M),$$

since it was proved above that

$$A + B + C - 3M : A' + B' + C' - 3M' :: \text{mass } \mu : \text{mass } \mu'.$$

Now, since the preceding expressions for the attraction along PO, and for the components parallel to a, b, c, are linear functions of μ, A, B, C, and hold true for *any* ellipsoid of small excentricities, it follows therefore that they hold true also for an ellipsoid which is not homogeneous, but composed of concentric ellipsoidal strata, having the same *principal* axes and variable but small excentricities.

If the ellipsoid become a spheroid, so that $c = b = a(1 + e)$, then the central attraction on P, along PO,

$$\frac{\mu}{d^2} + \frac{3}{2d^4}(A + B + C - 3M)$$

on account of $C = B$ becomes

$$\frac{\mu}{d^2} + \frac{3}{2d^4}\{A + 2B - 3(B\cos^2\lambda + A\sin^2\lambda)\}$$

$$= \frac{\mu}{d^2} + \frac{3}{2d^4}(A - B)(1 - 3\sin^2\lambda) = \frac{\mu}{d^2} + T(1 - 3\sin^2\lambda),$$

where $T = \frac{3}{2d^4}(A-B)$ and $d = OP$, the distance of the attracted point P (now supposed on the surface of the spheroid) from the centre, and λ equals the angle d makes with the equator, *i.e.* equals the latitude of P.

The component parallel to a of the force perpendicular to OP, was $= \frac{3\cos\alpha}{d^4}(M-A)$; and since

$$M = B\cos^2\lambda + A\sin^2\lambda, \text{ and } \therefore M - A = (B-A)\cos^2\lambda,$$

therefore the said component

$$= -\frac{3}{d^4}(A-B)\sin\lambda\cos^2\lambda;$$

and in like manner the component parallel to b is

$$= \frac{3}{d^4}(A-B)\cos\lambda\sin^2\lambda:$$

therefore the whole force or attraction on P perpendicular to PO, being the square root of the sum of the squares of these components, is

$$= \frac{3}{d^4}(A-B)\sin\lambda\cos\lambda, \therefore = T\sin 2\lambda,$$

and tends to urge P towards the equator (on account of the

trifugal force (acting on P parallel to b), and of the force perpendicular to PO; which condition gives the equation

$$\left\{\frac{\mu}{d^2} + T(1 - 3\sin^2\lambda)\right\}\sin\theta = T\sin 2\lambda\cos\theta + g\phi\cos\lambda\sin\lambda,$$

where g is the mean force of gravity at the surface, and $g\phi$ is the centrifugal force at the equator; but since T is small compared to $\frac{\mu}{d^2}$ or g, and since $\cos\theta = 1$, and $\sin\theta = e\sin 2\lambda$ (rejecting e^2, e^3, &c.), the preceding equation becomes therefore

$$ge\sin 2\lambda = T\sin 2\lambda + \tfrac{1}{2}g\phi\sin 2\lambda,$$

or therefore
$$e = \frac{T}{g} + \frac{\phi}{2}.$$

Now, the formula $\frac{\mu}{d^2} + T(1 - 3\sin^2\lambda)$ gives $\frac{\mu}{a^2} - 2T$ for the polar gravity, and $\frac{\mu}{a^2(1+e)^2} + T - g\phi$ for the equatorial gravity, their difference is $\frac{\mu}{a^2} \times 2e + g\phi - 3T$: if this difference, divided by the mean gravity g, be denoted by n, then

$$n = 2e + \phi - \frac{3T}{g} \text{ nearly};$$

and eliminating $\frac{T}{g}$ from this and the preceding equation $e = \frac{T}{g} + \frac{\phi}{2}$, we get $n + e = \frac{5\phi}{2}$, which is Clairaut's Theorem. If the earth be homogeneous, then $n = e$, $\therefore = \frac{5\phi}{4}$. See the *Principia*, Prop. 20, Book III.

THE EQUATIONS OF SURFACES CONSIDERED AS SUMS OF HOMOGENEOUS FUNCTIONS.

By the Rev. Robert Carmichael,
Fellow of Trinity College, Dublin.

The theorems contained in the following paper are principally generalizations of others long familiar. Although, so far as I am aware, they are, with the exception of that

given in the fifth article, new; yet, from slight acquaintance with the theory of determinants, I am unable to say whether any of them have been previously published.* The paper then is mainly intended for the student, not for the advanced mathematician. It may, however, serve to systematize many results hitherto regarded as isolated, and to exhibit the convenience of adopting a more symmetrical notation in mathematical works than has been hitherto employed—unless in detached cases. I may observe that the principal results were obtained in the month of December 1851, and communicated to the Dublin University Philosophical Society in the early part of the year 1852.

1. It is easily seen that if we express any rational and integer function of x, y, z, as a sum of a number of homogeneous functions,

$$U = u_n + u_{n-1} + \ldots + u_1 + u_0;$$

then

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\right) U = nu_n + (n-1)\,u_{n-1} + \ldots + u_1.$$

Hence if $$U = 0 \ldots\ldots\ldots\ldots\ldots\ldots (\text{I.})$$

be the general equation of the surface of the n^{th} degree, and α, β, γ the coordinates of any point

$$\Big(\quad d \qquad d \qquad d\Big) \; -$$

equation of this plane is, generally,

$$\alpha \frac{dU}{dx} + \beta \frac{dU}{dy} + \gamma \frac{dU}{dz} + u_{n-1} + 2u_{n-2} + \ldots + nu_0 = 0,$$

and that the general expression for the perpendicular from the origin on the tangent plane is

$$P = -\frac{u_{n-1} + 2u_{n-2} + \ldots + nu_0}{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 + \left(\frac{dU}{dz}\right)^2\right\}^{\frac{1}{2}}}.$$

Hence the theorem—"Given a surface of the n^{th} degree, the points thereon for which the perpendicular on tangent plane is constant (k), lie on another surface of the degree $2(n-1)$, whose equation is

$$k^2 \left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 + \left(\frac{dU}{dz}\right)^2\right\} = (u_{n-1} + 2u_{n-2} + \ldots + nu_0)^2."$$

The general expression admits of instant verification in the simple case

$$U = u_n - c = 0.$$

It has been remarked to me by Mr. Spottiswoode that the polar surface to the origin passes through all the singular points of the surface $U = 0$ at which the perpendicular from the origin on the tangent plane is not infinite. In fact, at a singular point

$$\frac{dU}{dx} = 0, \quad \frac{dU}{dy} = 0, \quad \frac{dU}{dz} = 0,$$

whence $$P = \infty,$$

unless $$u_{n-1} + 2u_{n-2} + 3u_{n-3} + \ldots + nu_0 = 0$$
be satisfied.

3. Let now the point (α, β, γ) be supposed capable of motion on the surface of the m^{th} degree

$$V = v_m + v_{m-1} + \ldots + v_1 + v_0 = 0 \ldots\ldots\ldots\ldots(\text{II}),$$

and let it assume various consecutive positions on this surface. The corresponding successive polars, taken with respect to $U = 0$, will by their intersections generate a third surface, whose relation to (II.) is commonly expressed by the distinctive appellation of Reciprocal Polar, for the case in which (I.) is of the second order.

To find the equation of this third surface, differentiating (II.) and the general equation of the polar surf

respect to α, β, γ, we get

$$\frac{dV}{d\alpha}\,d\alpha + \frac{dV}{d\beta}\,d\beta + \frac{dV}{d\gamma}\,d\gamma = 0,$$

$$\frac{dU}{dx}\,d\alpha + \frac{dU}{dy}\,d\beta + \frac{dU}{dz}\,d\gamma = 0.$$

Multiplying the latter equation by the indeterminate quantity λ, adding, and putting the coefficients of $d\alpha$, $d\beta$, $d\gamma$, respectively, equal to zero, there results the system

$$\left.\begin{aligned} \frac{dV}{d\alpha} + \lambda\frac{dU}{dx} &= 0 \\ \frac{dV}{d\beta} + \lambda\frac{dU}{dy} &= 0 \\ \frac{dV}{d\gamma} + \lambda\frac{dU}{dz} &= 0 \end{aligned}\right\};$$

and between this, the equation of the polar surface, and (II), we have to eliminate α, β, γ, and λ.

To accomplish this, we multiply the three equations of the last system by α, β, γ, respectively, and remembering that

$$\alpha\frac{dV}{d\alpha} + \beta\frac{dV}{d\beta} + \gamma\frac{dV}{d\gamma} = -\{v_{m-1} + 2v_{m-2} + \ldots + mv_0\},$$

where the left-hand members contain only α, β, and γ, and the right-hand only x, y, and z.

Such an elimination, in the present state of analysis, is I believe impossible, and the general question therefore insoluble. Thus the only general representation of the envelope of the successive polar surfaces is the *system* of four equations last mentioned.

4. Upon communicating the above result to Mr. Spottiswoode, it was observed by him that the three last equations may be written in a new form, possibly leading to interesting consequences, and I am indebted to the Rev. R. Townsend for a valuable modification of his suggestions.

If we remember that the point (α, β, γ) lies on the surface $V = 0$, it is obvious that, P being the perpendicular from the origin on the tangent-plane at this point, and l, m, n the angles made by it with the coordinate axes, we may write those three last equations in the form

$$\left.\begin{aligned} -\frac{\cos l}{P} &= \frac{1}{(U)} \cdot \frac{dU}{dx} \\ -\frac{\cos m}{P} &= \frac{1}{(U)} \cdot \frac{dU}{dy} \\ -\frac{\cos n}{P} &= \frac{1}{(U)} \cdot \frac{dU}{dz} \end{aligned}\right\}.$$

It is evident that the right-hand members of the system do not admit of a modification similar to that which we have employed on the left hand, since the point (x, y, z) is *not* necessarily on the surface $U = 0$.

5. In one case, the general question of the envelope of the successive polars not only admits of solution, but the resultant equation of the envelope appears to possess both elegance and utility. It is that in which (II) assumes the symmetrical form

$$\frac{\alpha^m}{a^m} + \frac{\beta^m}{b^m} + \frac{\gamma^m}{c^m} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(II)}',$$

while (I) still retains all its generality. (The ordinary reciprocal of (II′) was given many years ago by Mr. Salmon.)

The three last equations of the third article, in this case, become

respect to α, β, γ, we get

$$\frac{dV}{d\alpha}\,d\alpha + \frac{dV}{d\beta}\,d\beta + \frac{dV}{d\gamma}\,d\gamma = 0,$$

$$\frac{dU}{dx}\,d\alpha + \frac{dU}{dy}\,d\beta + \frac{dU}{dz}\,d\gamma = 0.$$

Multiplying the latter equation by the indeterminate quantity λ, adding, and putting the coefficients of $d\alpha$, $d\beta$, $d\gamma$, respectively, equal to zero, there results the system

$$\left.\begin{aligned} \frac{dV}{d\alpha} + \lambda\frac{dU}{dx} &= 0 \\ \frac{dV}{d\beta} + \lambda\frac{dU}{dy} &= 0 \\ \frac{dV}{d\gamma} + \lambda\frac{dU}{dz} &= 0 \end{aligned}\right\};$$

and between this, the equation of the polar surface, and (11), we have to eliminate α, β, γ, and λ.

To accomplish this, we multiply the three equations of the last system by α, β, γ, respectively, and remembering that

$$\alpha\frac{dV}{d\alpha} + \beta\frac{dV}{d\beta} + \gamma\frac{dV}{d\gamma} = -\left\{v_{m-1} + 2v_{m-2} + \ldots + mv_0\right\},$$

$$\alpha\frac{dU}{dx} + \beta\frac{dU}{dy} + \gamma\frac{dU}{dz} = -\left\{u_{n-1} + 2u_{n-2} + \ldots + nu_0\right\},$$

we find that

$$\lambda = -\frac{v_{m-1} + 2v_{m-2} + \ldots + mv_0}{u_{n-1} + 2u_{n-2} + \ldots + nu_0} = -\frac{(V)}{(U)}.$$

Thus it remains for us to eliminate α, β, γ between the four equations

$$V = 0,$$

and

$$\left.\begin{aligned} \frac{1}{(V)}\cdot\frac{dV}{d\alpha} &= \frac{1}{(U)}\cdot\frac{dU}{dx} \\ \frac{1}{(V)}\cdot\frac{dV}{d\beta} &= \frac{1}{(U)}\cdot\frac{dU}{dy} \\ \frac{1}{(V)}\cdot\frac{dV}{d\gamma} &= \frac{1}{(U)}\cdot\frac{dU}{dz} \end{aligned}\right\},$$

$$\frac{\alpha^{m-1}}{a^m} + \frac{1}{(U)} \cdot \frac{dU}{dx} = 0,$$

$$\frac{\beta^{m-1}}{b^m} + \frac{1}{(U)} \cdot \frac{dU}{dy} = 0,$$

$$\frac{\gamma^{m-1}}{c^m} + \frac{1}{(U)} \cdot \frac{dU}{dz} = 0,$$

and eliminating α, β, γ between these equations and (11), we get the equation of the envelope required, in the symmetrical form

$$\left(a\frac{dU}{dx}\right)^{\frac{m}{m-1}} + \left(b\frac{dU}{dy}\right)^{\frac{m}{m-1}} + \left(c\frac{dU}{dz}\right)^{\frac{m}{m-1}} = [-(U)]^{\frac{m}{m-1}},$$

where $\qquad (U) = u_{n-1} + 2u_{n-2} + 3u_{n-3} + \ldots + nu_0.$

The discussion of some particular cases will be found to lead to interesting results.

(1) When $m = 2$, or when the pole is confined to a central surface of the second degree, then will the degree of the envelope of the successive polars with respect to a surface of the n^{th} degree be, in general,

$$2(n-1).$$

(2) When moreover the surface, with respect to which the polar is taken, is also of the second degree and central

NOTE ON THE TRANSFORMATION OF A TRIGONOMETRICAL EXPRESSION.

By Arthur Cayley.

The differential equation

$$\frac{dx}{(a+x)\sqrt{(c+x)}}+\frac{dy}{(a+y)\sqrt{(c+y)}}+\frac{dz}{(a+z)\sqrt{(c+z)}}=0,$$

integrated so as to be satisfied when the variables are simultaneously infinite, gives by direct integration

$$\tan^{-1}\sqrt{\left(\frac{a-c}{c+x}\right)}+\tan^{-1}\sqrt{\left(\frac{a-c}{c+y}\right)}+\tan^{-1}\sqrt{\left(\frac{a-c}{c+z}\right)}=0.$$

And, by Abel's theorem,

$$\begin{vmatrix} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{vmatrix}=0.$$

To shew *a posteriori* the equivalence of these two equations, I represent the determinant by the symbol $\square$, and expressing it in the form

$$\square=\begin{vmatrix} 1, & a+x, & (a+x)\sqrt{(c+x)} \\ \vdots & & \end{vmatrix}$$

I write for the moment $\xi=\sqrt{\left(\frac{a-c}{c+x}\right)}$ &c., this gives

$$\square=\begin{vmatrix} 1, & (a-c)\left(1+\frac{1}{\xi^2}\right), & (a-c)^{\frac{3}{2}}\left(\frac{1}{\xi}+\frac{1}{\xi^3}\right) \\ \vdots & & \end{vmatrix}$$

$$=\frac{(a-c)^{\frac{5}{2}}}{\xi^3\eta^3\zeta^3}\begin{vmatrix} \xi^3, & \xi^3+\xi, & \xi^2+1 \\ \vdots & & \end{vmatrix}$$

$$=\frac{(a-c)^{\frac{5}{2}}}{\xi^3\eta^3\zeta^3}\begin{vmatrix} \xi^3, & \xi, & \xi^2+1 \\ \vdots & & \end{vmatrix}$$

$$=-\frac{(a-c)^{\frac{5}{2}}}{\xi^3\eta^3\zeta^3}\left\{\begin{vmatrix} 1, & \xi, & \xi^2 \\ \vdots & & \end{vmatrix}-\xi\eta\zeta\begin{vmatrix} 1, & \xi, & \xi^3 \\ \vdots & & \end{vmatrix}\right\}$$

$$=-\frac{(a-c)^{\frac{5}{2}}}{\xi^3\eta^3\zeta^3}(\xi+\eta+\zeta-\xi\eta\zeta)\begin{vmatrix} 1, & \xi, & \xi^2 \\ \vdots & & \end{vmatrix}$$

... identically

$$\ldots \frac{\ldots-c}{\ldots+x}+\sqrt{\frac{a-c}{c+y}}+\sqrt{\frac{a-c}{c+z}}-\sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}}\Bigg\} \left| \begin{matrix} 1, & \sqrt{\frac{a-c}{c+x}}, & \frac{a-c}{c+x} \\ 1, & \sqrt{\frac{a-c}{c+y}}, & \frac{a-c}{c+y} \\ 1, & \sqrt{\frac{a-c}{c+z}}, & \frac{a-c}{c+z} \end{matrix} \right|$$

$$\ldots \frac{a-c}{c+z}}-\sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}}=0$$

... equation

$$\ldots^{-1}\sqrt{\frac{a-c}{c+y}}+\tan^{-1}\sqrt{\frac{a-c}{c+z}}=0,$$

...ions in question.

REMARQUES SUR UN MEMOIRE DE M. CAYLEY RELATIF AUX DETERMINANTS GAUCHES.

Par M. Hermite.

Mr. Cayley a nommé système gauche symétrique, un système de n^2 quantités représentées par $\lambda_{r,s}$ en attribuant aux indices toutes les valeurs entières depuis 1 jusqu' à n, lorsqu' on a la condition générale

$$\lambda_{r,s} = -\lambda_{s,r},$$

d' où resulte, $\lambda_{r,r} = 0.$

De pareils systèmes jouissent de propriétés importantes qui jouent au grand rôle dans les diverses circonstances analytiques ou ils se présentent, et Mr. Cayley en a fait lui même un nouvel usage pour la solution de cette question—

Obtenir toutes les transformations d' une forme quadratique en elle-même lorsque cette forme est une somme de carrés.

Je me propose de donner içi des formules analogues à celles de M. Cayley, pour la transformation en elle-même d'une forme quadratique quelconque. Le problème peut-être posé ainsi $f(x_1, x_2, \ldots x_n)$ designant la forme quadratique proposée ; trouver l'expression la plus générale des quantités $X_1, X_2, \ldots X_n$, qui donnent

$$f(X_1, X_2, \ldots X_n) = f(x_1, x_2, \ldots x_n).$$

Pour cela, j'imagine que les quantités X et x soient exprimées par des indéterminées auxiliaires ξ, de sorte qu' on ait en general

$$X_r + x_r = 2\xi_r,$$

et sous cette condition on va voir qu' il est très facile d' obtenir l' expression générale de X et x en ξ. On a en effet

$$X_r = 2\xi_r - x_r,$$

donc $\quad f(X_1, X_2, \ldots) = f(2\xi_1 - x_1,\ 2\xi_2 - x_2, \ldots) \ldots\ldots(1),$

ou en developpant le second membre

$$f(X_1, X_2, \ldots) = 4f(\xi_1, \xi_2, \ldots) - 2\left\{x_1 \frac{df}{d\xi_1} + x_2 \frac{df}{dx_2} + \ldots\right\}$$
$$+ f(x_1, x_2, \ldots) \ldots\ldots(2).$$

Donc par la condition supposée

$$f(X_1, X_2, \ldots) = f(x_1, x_2, \ldots),$$

cette équation se reduit à

$$x_1 \frac{df}{d\xi_1} + x_2 \frac{df}{d\xi_2} + \ldots = 2f \ldots\ldots\ldots\ldots\ldots (3).$$

Or la manière le plus générale de la vérifier en exprimant les quantités x en ξ, sera de faire

$$x_r = \xi_r + \tfrac{1}{2}\sum_1^n{}_s \lambda_{r,s} \frac{df}{d\xi_s} \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

les indéterminées λ étant assujetées à la condition

$$\lambda_{r,s} = -\lambda_{s,r}.$$

Ou en conclut

$$X_r = 2\xi_r - x_r = \xi_r - \tfrac{1}{2}\sum_1^n{}_s \lambda_{r,s} \frac{df}{d\xi_s},$$

et il est facile de reconnaitre *a posteriori* que ces expressions de X et x en ξ, donnent bien

$$f(X_1, X_2, \ldots) = f(x_1, x_2, \ldots) \ldots\ldots\ldots\ldots (5).$$

Reprennant en effet l'équation (1) et l'équation (2), on verra par l'équation (3), équation satisfaite d'elle-même, qu'on retombe précisément sur l'équation (5) qui était à vérifier. Donc enfin les expressions cherchées de X et x, qui donnent la transformation en elle-même d'une forme quelconque, s'obtiendront en résolvant par rapport aux quantités ξ les équations (4) et substituant les valeurs en x

Soit pour abréger

$$b^2 - ac = D,$$

on trouvera

$$X = 2\xi - x = \frac{(1 - 2\lambda b + \lambda^2 D)\, x - 2\lambda cy}{1 - \lambda^2 D},$$

$$Y = 2\eta - y = \frac{2\lambda ax + (1 + 2\lambda b + \lambda^2 D)\, y}{1 - \lambda^2 D}.$$

Or ces formules en posant,

$$t = \frac{1 + \lambda^2 D}{1 - \lambda^2 D},$$

$$u = \frac{2\lambda}{1 - \lambda^2 D};$$

ce qui donne

$$t^2 - Du^2 = 1,$$

deviendront

$$X = x(t - bu) - cuy,$$

$$Y = xau + (t + bu)\, y.$$

C'est la forme analytique obtenue par M. Gauss pour la question arithmétique où l'on veut que les coefficients de la substitution soient des nombres entiers.

Enfin si l'on fait l'application de la même méthode à une forme quadratique d'un nombre quelconque d'indéterminées dans le cas ou elle est une somme de carrés, on se trouvera immédiatement les résultats que M. Cayley a obtenus dans son beau mémoire, et je m'empresse de dire que je dois à l'étude de ce mémoire, l'analyse que je viens d'expresser en peu de mots. J'ajouterai cependant encore les théorèmes suivants qui servent de lemmes à une recherche arithmétique importante.

I. Ayant ramené à une somme de carrés de fonctions linéaires une forme quadratique quelconque, de sorte qu'on ait *p.ex.*

$$f = A^2 + B^2 + C^2 + \&c.$$

Si l'on désigne par $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c., ce que deviennent respectivement $A, B, C, \ldots$ lorsqu'on fait dans f, une substitution quelconque qui la change en elle-même, on aura évidemment

$$\mathfrak{A} = \alpha A + \beta B + \gamma C + \ldots$$

$$\mathfrak{B} = \alpha' A + \beta' B + \gamma' C + \ldots$$

$$\mathfrak{C} = \alpha'' A + \beta'' B + \gamma'' C + \ldots$$

....

les quantités α, β, γ, &c., etant des constantes convenablement choisies. Cela posè, à une substitution qui change f en elle-même, ou pourra toujours faire correspondre une telle représentation de f par la forme $A^2 + B^2 + C^2 + \ldots$, que l'expression

$$A\mathfrak{A} + B\mathfrak{B} + C\mathfrak{C} + \&c.,$$

ne contienne aucun des rectangles AB, AC, &c.

Pour donner une application de ce théorème, nous allons considérer le cas des formes quadratiques ternaires

$$f = A^2 + B^2 + C^2.$$

Alors les constantes α, β, γ, &c., devant être telles que

$$\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2 = A^2 + B^2 + C^2,$$

auront d'après M. Cayley, les valeurs suivantes,

$$\begin{array}{lll} \kappa\alpha = 1 + \lambda^2 - \mu^2 - \nu^2, & \kappa\alpha' = 2(\lambda\mu - \nu), & \kappa\alpha'' = 2(\lambda\nu + \mu), \\ \kappa\beta = 2(\mu\lambda + \nu), & \kappa\beta' = 1 - \lambda^2 + \mu^2 - \nu^2, & \kappa\beta'' = 2(\mu\nu - \lambda), \\ \kappa\gamma = 2(\nu\lambda - \mu), & \kappa\gamma' = 2(\nu\mu + \lambda), & \kappa\gamma'' = 1 - \lambda^2 - \mu^2 + \nu^2, \end{array}$$

où

$$\kappa = 1 + \lambda^2 + \mu^2 + \nu^2,$$

et à toute substitution S qui change f en elle-même, on pourra toujours faire correspondre un système de fonctions linéaires A, B, C, jouissant de la propriété, qu'en devenant

une substitution qui change en elle-même une forme ternaire quelconque; l'une des racines λ de l'équation

$$\Lambda = \begin{vmatrix} p-\lambda & p' & p'' \\ q & q'-\lambda & q'' \\ r & r' & r''-\lambda \end{vmatrix} = 0\,;$$

sera égale à ± 1, et les deux autres seront réciproques.

III. Il existe une infinité de formes ternaires différentes de f que la substitution ci-dessus change en elles-mêmes, ces formes seront toutes données par l'expression

$$F = kA^2 + l(B^2 + C^2),$$

k et l étant des constantes arbitraires. Cependant le cas des racines égales dans l'équation $\Lambda = 0$, doit être traité à part et exige une discussion spéciale, que nous laisserons faire au lecteur.

ON A GEOMETRICAL INTERPRETATION OF THE FUNCTION $at^2 + bu^2 + ctu$, WITH SOME APPLICATIONS.

By THOMAS WEDDLE.

In the expression $at^2 + bu^2 + ctu$, let a, b, c be constants and t, u linear functions of the variable coordinates x, y, z; so that $t = 0$, and $u = 0$, are the equations to two planes intersecting in a straight line (tu). There is no difficulty in interpreting the given function when the roots of the quadratic, $at^2 + bu^2 + ctu = 0$, are real; for if the roots are real and unequal, the value which $at^2 + bu^2 + ctu$ has at any point will be proportional to the rectangle of the perpendiculars from that point on two fixed planes; and if the roots are real and equal, it will be proportional to the square of the perpendicular from the point on a certain fixed plane. These cases therefore present no difficulty, and I shall consequently confine myself to the consideration of the case in which the roots are unreal.

If we remove the origin of coordinates to a point in the line (tu), the new plane of xy being perpendicular to the line (tu), and those of yz and xz coinciding with t and u respectively, then, h and k being certain constants,

$$t = hx \text{ and } u = ky\,;$$

and putting $\alpha = ah^2$, &c.,

$$at^2 + bu^2 + ctu = \alpha x^2 + \beta y^2 + \gamma xy.$$

Now, by turning the coordinate planes yz and xz round the axis of z, we can reduce $\alpha x^2 + \beta y^2 + \gamma xy$ to the form $mx'^2 + ny'^2$ in an infinite number of ways; and since $at^2 + bu^2 + ctu$, that is $\alpha x^2 + \beta y^2 + \gamma xy$, cannot be resolved into real factors, m and n must have the same sign (which I shall suppose to be positive).

Now, of the infinite variety of ways in which $at^2 + bu^2 + ctu$ can be transformed into $mx'^2 + ny'^2$, there is one in which $m = n$; in this case

$$at^2 + bu^2 + ctu = m(x'^2 + y'^2);$$

but if p and q be the perpendiculars from any point on the planes of $y'z$ and $x'z$, and θ be the inclination of these planes, then $p = x' \sin\theta$, and $q = y' \sin\theta$, therefore

$$at^2 + bu^2 + ctu = \frac{m}{\sin^2\theta}(p^2 + q^2).$$

Hence

(A) *If the roots of the quadratic $at^2 + bu^2 + ctu = 0$ be unreal, two planes may be drawn through the straight line (tu) such that the value which the function $at^2 + bu^2 + ctu$ has at any point shall be proportional to the sum of the squares of the perpendiculars from that point on these two planes.*

Again, of the innumerable ways in which $at^2 + bu^2 + ctu$ can be transformed into $mx'^2 + ny'^2$, there is one in which the planes of $y'z$ and $x'z$ are perpendicular to each other. In this case let δ denote the distance of any point $(x'y'z)$ from the axis of z (that is, the straight line (tu)), this distance being measured parallel to either of the planes,

$$z = \pm y'\sqrt{\left(\frac{n-m}{m}\right)},$$

(m being supposed less than n), then it is easy to shew that

$$\delta^2 = x'^2 + y'^2 + \frac{n-m}{m}\,y'^2 = \frac{mx'^2 + ny'^2}{m} = \frac{at^2 + bu^2 + ctu}{m},$$

therefore $\qquad at^2 + bu^2 + ctu = m.\delta^2.$

Moreover it will be observed that the circular sections of the elliptic cylinder

$$mx'^2 + ny'^2 = \text{constant},$$

that is, $\qquad at^2 + bu^2 + ctu = \text{constant},$

are parallel to the same planes,

$$z = \pm y' \sqrt{\left(\frac{n-m}{m}\right)}; \text{ hence,}$$

(B) *If the roots of the quadratic* $at^2 + bu^2 + ctu = 0$ *be unreal, the value which the function* $at^2 + bu^2 + ctu$ *takes at any point is proportional to the square of the distance of that point from the line* (tu), *this distance being measured parallel to a fixed plane. Moreover this fixed directive plane is parallel to a circular section of the elliptic cylinder* $at^2 + bu^2 + ctu$ = constant.*

Hence to find the directive plane, we have only to draw a plane to cut the elliptic cylinder $at^2 + bu^2 + ctu$ = constant in a circle; there are of course various ways of doing this, the following being one.

Perpendicular to the generators of the cylinder draw a plane which will cut the cylinder in an ellipse and the straight line (tu) in a point, the centre (O) of the ellipse. Draw the semimajor and semiminor axes OA and OB, and find one of the foci F. Along the axis of the cylinder (*i.e.* the straight line (tu)), and on each side of the plane of the ellipse, set off the distances Of and Of' each $=OF$; through B draw a straight line BC parallel to OA; and finally through BC and the points f and f' respectively draw planes; either of these may be taken for the directive plane. Also, if we find the equal conjugate diameters of the preceding ellipse and draw a plane through the straight line (tu), and each of these diameters, we shall have the two planes mentioned in (A).

It thus appears that if we can find the magnitude and position of the axes of the ellipse in which the cylinder $at^2 + bu^2 + ctu$ = constant is cut by a plane perpendicular to the straight line (tu), we can find both the two planes mentioned in (A) and the directive plane mentioned in (B).

It is clear that the converse of (A) is true; that is, if $t = 0$ and $u = 0$ be the equations to any two planes through the straight line (tu), then the sum of the squares of the perpendiculars from any point on any two fixed planes passing through this straight line is of the form $at^2 + bu^2 + ctu$, the roots of the quadratic $at^2 + bu^2 + ctu = 0$ being unreal; for

* Mr. Salmon informs me that he has already made use of this interpretation of the function $at^2 + bu^2 + ctu$ in connexion with the late Professor MacCullagh's method of generating surfaces of the second degree, but I have not seen Mr. Salmon's investigations.

these perpendiculars will be of the forms $\alpha t+\beta u$ and $\alpha' t+\beta' u$, and $(\alpha t+\beta u)^2+(\alpha' t+\beta' u)^2$ is of the form at^2+bu^2+ctu.

Also, the square of the distance of any point from a fixed straight line, measured parallel to a fixed directive plane, is of the form at^2+bu^2+ctu, where $t=0$ and $u=0$ are the equations to any two planes passing through the straight line, and the roots of $at^2+bu^2+ctu=0$ are unreal. For transform the coordinate axes so that the planes t and u may be taken for those of yz and xz, and the directive plane for that of xy; also let θ be the inclination of the axes of x and y, that is, of the straight lines in which the planes t and u intersect the directive plane. The square of the distance (δ) of any point (xyz) from the axis of z, measured parallel to the plane of xy, is

$$\delta^2=x^2+y^2+2xy\cos\theta.$$

Let us now restore the axes to their original position, then h and k being certain constants,

$$x=ht \text{ and } y=ku;$$

substituting these in the value of δ^2, we get a result of the form

$$\delta^2=at^2+bu^2+ctu,$$

in which the roots of $at^2+bu^2+ctu=0$ are evidently unreal.

When $c=0$, so that δ^2 is of the form at^2+bu^2, the planes t and u may be termed *conjugate* planes; and the preceding values of δ^2 inform us under what condition the planes t and u are conjugate, for if $c=0$, then we must have $\cos\theta=0$, or $\theta=90°$, so that

*If the two planes t and u be conjugate, they will cut the directive plane in straight lines at right angles to each other; and conversely, if two planes cut the directive plane in straight lines at right angles, they will be conjugate.**

Given the directive plane, we can therefore find the rectangular conjugate planes in the following manner. Draw a plane through the given straight line (tu) perpendicular to the directive plane, and through the same straight line draw another plane perpendicular to that just drawn, the two planes thus found are those required; for they are perpendicular to each other, and cut the directive plane in straight lines also perpendicular to each other.

* It is plain from what has been previously said that conjugate planes pass through conjugate diameters of the ellipse in which the cylinder $at^2+bu^2+ctu=$constant is cut by a plane perpendicular to its generators, the *rectangular* conjugate planes passing through the axes.

Also, since the two planes mentioned in (A) are equally inclined to the planes just drawn, and cut the directive planes in straight lines at right angles to one another, we can construct them as follows:—Project the line (tu) orthogonally on the directive plane and let this projection be OP, O being the point in which the straight line (tu) intersects the directive plane; in this plane draw the straight lines OQ and OR on opposite sides of OP, each making half a right angle with it; and finally through the straight line (tu), and each of the lines OQ and OR, draw a plane. The two planes thus found are evidently those required.

Conversely, when the two planes mentioned in (A) are given, we can find the directive plane mentioned in (B). It is clear that the rectangular conjugate planes bisect the dihedral angles (one acute and the other obtuse) contained by the two given planes, and that the directive plane is perpendicular to that which bisects the obtuse angle; consequently the directive plane must cut the plane (Z) which bisects the acute angle in a straight line (OQ) at right angles to the line (OP) of intersection of the two given planes. Take O for the origin of rectangular axes, OP and OQ being the axes of z and x; also let 2θ denote the acute dihedral angle contained by the two given planes, then it is clear that the equations to these planes are

$$y = \tan\theta . x, \text{ and } y = -\tan\theta . x.$$

Also, since the directive plane passes through the axis of x, its equation will be

$$y = \tan\phi . z,$$

where ϕ is the inclination of the directive plane to the plane of xz. Hence the equations to the straight lines in which the directive plane intersects the given planes are

$$\frac{x}{\cot\theta} = y = \frac{z}{\cot\phi},$$

and
$$\frac{x}{-\cot\theta} = y = \frac{z}{\cot\phi};$$

but these two straight lines are at right angles (since the given planes are conjugate), therefore

$$-\cot^2\theta + 1 + \cot^2\phi = 0,$$

which reduces to $\quad \sin\phi = \pm \tan\theta.$

Hence we can find the directive plane by the following construction:—Draw a plane z bisecting the acute dihedral angle (2θ) of the given planes, in this plane draw a straight

line OQ perpendicular to the line of intersection of the said planes; and through OQ draw a plane on each side of z, the sine of whose inclination to z shall be equal to $\tan\theta$; then either of the planes thus drawn may be taken as the directive plane.

As a first application of these principles, let us take the equation to an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(c+z)(c-z)}{c^2}.$$

Now, by (B), $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ is proportional to the square of the distance of the point (xyz) from the axis of z, this distance being measured parallel to a circular section of the cylinder

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Hence

I. *If from any point in an ellipsoid a perpendicular be drawn to any of the axes, the rectangle of the segments of the axis is proportional to the square of the distance of the point from the axis, this distance being measured parallel to a circular section of that circumscribed cylinder whose generators are parallel to the axis.*

It is clear that this theorem will still be true, if instead of the axis we substitute any diameter of the ellipsoid, providing the segments of the diameter be made, not by a perpendicular but by a plane drawn through the point conjugate to the diameter. If the diameter be not an axis, the most interesting case is when it is conjugate to a circular section of the ellipsoid, for this being also one of the circular sections of the corresponding circumscribed cylinder, the preceding theorem will then take this form:

II. *If from any point in an ellipsoid a straight line be drawn parallel to a circular section so as to meet the diameter conjugate to that section, then shall the square of the straight line be proportional to the rectangle of the segments of the diameter.*

If we had applied (A) instead of (B) to interpret the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(c+z)(c-z)}{c^2},$$

we should have got the following theorem instead of (1).

If from any point in an ellipsoid a perpendicular be drawn to any axis, the rectangle of the segments of the axis is proportional to the sum of the squares of the perpendiculars from the point on two fixed planes passing through the axis.

The two fixed planes can, as we have seen, be constructed by means of the circumscribed cylinder. The interpretation (A), however, is in general so much less interesting than (B), that I shall take no further notice of the former, but shall leave the reader to modify each of the following theorems by using (A) instead of (B), it being remembered that the same cylinder or ellipse that determines the directive plane, also enables us to find the two planes mentioned in (A).

Again, the equation to the umbilical hyperboloid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(z+c)(z-c)}{c^2}.$$

III. *If from any point in an umbilical hyperboloid a straight line be drawn perpendicular to the real axis (produced), the rectangle of the segments of the axis is proportional to the square of the distance of the point from the axis, this distance being measured parallel to a certain fixed plane.*

This of course may be modified for any (real) diameter, as in the case of the ellipsoid. As z = constant, in the equation to the hyperboloid, implies $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ = constant, it follows that the cylinder of which the 'certain fixed plane' is a circular section will have for its base any section of the hyperboloid made by a plane perpendicular to the axis (or, more generally, conjugate to the diameter), its generators being parallel to the axis (or diameter, as the case may be). The following is evident.

IV. *If from any point in an umbilical hyperboloid a straight line be drawn parallel to a circular section, so as to meet the diameter (produced) conjugate to that section, then shall the square of the straight line be proportional to the rectangle of the segments of the diameter.*

Again, the equation to the elliptic paraboloid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{p},$$

we have the following theorem:

V. *If from any point in an elliptic paraboloid a straight line be drawn perpendicular to the axis, the portion of the axis intercepted between its vertex and the perpendicular is proportional to the square of the distance of the point from the axis, this distance being measured parallel to a certain fixed plane.*

This may be generalized for any diameter as in the ellipsoid and umbilical hyperboloid; and the cylinder which determines the fixed plane may be constructed precisely as in the case of the hyperboloid. When the diameter is conjugate to either series of circular sections, we have the following theorem:

VI. *If from any point in an elliptic paraboloid a straight line be drawn parallel to a circular section, so as to meet the diameter conjugate to that section, then shall the square of this straight line be proportional to the segment of the diameter intercepted between its vertex and the aforesaid straight line.*

It is scarcely necessary to remark that theorems (I), (III) and (V), (and consequently (II), (IV) and (VI),) are analogous to familiar and fundamental properties of the ellipse, hyperbola, and parabola, respectively.

The equation to a cone of the second degree is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2},$$

where the plane of xy is any plane through the vertex not intersecting the cone in any other real point, and the axis of z is the conjugate or reciprocal line. Hence

VII. *The distance of any point in a cone of the second degree from any plane passing through the vertex but not intersecting the cone in any other real point, has a constant ratio to its distance from the line reciprocal to the plane, this distance being measured parallel to a certain fixed plane.*

The cylinder which determines the last-mentioned plane has for base any section of the cone by a plane parallel to the plane through the vertex, and its generators are parallel to the reciprocal line.

When the plane through the vertex of the cone is parallel to a circular section, it is clear that the distance of the point from the line is to be measured parallel to that section.

Let there be a ruled surface of the second degree, and AB a straight line not intersecting the surface in real points. In this straight line take any point A, and draw its polar

plane intersecting AB in B: draw the polar plane of B which will pass through A and cut the former plane in a straight line CD; in this line take any point C and find its polar plane which will pass through AB and cut CD in some point D; then the tetrahedron $ABCD$ will be such that each angle is, with respect to the ruled surface, the pole of the opposite face. Hence if the faces BCD, ACD, ABD, and ABC, be denoted by $t = 0$, $u = 0$, $v = 0$, and $w = 0$, respectively, the equation to the surface of the second degree will be of the form

$$at^2 + bu^2 + cv^2 + ew^2 = 0;$$

but since the surface is *ruled*, two of the quantities a, b, c, e must be positive and two negative; also since the straight line AB or (vw) does not intersect the surface in real points, a and b must have the same sign; hence the preceding equation may be written

$$at^2 + bu^2 = fv^2 + gw^2,$$

where the constants a, b, f, and g are positive; or supposing these constants to be implicitly contained in t, u, v, and w, the equation may be written

$$t^2 + u^2 = v^2 + w^2.$$

Recollecting that the straight lines (tu) and (vw) are reciprocal, this equation gives us the following theorem:

VIII. *The distance of any point in a ruled surface of the second degree from a straight line which does not intersect the surface in real points is proportional to its distance from the reciprocal line, each distance being measured parallel to a fixed plane.*

We have now to determine the fixed planes (one corresponding to each line). Having constructed the planes t, u, v, and w, as before, we have

$$t^2 + u^2 = v^2 + w^2$$

for the equation to the surface; put $v = 0$, and we get

$$t^2 + u^2 = w^2,$$

the equation to the enveloping cone having its vertex at the angular point (tuw); next let w = constant = c, and we get

$$t^2 + u^2 = c^2,$$

which is the equation to the cylinder by means of which we have to find the fixed plane corresponding to the straight line (tu). Hence we may proceed as follows.

Find the sides in which the enveloping cone whose vertex is at (tuw), cuts the planes t and u, (or, in other words, join the point (tuw) to each of the points in which the straight lines (tv) and (uv) intersect the given surface); cut these sides by a plane parallel to w, and through the four points of section draw parallels to the straight line (tu); then a plane drawn perpendicular to (tu) will cut these parallels in points which are the extremities of two conjugate diameters of the ellipse in which the cylinder, $t^2 + u^2 = c^2$, is cut by a plane perpendicular to its generators. Having thus two conjugate diameters of this ellipse, the axes can be constructed and the directive plane found (as at p. 69). In a similar manner the directive plane corresponding to the straight line (vw) may be obtained.

If the surface of the second degree instead of being ruled be umbilical, then, having found the planes t, u, v, and w, as before, three of these planes will intersect the surface, and the fourth will not; let this last be the plane w, then the equation to the surface will be

$$t^2 + u^2 + v^2 = w^2,$$

therefore $$t^2 + u^2 = (w + v)(w - v).$$

Now $w + v = 0$ and $w - v = 0$ are the equations to the two planes touching the surface at the extremities of the chord (tu); hence the following theorem, of which, by-the-by, (I), (III), and (V) may be considered particular cases:

IX. *The rectangle of the perpendiculars from any point in an umbilical surface of the second degree on two tangent planes is proportional to the square of the distance of the point from the chord of contact, this distance being measured parallel to a certain fixed plane.**

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* In an ingenious paper in this *Journal* Mr. Walker has given the following theorem (see vol. VII. p. 27):

"If two planes touch a surface of the second order in two points, the rectangle under perpendiculars let fall from any point on the surface on the planes is to the square of the perpendicular let fall on the chord of contact in a constant ratio."

By comparing this theorem with (IX) in the text, it is evident however that it is true only for those positions of the tangent planes in which the fixed (or directive) plane is perpendicular to the chord; (in this case the cylinder $t^2 + u^2 = c^2$ is of revolution). A more general theorem given by Mr. Walker is also liable to the same objection, and it therefore requires a little modification. One way of making this modification is as follows:

Let two surfaces of the second degree have double contact, and let $P = 0$ be the equation to one of them; then, by suitably drawing two planes $t = 0$ and $u = 0$ through the chord of contact, the equation to the

The fixed plane may be constructed precisely as in the case of theorem (VIII), taking care to make (tuw) and not (tuv) the vertex of the enveloping cone (for the cone $t^2 + u^2 + v^2 = 0$ is imaginary).

This theorem is perfectly analogous to a property of the conic sections, but it must be carefully noted that it applies to umbilical surfaces only, for the fixed or directive plane becomes imaginary in the case of the ruled surfaces.

It would not be difficult to give other applications of (B), but enough has been done for my present purpose, and I shall therefore conclude with one or two inferences from theorem (IX), which are analogous to certain plane properties.

Let $A_1 A_2 A_3 \ldots\ldots A_n$ be any polygon, plane or twisted, inscribed in an umbilical surface of the second degree; also let $p_1, p_2, p_3 \ldots \ldots p_n$ be the perpendiculars drawn from any point in the surface on the tangent planes at $A_1, A_2, A_3 \ldots A_n$; and $\delta_{12}, \delta_{23}, \delta_{34} \ldots \delta_{n1}$, the distances of the same point from the chords or sides $A_1A_2, A_2A_3, A_3A_4 \ldots A_nA_1$, these distances being measured parallel to certain planes which are to be constructed in the manner previously shewn. Then, $a, b, c, \ldots l$ being certain constants, we have, by (IX),

$$\left.\begin{aligned} p_1p_2 &= a.\delta_{12}^2, \\ p_2p_3 &= b.\delta_{23}^2, \\ p_3p_4 &= c.\delta_{34}^2, \\ &\vdots \\ p_np_1 &= l.\delta_{n1}^2, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

Multiplying these equations and extracting the square root, we have a result of the form

$$p_1 p_2 p_3 \cdots p_n = m \,.\, \delta_{12}.\delta_{23}.\delta_{34} \ldots \delta_{n1},$$

which gives this theorem:

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other surface may be exhibited under one of these forms,

$$P + t^2 + u^2 = 0 \text{ or } P + t^2 - u^2 = 0.$$

Now in the former case the two surfaces intersect in imaginary conics in the imaginary planes $t + u\sqrt{(-1)} = 0$ and $t - u\sqrt{(-1)} = 0$; and in the latter they intersect in two conics, real or imaginary, in the real planes $t + u = 0$, and $t - u = 0$; hence, calling these planes (as M. Chasles has done) the *symptotic* planes of the two surfaces, we have the following theorem:

Let two surfaces of the second degree have double contact; when their symptotic planes are real, the rectangle of the segments of the chord (or *secant*) *to one of these surfaces drawn through any point in the other parallel to a fixed line, is proportional to the rectangle of the perpendiculars from that point on the symptotic planes; but when the symptotic planes are imaginary, the first rectangle is proportional to the square of the distance of the point from the chord of contact, this distance being measured parallel to a certain fixed plane.*

X. *If any polygon, either plane or twisted, be inscribed in an umbilical surface of the second degree, then planes may be found (one corresponding to each side) such that the continued product of the perpendiculars from any point in the surface on the tangent planes at the angles is proportional to the continued product of the distances of the said point from the sides, these distances being measured parallel to the planes previously found.*

Again, when the number of sides of the polygon is even, multiplying the first, third, fifth, &c. of equations (α), and also the second, fourth, sixth, &c., it is easy to see that we get a result of the form

$$\delta_{12}.\delta_{34}.\delta_{56}\ldots\ldots = m.\delta_{23}.\delta_{45}.\delta_{67}\ldots\ldots$$

Hence,

XI. *If an even-sided polygon, either plane or twisted, be inscribed in an umbilical surface of the second degree, planes may be found (one corresponding to each side) such that the continued product of the distances of any point in the surface from one set of alternate sides is proportional to the continued product of the distances of the same point from the other set of alternate sides, these distances being measured parallel to the planes previously found.*

In a plane the analogous theorem may be extended to an odd-sided polygon, provided we suppose the side wanting to be supplied by a tangent at one of the angles; and exactly in the same way (XI) may be extended to an odd-sided polygon, provided we substitute the perpendicular from the point in the surface on the tangent plane at one of the angles instead of the distance of the said point from the deficient side. Thus, suppose the polygon to have five sides we shall, from equations (α), get a result of the form

$$\delta_{12}.\delta_{34}.\delta_{51} = m.\delta_{23}.\delta_{45}.p_1,$$

so that $\delta_{12}.\delta_{34}.\delta_{51}$ is proportional to $\delta_{23}.\delta_{45}.p_1$.

It is worthy of observation that the plane theorem analogous to (XI) is convertible in the case of the quadrilateral, but that (XI) is not; for the locus of a point the product of whose distances from two opposite sides of a quadrilateral has a constant ratio to the product of its distances from the other two opposite sides is, in a plane, a conic circumscribed about the quadrilateral; while in three dimensions it is in general a surface of the fourth degree.

York Town, near Bagshot,
December 9, 1852.

ON A NEW AND SIMPLE RULE FOR APPROXIMATING TO THE AREA OF A FIGURE BY MEANS OF SEVEN EQUIDISTANT ORDINATES.

By Thomas Weddle.

Let $\beta, \beta_1, \beta_2, \ldots \beta_7$ be the values of the ordinate y when $x = 0, h, 2h, \ldots 6h$ respectively, or when $z = 0, 1, 2, \ldots 6$ respectively, where $z = \frac{x}{h}$. By a well-known formula we have

$$y = \beta + z.\Delta\beta + z(z-1).\frac{\Delta^2\beta}{2} + z(z-1)(z-2).\frac{\Delta^3\beta}{2.3} + \&c.$$

$$= \beta + z.\Delta\beta + (z^2 - z).\frac{\Delta^2\beta}{2} + (z^3 - 3z^2 + 2z).\frac{\Delta^3\beta}{2.3} + \&c.$$

Multiply by $\frac{dx}{h} = dz$, and integrate, therefore

$$\begin{aligned}\frac{1}{h}\int y\,dx &= z\beta + \frac{z^2}{2}.\Delta\beta + \left(\frac{z^3}{3} - \frac{z^2}{2}\right)\frac{\Delta^2\beta}{2} \\
&+ \left(\frac{z^4}{4} - \frac{3z^3}{3} + \frac{2z^2}{2}\right).\frac{\Delta^3\beta}{2.3} + \left(\frac{z^5}{5} - \frac{6z^4}{4} + \frac{11z^3}{3} - \frac{6z^2}{2}\right).\frac{\Delta^4\beta}{2.3.4} \\
&+ \left(\frac{z^6}{6} - \frac{10z^5}{5} + \frac{35z^4}{4} - \frac{50z^3}{3} + \frac{24z^2}{2}\right).\frac{\Delta^5\beta}{2.3.4.5} \\
&+ \left(\frac{z^7}{7} - \frac{15z^6}{6} + \frac{85z^5}{5} - \frac{225z^4}{4} + \frac{274z^3}{3} - \frac{120z^2}{2}\right).\frac{\Delta^6\beta}{2.3\ldots6} \\
&+ \left(\frac{z^8}{8} - \frac{21z^7}{7} + \frac{175z^6}{6} - \frac{735z^5}{5} + \frac{1624z^4}{4} - \frac{1764z^3}{3} + \frac{720z^2}{2}\right).\frac{\Delta^7\beta}{2.3\ldots7} \\
&+ \left(\frac{z^9}{9} - \frac{28z^8}{8} + \frac{322z^7}{7} - \frac{1960z^6}{6} + \frac{6769z^5}{5} - \frac{13132z^4}{4}\right. \\
&\qquad\qquad \left. + \frac{13068z^3}{3} - \frac{5040z^2}{2}\right).\frac{\Delta^8\beta}{2.3\ldots8} \\
&+ \&c. \ldots\ldots\ldots\ldots\ldots\ldots (\text{A}).\end{aligned}$$

Take this integral between the limits $z = 0$ and $z = 6$ (which correspond to the limits $x = 0$ and $x = 6h$), and multiply by h, therefore

$$\begin{aligned}\int_0^{6h} y\,dx = \{6\beta + 18.\Delta\beta + 27.\Delta^2\beta + 24.\Delta^3\beta + \frac{123}{10}.\Delta^4\beta + \frac{33}{10}.\Delta^5\beta \\
+ \frac{41}{140}\Delta^6\beta \ldots - \frac{9}{1400}\Delta^8\beta + \&c.\}\, h \ldots\ldots (\text{B}).\end{aligned}$$

Now in finding the area of a figure by means of seven equidistant ordinates, we must suppose sixth differences constant, hence all the terms in (B) after $\Delta^6\beta$ will vanish; also since $\frac{41}{140}$ differs from $\frac{42}{140}=\frac{3}{10}$ by the small fraction $\frac{1}{140}$ only, and $\Delta^6\beta$ will usually be small, we may write $\frac{3}{10}\Delta^6\beta$ instead of $\frac{41}{140}\ \Delta^6\beta$, without material error; and then (B) takes the form

$$\int_0^{6h} y.dx = \{6\beta + 18.\Delta\beta + 27.\Delta^2\beta + 24.\Delta^3\beta + \frac{123}{10}.\Delta^4\beta + \frac{33}{10}.\Delta^5\beta + \frac{3}{10}\ \Delta^6\beta\}\ h \ \ldots\ldots(\mathrm{C}).$$

For $\Delta\beta$, $\Delta^2\beta$, &c. write their values $\beta_1 - \beta$, $\beta_2 - 2\beta_1 + \beta$, &c., and (C) reduces to

$$\int_0^{6h} y.dx = \frac{3h}{10}\{\beta + \beta_2 + \beta_4 + \beta_6 + 5(\beta_1 + \beta_5) + 6\beta_3\}\ldots(\mathrm{D}),$$

from which the value of the integral $\int_0^{6h} y.dx$, that is, of the area required, can be computed; but it will be better to throw the right-hand member into the form

$$\frac{3h}{10}\{\beta_3 + \beta + \beta_2 + \beta_4 + \beta_6 + 5(\beta_1 + \beta_3 + \beta_5)\},$$

which gives us the following very simple rule for approximating to the area of a figure by means of seven equidistant ordinates.

To five times the sum of the even ordinates, add the fourth (or middle) ordinate, and all the odd ordinates, multiply this sum by the common distance between the ordinates and three tenths of the product will be the area required.

It appears from the preceding investigation that this rule will give the *exact* area when *fifth* differences are constant (so that the sixth and subsequent differences vanish); while it differs (in excess) from the true value by only $\frac{1}{140}\ \Delta^6\beta$, when sixth (or even seventh) differences are constant. In other cases it will give the area very nearly *providing* the differences beginning at the sixth are small.

York Town, near Bagshot,
June 15, 1853.

ON THE TRANSFORMATION OF DIFFERENTIAL EQUATIONS.

By W. H. L. Russell, B.A.

In the differential equation

$$S\frac{dy}{dx} = P + Qy + Ry^2,$$

let P, Q, R be functions of (x) and contain moreover a certain quantity (n). If we put $y = \frac{u_0 + v_0 z}{u + vz}$ where u_0, v_0, u, v are functions of x and (n), the transformed equation will be of the form

$$S_0 \frac{dz}{dx} = P_0 + Q_0 z + R_0 z^2,$$

P_0, Q_0, R_0, S_0 being functions of (x) and (n). Now we know that in certain specific cases S_0 after the transformation remains the same as S, while P_0, Q_0, R_0 are formed from P, Q, R, by merely increasing or diminishing (n) by unity. It becomes therefore interesting to ascertain what forms of the functions P, Q, R, S admit of this transformation. I propose in the following paper to shew how this may be done by certain examples. The transformations which I am about to give, besides being elegant in themselves, derive interest from their connexion with the theory of continued fractions, by which the functions which satisfy the differential equations which we are about to consider may always be expressed. An analogous method of investigation will also, I apprehend, apply to other classes of differential equations. I shall not in this paper write down the transformed differential equations, as well as those from which they were derived, but shall suppose the former always formed from the latter by changing (y) into (z), and (n) into $(n+1)$. Moreover, I shall always suppose the continued fractions I shall have occasion to refer to, to be of this form,

$$\frac{Ax^\mu}{1+} \quad \frac{Bx^\mu}{1+} \quad \frac{Cx^\mu}{1+} \cdots$$

In the differential equation

$$s\frac{dy}{dx} = p + qy + ry^2,$$

let $y = \frac{x}{\alpha + \beta z}$, where α and β do not contain either (x) or

(z). Then the transformed equation will be

$$s\frac{dz}{dx} = \frac{s\alpha}{\beta x} - \frac{p\alpha^2}{\beta x} - \frac{q\alpha}{\beta} - \frac{rx}{\beta}$$
$$+ \left(\frac{s}{x} - \frac{2p\alpha}{x} - q\right)z - \frac{\beta p}{x}z^2.$$

Let
$$p = (A + Bx) \quad + n(C + Ex),$$
$$q = (A' + B'x) \ + n(C' + E'x),$$
$$r = (A'' + B''x) + n(C'' + E''x),$$
$$s = L + Mx + Nx^2.$$

The transformed equation is to be formed from the original equation by changing (n) into $(n+1)$. Hence, we easily see that $\quad L = 0, \quad A = 0, \quad C = 0, \quad B'' = 0, \quad E'' = 0.$

For the right-hand side of the transformed equation must contain no negative power of (x), and no positive power greater than unity. We shall, moreover, have the following equations obtained by equating like powers of (x), (we may put $M = 1$):

$$\frac{\alpha}{\beta} - \frac{\alpha^2}{\beta}(B + nE) - \frac{\alpha}{\beta}(A' + nC') = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

Also from (2) we shall have the following equation,

$$B'(1+n-A') - (B+nE)(A''+nC'')$$
$$= -\{B+(n+1)E\}\{A''+(n+1)C''\};$$

$$\therefore\ (1-A')B' + C''B + EA'' + EC'' = 0, \quad B' + 2EC'' = 0.$$

Hence, if $A'' = 0$, we have

$$C'' = -\frac{B'}{2E}, \quad A' = \frac{E-B}{2E}.$$

Hence the equation may be written as follows:

$$x(1+2mx)\frac{dy}{dx} = (a+en)x + \left\{\frac{e-a}{2e} + mx - n\right\}y - \frac{mn}{2e}y^2,$$

$$y = \frac{x}{\alpha+\beta z}, \quad \text{where } \alpha = \frac{(2n+1)e+a}{2e(a+en)}, \quad \beta = \frac{(n+1)m}{2e(a+en)},$$

This equation can be completely integrated when (n) is an integer.

The following is obtained by assuming p, q, r to be of two dimensions in both x and (n), and proceeding as before,

$$x(1+mx^2)\frac{dy}{dx} = (a+bn+cn^2)x^2 + \left(\frac{c-b}{c} + mx^2 - 2n\right)y - \frac{m}{c}y^2,$$

$$y = \frac{x^2}{\alpha+\beta z}, \quad \text{when } \alpha = \frac{(2n+1)c+b}{c(a+bn+cn^2)}, \quad \beta = \frac{m}{c(a+bn+cn^2)},$$

The known equations by which $\tan x$ and $\tan^{-1}x$ are expanded into continued fractions are particular cases of this. More generally,

$$x(1+mx^\mu)\frac{dy}{dx} = (a+bn+cn^2)x^\mu + \left(\frac{c-b}{2c} + \frac{mx^\mu}{2} - n\right)\mu y - \frac{\mu^2 m}{4c}y^2,$$

$$y = \frac{x^\mu}{\alpha+\beta z}, \quad \alpha = \frac{\mu(2cn+c+b)}{2c(a+bn+cn^2)}, \quad \beta = \frac{\mu^2 m}{4c(a+bn+cn^2)}.$$

This equation can be integrated whenever the equation $a + bx + cx^2 = 0$ has a positive entire root.

Let $\alpha = \phi(n)$, $\beta = \psi(n)$, then the continued fraction into which y can be expanded has only one period, and the coefficient of x^μ in it $= \dfrac{\psi(n)}{\phi(n)\,\phi(n+1)}$.

I shall now give an instance of a differential equation leading to a continued fraction with a double period, obtained by an analogous process:

$$x(1+mx)\frac{dy}{dx} = anx + \left(cx - 2n + \frac{ab+c}{m}\right)y + \left(b - \frac{mn}{a}\right)y^2,$$

$$y = \frac{\alpha x(1+\gamma z)}{1+\beta x+\gamma z}, \quad \alpha = \frac{amn}{(2n+1)m-(ab+c)}, \quad \gamma = \frac{m\{ab-m(n+1)\}}{a\{ab+c-2m(n+1)\}},$$

$$\beta = \frac{m\{(n+1)m-c\}\{(n+1)m-(ab+c)\}}{\{(ab+c)-(2n+1)m\}\{(ab+c)-2m(n+1)\}}.$$

This equation is always integrable when (n) is an integer. The known equation by which ε^x is expanded into a continued fraction is a particular case of it. Let

$$\alpha = \phi(n), \quad \beta = \psi(n), \quad \gamma = \theta(n).$$

Then the coefficients of x in the continued fraction by which (y) may be expressed are alternately of the forms $\psi(n)$ and $\theta(n)\,\phi(n+1)$.

The following is an example of a similar equation:

$$mx^2\frac{dy}{dx} = (k-an)x + (cx-2)y + by^2, \quad \text{when } m = \frac{ab}{2},$$

$$y = \frac{\alpha x(1+\gamma z)}{1+\beta x+\gamma z}, \quad \alpha = \frac{k-an}{2}, \quad \gamma = -\frac{b}{2},$$

$$\beta = \frac{(n+1)ab - (2c+bk)}{}.$$

and we derive the $(p-1)$ congruences to modulus (p):

$$A_0 + A_1 + A_2 + A_3 \ldots\ldots\ldots\ldots\ldots + A_{p-2} \equiv 0,$$
$$A_0 + 2^{p-2}A_1 + 2^{p-3}A_2 + 2^{p-4}A_3 \ldots\ldots + 2A_{p-2} \equiv 0,$$
$$A_0 + 3^{p-2}A_1 + 3^{p-3}A_2 + 3^{p-4}A_3 \ldots\ldots + 3A_{p-2} \equiv 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_0 + (p-1)^{p-2}A_1 + (p-1)^{p-3}A_2 + (p-1)^{p-4}A_3 \ldots + (p-1)A_{p-2} \equiv 0.$$

Now the determinant formed by the coefficients of

$$A_0,\ A_1,\ A_2,\ \ldots\ldots\ A_{p-2}$$

is $1.2.3\ldots(p-1)$ multiplied into the product of the differences of $1, 2, 3, \ldots (p-1)$, and is therefore incongruent to zero for the modulus p. Hence, there being $(p-1)$ *independent* homogeneous congruences between $(p-1)$ quantities, each of these quantities must be congruent to zero, that is

$$A_0 \equiv 0, \quad A_1 \equiv 0, \ldots\ldots A_{p-2} \equiv 0 \ [\text{mod.}\ p].$$

The congruence $A_0 \equiv 0$, that is $1 + 1.2.3\ldots(p-1) \equiv 0$ [mod p], is evidently Sir John Wilson's Theorem. We see also (by virtue of the remaining equations) at the same time, that the sums of the binary, ternary, &c., up to the $(p-2)^{\text{ary}}$ combinations of the numbers $1, 2, 3 \ldots (p-1)$, are all severally congruent to zero to the modulus p; *i.e.* are all divisible by that number.

J. J. S.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

By J. J. SYLVESTER.

Section VII. Continued.

BEFORE proceeding further I must guard against a misconception as to my meaning to which the modification of the title of this memoir might give birth; it is not to be understood that I regard the Theory of Invariants as coextensive with the Calculus of Forms, but only with a certain portion of that Calculus which is here exclusively treated of; the Calculus of Forms itself has for its subject-matter the whole theory of the Composition, Decomposition, and Comparison of Forms. In the theory of invariants the composition of single forms with sets of linear forms is alone considered,

and the idea of invariance must be regarded as a transient idea arising out of an artificial mode of viewing the effects of composition, so as to ignore the presence in the result of factors which depend on the resultants of the linear forms employed, which resultants, although in this portion of the subject treated as a mere moduli and as such generally supposed to be reduced to unity, yet in regard to the general theory are as important as the factors which are retained as the sole objects of contemplation; so that in fact the idea of invariance is but a special and it may be said accidental notion which merges in the more general notion of permanency of character in the result of forms compounded in a given manner out of given forms. Again, as to combinants, the idea contained in this word may, by a change in the mode of statement of the definition, be extended to functions of unlike degrees. A combinant of $U, V, W, \ldots\ldots$ all functions of the same system or systems of variables is in fact only another name for invariants of the function $\lambda U + \mu V + \nu W + \&c.$, where, over and above the sets of variables contained in $U, V, W, \ldots$ there is a new correlated set of variables $\lambda, \mu, \nu, \&c.$ So now, more generally, if $U, V, W, \ldots\ldots$ are of $p, q, r, \ldots\ldots$ dimensions in one set of variables of which the highest number is I if λ is taken of $I-p$, μ of $I-q$, W of $I-r$, &c. dimensions in the same, the functions $\lambda, \mu, \nu, \&c.$ being each the most general of their kind, any invariant of $\lambda U + \mu V + \nu W + \ldots\ldots$ which is such as well in respect to the coefficients in $\lambda, \mu, \nu, \ldots\ldots$ which must be considered as forming a set among themselves, as also in respect to the set of variables in $U, V, W, \ldots$ will be a combinant to the system $U, V, W, \ldots$; and so, more generally, if $U, V, W, \ldots\ldots$ contain several (say i) unrelated sets or systems of sets of variables, we must form in an analogous manner

$$\lambda_1\lambda_2\ldots\lambda_i U + \mu_1\mu_2\ldots\mu_i V + \nu_1\nu_2\ldots\nu_i W + \&c.,$$

and then an invariant in respect to the i given sets in $U, V, W, \ldots$ and the i new sets contained in $(\lambda_1, \mu_1, \nu_1, \ldots)$, $(\lambda_2, \mu_2, \ldots \nu_2)$, &c. $(\lambda_i, \mu_i, \nu_i, \ldots)$ will be a combinant to the system $U, V, W. \ldots$ Perhaps, however, a more immediate extension of the idea of combinants to the case supposed of i unrelated sets or systems of sets would be to take, instead of $\lambda_1\lambda_2\ldots\lambda_i$, $\mu_1\mu_2\ldots\mu_i$, &c., the perfectly general forms of the same degrees in each set of the variables as these quantities are respectively of the same; to use these general forms, the coefficients of which will constitute not i new sets but a

single new set of variables, as the syzygetic multipliers to $U, V, W, \ldots\ldots$ and then the invariant of the corresponding conjunctive in respect to the i original sets or systems of sets, and the one new set of variables thus obtained will be a combinant to the given system of Functions.* As a matter of punctilio I may here take the opportunity of observing that the process for obtaining the relation between שׁ, ט (inadvertently written טּ), and R, would have been more perfectly symmetrical to the eye had the equation for W (p. 262) been written $\tau(z^2-y^2)=W$ in lieu of $\sigma(y^2-z^2)=W$. I now return to take up the subject from the point where it was brought to a close in the last number of the *Journal*.

Let us consider what the equation (A)† becomes when U, V, W becomes the first partial derivatives (quâ x, y, z) of a single homogeneous cubic function ψ, so that

$$U=\frac{d\psi}{dx}, \quad V=\frac{d\psi}{dy}, \quad W=\frac{d\psi}{dz},$$

ϑ then becomes the Hessian of ψ, and the S of this (like every other invariant of ψ)‡ may be expressed, as is well known, as a rational integral function of the S and T of ψ. The relation between the S of the H and the S and T may readily be obtained from the canonical form

$$(\psi) = x^3 + y^3 + z^3 + 6mxyz.$$

The Hessian of this is

$$(1 + 2m^3)\, xyz - m^2(x^3 + y^3 + z^3);$$

and making $-\dfrac{1+2m^3}{6m^2} = \mu$, the S of this Hessian will be

$$6^4 . m^8 \times (\mu - \mu^4),$$

---

* I propose to append at the end of the next or some subsequent Section what ought to have been given in this or previous place, viz. the general differential equations for any concomitant to any congeries of forms, comprising amongst them any number of various distinct (*i.e.* unrelated) classes of systems of sets of variables, the relations between the sets belonging to any one system being supposed to be either simple or compound, and after the manner of either cogredience or contragredience; in fact, to do this only requires a slight extension of the formulæ given by me with that object in the fifth section of my paper in the *Philosophical Transactions* for the year 1853, Part III., which see.

† *Vide* last number of this *Journal*, near the end of Author's paper therein.

‡ I have given a perfectly rigid demonstration in the *Philosophical Magazine*, in the early part of 1853, that every invariant to a cubic function of three variables is a rational integral function of the two Aronholdian invariants S and T.

which is $(1 + 2m^3)\{(1 + 2m^3)^3 + 216m^6\}$.

(See Calculus of Forms, *Camb. and Dub. Math. Journal*, Sect. 3, vol. VIII.)

that is

$$\begin{aligned} & 1 + 8m^3 + 240m^6 + 464m^9 + 16m^4, \\ & = (1 - 20m^3 - 8m^2)^2 + 48(m - m^4)^3, \\ & = (S)^3 + 48(T)^3, \end{aligned}$$

where (S) and (T) are respectively the S and T of (ψ). Hence we have in general

$$S.H.\psi = (S\psi)^3 + 48(T\psi)^3.$$

So that $\dot{\varpi}$ becomes $T^2 + 48S^3$, and ϖ evidently from Calculus of Forms (same page) becomes

$$\tfrac{16}{8}(1 - 20m^3 - 8m^6),$$

that is $2T$, so that

$$4\dot{\varpi} - \tfrac{1}{4}\varpi^2 = 3T^2 + 192S^3;$$

so that equation (A) becomes

$$R = T^2 + 64S^3,$$

the Aronholdian representation of the Discriminant of ψ.

We see from this numerical calculation that it is not $\Sigma\Omega$ but $\frac{1}{2}\Sigma\Omega$ which ought to receive the appellation of ϖ, making which modification the general equation, written A, becomes

which will be a cubic covariant to the system. The S of this will be another form of $\dot{\varpi}$. So too, again, if we border the matrix to the Jacobian determinant above written vertically and horizontally with ξ, η, ζ, and call the determinant of the matrix thus formed I', I' will be quadratic in the system x, y, z, in the system ξ, η, ζ, and in the system formed by the coefficients of U, V, W, and the result of affecting this with the operator Σ will be the same as the result of the operation upon Ω with the same symbol; that is to say, $\frac{1}{24}E.I'$ will be equal to ϖ, this latter symbol being so taken (as last explained) in such a way that $3R$ shall equal $4\dot{\varpi} - \varpi^2$, and each of the four lines in the operator Σ being supposed to go their complete number (6) of permutations.

The terms sextic and dodecadic combinants will not be sufficient *per se* to characterize $\dot{\varpi}$ or ϖ (to a numerical factor *près*), supposing that there exist combinants of the 3rd and 9th degree respectively in the coefficients, in which case the general sextic would contain two and the general dodecadic five arbitrary numerical parameters.

This makes so much the more remarkable and satisfactory the method above developed for finding $\dot{\varpi}$ and ϖ as undecompounded forms; the general dodecadic combinant at all events being rendered indeterminate by virtue of the existence of a sextic combinant above demonstrated.

It is interesting to evince the identity of the S of the Jacobian with that of the discriminant to the conjunctive of U, V, W, which latter has been called $\dot{\varpi}$.

Starting with the canonical forms of the system U, V, W, and neglecting the ρ and σ, which cannot influence the result of the intended comparison, we have

$$J(U,V,W) = \begin{vmatrix} x; & -y; & 0 \\ 0; & -y; & z \\ gz+hy; & hx+y+fz; & gz+hy \end{vmatrix}$$

$$= fx(y^2+z^2) + gy(x^2+z^2) + hz(x^2+y^2) + xyz.$$

And multiplying by 6 and adopting the same notation as before (from the *Higher Plane Curves*, p. 182), we have

$$\begin{aligned} &b_1 = 2f, \quad b_2 = 0, \quad b_3 = 2h, \\ &a_1 = 0, \quad a_2 = 2g, \quad a_3 = 2h, \\ &c_1 = 2f, \quad c_2 = 2y, \quad c_3 = 0, \\ &\qquad\quad d = 1. \end{aligned}$$

And the expression for S in *Higher Plane Curves*, p. 184, becomes, omitting every term containing a_1, b_2, or c_3,

$$d^4 - 2d^2(b_1c_1 + c_2a_2 + a_3b_3) + 3d(a_2b_3c_1 + a_3b_1c_2)$$
$$- (b_3c_2a_2a_3 + a_3c_1b_1b_3 + a_2b_1c_1c_2) + (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2),$$

that is $1 - 8(f^2 + g^2 + h^2) + 48fgh - 16(h^2y^2 + g^2f^2 + f^2h^2)$
$+ 16(f^4 + g^4 + h^4)$, so that

$$\underset{x,y,z}{S}\ J(U,V,W) = \dot{\text{ש}} = \underset{\lambda,\mu,\nu}{S}\ \underset{x,y,z}{\square}(\lambda U + \mu V + \nu W),$$

as was to be shewn. As observed above, the form first found has the advantage over the one just obtained in disclosing the elements (cubic invariants to U, V, W) of which the ש is a biquadratic function. So, analogously, the resultant of two quadratic functions (P, Q) of x and y may be exhibited either under the form of the discriminant in respect to the coefficients of conjunction of the discriminant in respect to the original variables of the conjunctive of P, Q, or under the form of the discriminant of the Jacobian of P, Q. The former discloses the invariantive composition of the resultant which remains latent in the latter. As regards the ט, the proof of its being capable of the second mode of generation above indicated must, on account of the tediousness of the calculation, be for the present reserved; nor can I assert the fact with entire confidence until I have made a more complete investigation into the combinants of the system U, V, W, the remarks concerning which, in p. 262, I wish to be considered as provisionally withdrawn.

The analogy between the invariants of a cubic form of three variables and a biquadratic of two has been frequently insisted upon in the foregoing pages; but we shall now see that this analogy has its foundation in the deeper-seated analogy which connects a ternary system of quadratics of three variables with a binary system of cubics of two variables.

We may suppose the two given functions so combined that the linear conjunctive $lP + mQ$ shall contain two equal roots, and so take the form x^2y; this may then be combined with either of the given functions so as to give a conjunctive of the form

$$ax^3 + 3xy^2 + dy^3,$$

and writing for x and y, $\dfrac{x}{\sqrt[3]{a}}$, $\dfrac{y}{\sqrt[3]{d}}$, respectively, and multiply-

ing $lP + mQ$ by $\sqrt[3]{a^2d}$, we obtain for our standard form

$$P = 3x^2y,$$
$$Q = x^3 + 3exy^2 + y^3.$$

The resultant of this system rejecting an universally-irrelevant numerical factor is 1.

Again, write

$$\lambda P + \mu Q = \lambda x^2y + \mu x^3 + 3\mu exy^2 + \mu y^3,$$

and operate upon this with the commutator (say ω),

$$\begin{vmatrix} \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \end{vmatrix}$$

Keeping one of the lines (*ex. gr.* the first) stationary, and, for greater brevity, writing $\delta_\lambda, \delta_\mu, \delta_x, \delta_y$ in place of $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{dx}, \frac{d}{dy}$, we obtain 8 positions, which, remembering that the order *in* the lines of these positions (and not the order *of* the lines) is the only thing to be attended to, are equivalent to

$$\begin{vmatrix} \delta_\lambda & \delta_\mu \\ \delta_x & \delta_y \\ \delta_x & \delta_y \\ \delta_x & \delta_y \end{vmatrix} - 3 \times \begin{vmatrix} \delta_\lambda & \delta_\mu \\ \delta_x & \delta_y \\ \delta_x & \delta_y \\ \delta_y & \delta_x \end{vmatrix} + 3 \times \begin{vmatrix} \delta_\lambda & \delta_\mu \\ \delta_x & \delta_y \\ \delta_y & \delta_x \\ \delta_y & \delta_x \end{vmatrix} - \begin{vmatrix} \delta_\lambda & \delta_\mu \\ \delta_y & \delta_x \\ \delta_y & \delta_x \\ \delta_y & \delta_x \end{vmatrix}$$

Hence we have $\frac{1}{36}\omega(\lambda P + \mu Q) = -e$.

[I need hardly observe, that in general for any two odd-degreed functions of the same degree in x, y, as

$$a_0x^m + ma_1x^{m-1}.y + m.\tfrac{1}{2}(m-1)\,a_2x^{m-2}.y^2 + \ldots + m(a_1)\,xy^{m-1} + (a_0)\,y^m,$$
$$b_0x^m + mb_1x^{m-1}.y + m.\tfrac{1}{2}(m-1)\,b_2x^{m-2}.y^2 + \ldots + m(b_1)\,xy^{m-1} + (b_0)\,y^m,$$

we may obtain, in an analogous manner, the combinant

$$a_0(b_0) - ma_1(b_1) + m.\tfrac{1}{2}(m-1)\,a_2(b_2) + \&c.$$

Moreover it is easily shewn that when m is an even integer

the above expression will remain invariant, although of course it is no longer a combinant.]

Again, the Hessian to $\lambda P + \mu Q$ will be

$$\begin{vmatrix} \mu x + \lambda y, & \lambda x + \mu e y \\ \lambda x + \mu e y, & \mu e x + \mu y \end{vmatrix},$$

which is equal to

$$e\mu^2 x^2 + \mu^2 xy - e^2\mu^2 y^2 + \lambda\mu y^2 - e\lambda\mu xy - \lambda^2 x^2,$$

which call $H.C$ (C meaning the conjunctive of P, Q). Let this be operated upon with the commutator

$$\begin{matrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2, \\ \delta_\lambda^2, & \delta_\lambda\delta_\mu, & \delta_\mu^2, \end{matrix}$$

which call Ω.

Since neither $y^2\lambda^2$ nor $xy\lambda^2$ does not enter $H.C$, we have only to consider out of the full number 6 of positions the two effective positions

$$\begin{vmatrix} \delta_x^2 & \delta_x\delta_y & \delta_y^2 \\ \delta_\lambda^2 & \delta_\lambda\delta_\mu & \delta_\mu^2 \end{vmatrix} - \begin{vmatrix} \delta_x^2 & \delta_x\delta_y & \delta_y^2 \\ \delta_\lambda^2 & \delta_\mu^2 & \delta_\lambda\delta_\mu \end{vmatrix}$$

Hence $\qquad \frac{1}{16} EHC(P,Q) = 1 - e^3.$

So that $\qquad \{\frac{-1}{36}\omega C(P,Q)\}^3 + \frac{1}{16} EHC(P,Q)\}$

$$= R(P,Q).$$

Thus R is expressed in terms of the cube of a simple quadratic combinant and a sextic compound combinant, which is made up of quadratic invariants. When P and Q become of the form $\frac{d\psi}{dx}$, $\frac{d\psi}{dy}$, respectively (ψ being a quartic in x and y), these become respectively (to numerical factors *près*) the quadrinvariant of the given function and the cube invariant of its Hessian, which latter is a linear function of the cube of the quadrinvariant and the square of the cubinvariant of the given function, as we know *à priori* from the fact of the fundamental scale of the quartic consisting of the quadrinvariant and cubinvariant (for a rigid demonstration of which fact see the *Philosophical Magazine* in the early part of 1853), and the expression for the resultant thus resolves itself into the known composite form of the sum of a square and cube.

The simple sextic combinant represented by $E.H.C(P,Q)$ may also, analogous to what has been observed concerning the ט, be expressed as a commutant (in fact the cubinvariant)

of the Jacobian to P and Q, but then the form will no longer disclose its invariantive sub-composition. So too, if it were thought worth while to push the analogies to an extreme, the quadri-combinant to P, Q might have been found, first by bordering the Hessian to the conjunctive to P, Q with ξ, η horizontally and vertically, and operating upon the result with the commutator

$$\left|\begin{array}{cc} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{array}\right|$$

or by bordering the Jacobian to P, Q with ξ, η, as before, and then operating upon the result with the commutator

$$\left|\begin{array}{cc} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{array}\right|$$

I propose hereafter to return to the consideration of the fundamental scale of combinants to the two systems, viz. of 3 quadratics in x, y, z, and of 2 cubics in x, y, which have been treated of in this section.

Section VIII.

On the Reduction of a Sextic Function of Two Variables to its Canonical Form.

In the *London and Edinburgh Philosophical Magazine* for Nov. 1851, after giving a simple method for representing any

function of two variables of an odd degree $(x, y)^{2m+1}$ under the form of

$$u_1^{2m+1} + u_2^{2m+1} + \ldots\ldots + u_{m+1}^{2m+1},$$

where $u_1, u_2 \ldots u_{m+1}$ are linear functions of x, y (which form, as appears from the method of obtaining it, is unique). I proceeded to shew how by a certain method therein explained the biquadratic and octavic function of x, y, $(x, y)^4$, $(x, y)^8$ could be thrown under the respective forms

$$u_1^4 + u_2^4 + mu_1^2.u_2^2,$$

$$u_1^8 + u_2^8 + u_3^8 + u_4^8 + mu_1^2.u_2^2.u_3^2.u_4^2,$$

the number of values of m in the first form being 3 and in the second form 5, the quantity m in the one case depending on the solution of the equation

$$\begin{vmatrix} a_0 & a_1 & a_2+\lambda \\ a_1 & a_2-\frac{1}{2}\lambda & a_3 \\ a_2+\lambda & a_3 & a_4 \end{vmatrix} = 0,$$

where a_0, a_1, a_2, a_3, a_4 are the coefficients of $(x, y)^4$ multiplied respectively by $1, \frac{1}{4}, \frac{1}{6}, \frac{1}{4}, 1$; and in the other case, on the solution of the equation

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4+\lambda \end{vmatrix}$$

be found entirely to fail. Here, however, considerations of a purely morphological character step in to our aid and immediately lead to the true canonical representation of the sextic function. Algebraically speaking, the only connexion between two identical forms F and F is through the equation $F=\psi^{-1}\psi F$; but, morphologically considered, a *form* F may admit of being derived by a series of entirely heterogeneous operations from *itself*. In general, supposing

$$F(x, y) = ax^n + nbx^{n-1}.y + \&c. \ldots + n(b)\,xy^{n-1} + (a)\,y^n,$$

the form $\xi^n \dfrac{d}{da} + \xi^{n-1}\eta \dfrac{d}{db} + \ldots + \xi\eta^{n-1}\dfrac{d}{d(b)} + \eta^n \dfrac{d}{d(a)}$,

operating upon any concomitant to F will, we know, (from the law of reciprocity in Section IV.) produce another concomitant. The operative form above written is termed the evector, and the result of operating therewith upon a concomitant is termed the evectant of the latter, which is said, when so operated upon, to be evected.* The polar reciprocal of the evector may be termed the contravector, and for two variables is of course of the form

$$y^n \frac{d}{da} - y^{n-1}.x \frac{d}{db} \pm \&c.$$

If we suppose n to be even, $F(x, y)$ will have the well-known quadrinvariant

$$a(a) - nb.(b) + n\,\tfrac{1}{2}(n-1)\,c.(c) \mp \&c.,$$

and if this be operated upon with the contravector, or if we like so to say, be contravected, we recover the original function F, so that any function of two variables of an even degree is the contravect of its quadrinvariant.

If now we return to the representation of $(x, y)^4$ under the form

$$u_1^4 + u_2^4 + m(u_1u_2)^2,$$

and make

$$u_1u_2 = F_2(x, y);$$

or to that of $(x, y)^8$ under the form

$$u_1^4 + u_2^4 + u_3^4 + u_4^4 + m(u_1u_2u_3u_4)^2,$$

* These terms "evector, evectant, contravectant, to evect and contravect," will of course admit of an immediate extension to functions of any number of variables. Evection gives rise to contravariants, contravection to covariants; but on this account to interchange the meanings respectively attached to the terms evector and contravector, and their respective allied terms, would be a simplification too dearly purchased at the expense of contravening the principle that the *word for the base should be the base for the word.*

and make $$u_1.u_2.u_3.u_4 = F_4(x, y),$$
the outstanding term multiplied by the parameter m may be regarded in each of these two cases as the squared contravects of the quadrintrivariants F_2 and F_4 respectively. Under this point of view we at once see a ground for the proved fact of $(x, y)^6$ not being capable of being thrown under the form
$$u_1^6 + u_2^6 + u_3^6 + m\{F_3(x, y)\}^2,$$
where $$u_1.u_2.u_3 = F_3(x, y),$$
because there exists no quadrinvariant to $F_3(x, y)$, the only invariant which it possesses being the discriminant which is of the fourth degree; if however instead of $m\{F_3(x, y)\}^2$ we write $mF_3(x, y)\,G_3(x, y)$, where $G_3(x, y)$ is the contravect of the discriminant of F_3, we shall find that the method applied to the reduction of $(x, y)^4$ and to $(x, y)^8$ will perfectly well succeed for $(x, y)^6$, as I proceed to demonstrate.

Let this function be written under the form
$$a_0x^6 + 6a_1x^5y + 15a_2x^4y^2 + 20a_3x^3y^3 + 15a_4x^2y^4 + 6a_5xy^5 + a_6y^6,$$
which suppose made equal to
$$(p_1x + q_1y)^6 + (p_2x + q_2y)^6 + (p_3x + q_3y)^6$$
$$+ (Ax^3 + 2Bx^2y + 3Cxy^2 + Dy^3)\,(Lx^3 + Mx^2y + Nxy^2 + Py^3);$$

*Let $\quad q_1 = p_1\lambda_1, \quad q_2 = p_2\lambda_2^4, \quad q_3 = p_3\lambda_3,$

$$\lambda_1 + \lambda_2 + \lambda_3 = 3s_1, \quad \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = 3s_2, \quad \lambda_1\lambda_2\lambda_3 = s_3,$$

$$p_1p_2p_3 = m.$$

Then $\quad A = m, \quad 3B = 3ms_1, \quad 3C = 3ms_2, \quad D = s_3.$

$$L = m^3(4s_1^3 - 6s_1s_2 + 2s_3),$$

$$M = m^3(6s_1^2s_2 + 6s_1s_3 - 12s_2^2),$$

$$N = m^3(12s_1^2s_3 - 6s_1^2s_2 - 6s_2s_3),$$

$$P = m^3(+\ 6s_1s_2s_3 - 4s_2^3 - 2s_3^2).$$

Let $(Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3)(Lx^3 + Mx^2y + Nxy^2 + Py^3)$

$$= K_0x^6 + K_1x^5y + K_2x^4y^2 + K_3x^3y^3 + K_4x^2y^4 + K_5xy^5 + K_6y^6$$

$$= T.$$

Then, equating this term for term with

$$\lambda_1^6(x + p_1y)^6 + \lambda_2^6(x + p_2y)^6 + \lambda_3^6(x + p_3y)^6 + \mu T,$$

we obtain the 7 equations following:

$$p_1^6 + p_2^6 + p_3^6 + \mu K_0 = a_0 \ldots\ldots\ldots (1),$$

$$p_1^6\lambda_1 + p_2^6\lambda_2 + p_3^6\lambda_3 + \frac{\mu}{6}K_1 = a_1 \ldots\ldots\ldots (2),$$

$$p_1^6\lambda_1^2 + p_2^6\lambda_2^2 + p_3^6\lambda_3^2 + \frac{\mu}{15}K_2 = a_2 \ldots\ldots\ldots (3),$$

$$p_1^6\lambda_1^3 + p_2^6\lambda_2^3 + p_3^6\lambda_3^3 + \frac{\mu}{20}K_3 = a_3 \ldots\ldots\ldots (4),$$

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* The reader will please to make the following correction in the text preceding this:—

All that comes between the word 'operator' (10th line from foot of p. 96) and the letters 'A, B, C, D' (3rd line from foot of same page) is to be omitted and the following inserted in its stead:

$$-y^3\frac{d}{dA} + y^2x\frac{d}{dB} - yx^2\frac{d}{dC} + x^3\frac{d}{dD},$$

and identifying the result with $Lx^3 + Mx^2y + Nxx^2 + Py^3$, we have

$$L = -6ABC + 2A^2D + 4B^3,$$

$$M = 6ABD - 12AC^2 + 6B^2C,$$

$$N = -6ACD + 12DB^2 - 6BC^2,$$

$$P = 6BCD - 2AD^2 - 4C^3.$$

$$p_1^6\lambda_1^4 + p_2^6\lambda_2^4 + p_3^6\lambda_3^4 + \frac{\mu}{15}K_4 = a_4 \ldots\ldots\ldots (5),$$

$$p_1^6\lambda_1^5 + p_2^6\lambda_2^5 + p_3^6\lambda_3^5 + \frac{\mu}{6}K_5 = a_5 \ldots\ldots\ldots (6),$$

$$p_1^6\lambda_1^6 + p_2^6\lambda_2^6 + p_3^6\lambda_3^6 + \mu K_6 = a_6 \ldots\ldots\ldots (7).$$

Eliminating linearly

$p_1^6,\quad p_2^6,\quad p_3^6$ between equations 1, 2, 3, 4,

$\lambda_1 p_1^6,\ \lambda_2 p_2^6,\ \lambda_3 p_3^6$ 2, 3, 4, 5,

$\lambda_1^2 p_1^6,\ \lambda_2^2 p_2^6,\ \lambda_3^2 p_3^6$ 3, 4, 5, 6,

$\lambda_1^3 p_1^6,\ \lambda_2^3 p_2^6,\ \lambda_3^3 p_3^6$ 4, 5, 6, 7,

we obtain the 4 equations following, viz.

$$a_0 s_3 - 3a_1 s_2 + 3a_2 s_1 - a_3 = \mu\vartheta_0,$$

$$a_1 s_3 - 3a_2 s_2 + 3a_3 s_1 - a_4 = \mu\vartheta_1,$$

$$a_2 s_3 - 3a_3 s_2 + 3a_4 s_1 - a_5 = \mu\vartheta_2,$$

$$a_3 s_3 - 3a_4 s_2 + 3a_5 s_1 - a_6 = \mu\vartheta_3,$$

where $\vartheta_0 = K_0 s_3 - \frac{3}{6}K_1 s_2 + \frac{3}{15}K_2 s_1 - \frac{1}{20}K_3$

$$= \tfrac{1}{60}(60K_0 s_3 - 30K_1 s_2 + 12K_2 s_1 - 3K_3),$$

$$\vartheta_1 = \tfrac{1}{6}K_1 s_3 - \tfrac{3}{15}K_2 s_2 + \tfrac{3}{20}K_3 s_1 - \tfrac{1}{15}K_4$$

$$\frac{1}{m^4}K_3 = \frac{AP+3BN+3CM+DL}{m^4} = (6s_1s_2s_3 - 4s_2^3 - 2s_3^2)$$
$$+ (36s_1^3s_3 - 18s_1^2s_2^2 - 18s_1s_2s_3)$$
$$+ (18s_1^2s_2^2 + 18s_1s_2s_3 - 36s_2^3)$$
$$+ (4s_1^3s_3 - 6s_1s_2s_3 + 2s_3^2)$$
$$= 40s_1^3s_3 - 40s_2^3,$$

$$\frac{1}{m^4}K_4 = \frac{3BP+3CN+DM}{m^4} = 18s_1^2s_2s_3 - 12s_1s_2^3 - 6s_1s_3^2$$
$$+ 36s_1^2s_2s_3 - 18s_1s_2^3 - 18s_2^2s_3$$
$$+ 6s_1^2s_2s_3 + 6s_1s_3^2 - 12s_2^2s_3$$
$$= 60s_1^2s_2s_3 - 30s_1s_2^3 - 30s_2^2s_3,$$

$$\frac{1}{m^4}K_5 = \frac{3CP+DN}{m^4} = 18s_1s_2^2s_3^2 - 12s_2^4 - 6s_2s_3^2$$
$$+ 12s_1^2s_3^2 - 6s_1s_2^2s_3 - 6s_2s_3^2$$
$$= 12s_1^2s_3^2 + 12s_1s_2^2s_3 - 12s_2^4 - 12s_2s_3^2,$$

$$\frac{1}{m^4}K_6 = \frac{DP}{m^4} = 6s_1s_2s_3^2 - 4s_2^3s_3 - 2s_3^3;$$

$$\therefore \quad \frac{60\vartheta_0}{\mu m^4} = 240s_1^3s_3 - 360s_1s_2s_3 + 120s_3^2$$
$$- 360s_1^4s_2 + 360s_1^2s_2^2 - 360s_1s_2s_3 + 360s_2^3$$
$$+ 360s_1^4s_2 + 360s_1^3s_3 - 720s_1^2s_2^2$$
$$- 120s_1^3s_3 + 120s_2^3$$
$$= 120(s_3^2 + 4s_2^3 + 4s_1^3s_3 - 3s_1^2s_2^2 - 6s_1s_2s_3),$$

i.e. $\vartheta_0 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)$.

Again,

$$\frac{\vartheta_1}{\mu m^4} = 120s_1^4s_3 - 120s_1^2s_2s_3 + 120s_1s_3^2 - 120s_2^2s_3$$
$$- 360s_1^3s_2^2 - 360s_1^2s_2s_3 + 720s_1s_2^3$$
$$360s_1^4s_3 - 360s_1s_2^2$$
$$- 240s_1^2s_2s_3 + 120s_1s_2^3 + 120s_1s_2^2s_3$$
$$= 120(s_1s_3^2 + 4s_1s_2^3 + 4s_1^4s_3 - 3s_1^3s_2^2 - 6s_1^2s_2s_3$$

$$\therefore \quad \vartheta_1 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6A$$

Again,

$$\frac{60\vartheta_2}{\mu m^4} = 120 s_1^3 s_2 s_3 + 120 s_1^2 s_3^2 - 240 s_1 s_2^2 s_3$$
$$- 360 s_1^3 s_2 s_3 + 360 s_2^4$$
$$+ 720 s_1^3 s_2 s_3 - 360 s_1^2 s_2^3 - 360 s_1 s_2^2 s_3$$
$$- 120 s_1^2 s_3^2 - 120 s_1 s_2^2 s_3 + 120 s_2^4 + 120 s_2 s_3^2$$
$$= 120 (s_2 s_3^2 + 4 s_2^4 + 4 s_1^3 s_2 s_3 - 3 s_1^2 s_2^3 - 6 s_1 s_2^2 s_3),$$

$$\therefore \quad \vartheta_2 = 2\mu (A^2 D^2 + 4AC^3 + 4DB^3 - 3B^2 C^2 - 6ABCD) s_2.$$

Finally,

$$\frac{60\vartheta_3}{\mu m^4} = 120 s_1^3 s_3^2 - 120 s_2^3 s_3$$
$$- 720 s_1^2 s_2^2 s_3 + 360 s_1 s_2^4 + 360 s_2^3 s_3$$
$$+ 360 s_1^3 s_3^2 + 360 s_1^2 s_2^2 s_3 - 360 s_1 s_2^4 - 360 s_1 s_2 s_3^2$$
$$- 360 s_1^2 s_2 s_3^2 + 240 s_2^3 s_3 + 120 s_3^3$$
$$= 120 (s_3^3 + 4 s_2^3 s_3 + 4 s_1^3 s_3^2 - 3 s_1^2 s_2^2 s_3 - 6 s_1 s_2 s_3^2),$$

$$\therefore \quad \vartheta = 2\mu (A^2 D^2 + 4AC^3 + 4DB^3 - 3B^2 C^2 - 6ABCD) s_3.$$

Hence, writing

$$2\mu (A^2 D^2 + 4AC^3 + 4DB^3 - 3B^2 C^2 - 6ABCD) = \rho,$$

equation for finding ρ, expanded out, becomes

$$\frac{\rho^4}{9}+\left(\frac{2}{3}\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} - \frac{2}{3}\begin{vmatrix} a_1 & a_3 \\ a_3 & a_5 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} a_0 & a_3 \\ a_3 & a_6 \end{vmatrix}\right)\rho^2,$$

$$-\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix} = 0;$$

that is
$$\rho^4 + (15a_2a_4 - 6a_1a_5 - 10a_3^2 + a_0a_6)\,\rho^2$$

$$+\begin{vmatrix} a_3 & a_2 & a_1 & a \\ a_4 & a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 & a_2 \\ a_6 & a_5 & a_4 & a_3 \end{vmatrix} = 0;$$

the coefficient of ρ^2 being the well-known quadrinvariant, and the final term the meiocatalecticizant of the given function. There will consequently be four different values of ρ and four different systems of values of s_1, s_2, s_3, expressible for each system respectively in terms of ρ by means of any three out of the four equations (R), and consequently there will be four systems of values of λ_1, λ_2, λ_3, each of which may be found separately by solving the cubic equation

$$\lambda^3 - 3s_1\lambda^2 + 3s_2\lambda - s_3 = 0;$$

also $K_0, K_1, K_2, K_3, K_4, K_5, K_6$ become known multiples of m^4, and finally, the values of any λ and K system being thus determined, we may then, by means of the identity

$$p_1^6(x+\lambda_1 y)^6 + p_2^6(x+\lambda_2 y)^6 + p_3^6(x+\lambda_3 y)^6$$

$$+\mu m^4\left(\frac{K_0}{m^4}x^6 + \frac{K_1}{m^4}x^5y + \text{\&c.} + \frac{K_6}{m^4}y^6\right) = a_0x^6 + 6a_1x^5y + \text{\&c.} + a_6y^6,$$

write down at will any 4 equations out of the 7 equations therefrom resulting, and these will serve to determine linearly the values of p_1^6, p_2^6, p_3^6, μm^4; and consequently, by means of the equations

$$q_1 = p_1\rho_1,\quad q_2 = p_2\rho_2,\quad q_3 = p_3\rho_3,$$

q_1, q_2, q_3 are known, and consequently every coefficient in

$$(p_1x+q_1y)^6 + (p_2x+q_2y)^6 + (p_3x+q_3y)^6 + \mu M$$

is completely determined. But we shall hereafter return to this theory, and seek for a direct method of finding the four values of the functions (p_1x+q_1y), (p_2x+q_2y), (p_3x+q_3y).

It appears from the above investigation that there are four modes of throwing $(x, y)^6$ under the assumed form which possess the remarkable property of separating into two pairs of modes, as is obvious from the fact of the resolving equation in ρ having two pairs of roots, those of the same pair being equal but of contrary signs. As this form will be of extreme value in studying the invariants of $(x, y)^6$, it may be well to consider the simplest shape to which it admits of being reduced.

We may suppose $(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$ thrown under the form of $u^3 + v^3$, the contravectant of the discriminant to which in respect to u and v is $v^3 - u^3$, so that we may use for the canonical form the expression

$$a(u+v)^6 + b(u+\rho v)^6 + c(u+\rho^2 v)^6 + \mu(u^6 - v^6),$$

where $\rho^3 = 1$; or if we please, more simply

$$a(u+v)^6 + b(u+\rho v)^6 + c(u+\rho^2 v)^6 + u^6 - v^6.$$

I may take this occasion to observe that there are generally two modes of a distinct kind for obtaining any simple concomitant; the difference (a most important practical one) consisting in the circumstance that in the one mode there are differentiations to be performed in respect to the coefficients, the consequence of which is that the whole of the operations must be gone through for obtaining the concomi-

will be

$$\begin{vmatrix} 0 & w & v & 1 \\ w & 0 & u & 1 \\ v & u & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

that is to say, $u^2 + v^2 + w^2 - 2uv - 2uw - 2vw$.

The two emanants will then be

$$vwu' + wuv' + uvw'$$

$$(u - v - w)\,u' + (v - w - u)\,v' + (w - u - v)\,w',$$

subject to the relation

$$u' + v' + w' = 0\,;$$

and taking the resultant of these three equations, or, which is the same thing, of

$$\begin{aligned} &vwu' + wuv' + uvw', \\ &uu' + vv' + ww', \\ &u' + v' + w', \end{aligned}$$

we obtain the determinant

$$\begin{vmatrix} vw & wu & uv \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}$$

which is equal to

$$vw(v - w) + wu(w - u) + uv(u - v),$$

that is to say $\qquad (u - v)(v - w)(w - u).$

Hence another variety of the external shape to which the canonical form for the sextic function of x, y may be reduced will be

$$au^6 + bv^6 + cw^6 + \mu uvw\,(u - v)\,(v - w)\,(w - u).$$

I shall presently revert to the theory of the corresponding mode of reducing to their canonical forms the biquadratic and octavic functions of x, y, the number of solutions for which will be respectively 3 and 5, and the discovery of which, as shewn by me in the Number of the *Phil. Mag.* before adverted to, depend upon the solution of equations of the third and fifth degrees in ρ expressed by means of determinants of the third and fifth orders formed in precise correspondence with that of the fourth order, upon which, as we have found above, the reduction of the sextic function to its canonical form depends.

[*To be Continued.*]

ON THE INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS.

By W. H. L. Russell.

I propose in the following paper to investigate a method for solving an extensive class of differential equations by means of definite integrals. I shall commence with some theorems relative to the summation of series, of which I shall make continual use in the analysis on which I am about to enter.

To determine the sum of the series

$$1+\frac{\alpha}{\beta}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\cdot\frac{x^2}{1.2}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\cdot\frac{x^3}{1.2.3}+\&c.,$$

$$\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\ldots(\beta+n-1)}=\frac{\Gamma\beta}{\Gamma\alpha\,\Gamma(\beta-\alpha)}\int_0^1 v^{\alpha+n-1}(1-v)^{\beta-\alpha-1}\,dv.$$

Consequently we find the sum of the series equal to

$$\frac{\Gamma\beta}{\Gamma\alpha\,\Gamma(\beta-\alpha)}\int_0^1 v^{\alpha-1}(1-v)^{\beta-\alpha-1}\,\varepsilon^{vx}\,dv.$$

Again, if the general term be

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{\beta(\beta+1)\ldots(\beta+n-1)}\cdot\frac{\alpha'(\alpha'+1)\ldots(\alpha'+n-1)}{\beta'(\beta'+1)\ldots(\beta'+n-1)}\cdot\frac{x^n}{1.2.3\ldots n},$$

the sum of the series will be, in like manner,

$$\frac{\Gamma\beta}{\Gamma\alpha\,\Gamma(\beta-\alpha)},\ \frac{\Gamma\beta'}{\Gamma\alpha'\,\Gamma(\beta'-\alpha')}\ldots\int_0^1\int_0^1\ldots v^{\alpha-1}z^{\alpha'-1}\ldots(1-v)^{\beta-\alpha-1}(1-z)^{\beta'-\alpha'-1}\ldots\,\varepsilon^{vzx\ldots}\,dv\,dz\ldots$$

To sum the series

$$1+\frac{x}{\beta}+\frac{1}{\beta(\beta+1)}\cdot\frac{x^2}{1.2}+\frac{1}{\beta(\beta+1)(\beta+2)}\cdot\frac{x^3}{1.2.3}+\ldots,$$

$$\frac{1}{\beta(\beta+1)\ldots(\beta+n-1)}=\frac{\Gamma\beta}{\Gamma(\beta+n)}=\frac{\Gamma\beta.\varepsilon}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^{iz}\,dz}{(1+iz)^{\beta+n}};$$

consequently the required sum is equal to

$$\frac{\Gamma\beta.\varepsilon}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^{iz}dz}{(1+iz)^{\beta}}\,\varepsilon^{\frac{x}{1+iz}}\,dz.$$

If it be required to sum the series whose general term is

$$\frac{1}{\beta(\beta+1)\ldots(\beta+n-1)\,\beta'(\beta'+1)\ldots(\beta'+n-1)\ldots}\cdot\frac{x^n}{1.2.3\ldots n},$$

we easily see that the sum is equal to

$$\frac{\Gamma\beta.\varepsilon}{2\pi}\cdot\frac{\Gamma\beta'.\varepsilon}{2\pi}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz\,dz'\ldots\frac{\varepsilon^{i(z+z'+\ldots)}}{(1+iz)^{\beta}(1+iz)^{\beta'}\ldots}\,\varepsilon^{\frac{x}{(1+iz)(1+iz')\ldots}}.$$

We may easily reduce this to a possible form by putting $z=\tan\theta$, $z'=\tan\theta'\ldots$.

If the series to be summed is of the nature of both the kinds of series we have been discussing, we must combine the two methods of summation together; as will be evident in the examples which I shall now give of the application of these series to the solution of differential equations.

For this purpose, I shall avail myself of the methods for expressing the Integrals of Differential Equations by means of series, which are given in the *Philosophical Transactions* for the year 1844.

Let the equation be

$$x^2\frac{d^2u}{dx^2}-\{(a+b-1)x+\mu x^2\}\frac{du}{dx}+\{ab+(e-1)\mu x\}u=0.$$

The symbolical form of this equation is

$$(D-a)(D-b)u-\mu(D-e)\varepsilon^{\theta}u=0;$$

and the solution in series will be

$$u=Ax^a\left\{1+\frac{a-e+1}{a-b+1}\mu x+\frac{(a-e+1)(a-e+2)}{(a-b+1)(a-b+2)}\cdot\frac{\mu^2x^3}{1.2}+\&c.\right\},$$

$$+Bx^b\left\{1+\frac{b-e+1}{b-a+1}\mu x+\frac{(b-e+1)(b-e+2)}{(b-a+1)(b-a+2)}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\}.$$

Hence we have

$$u=Ax^a\int_0^1 v^{a-e}(1-v)^{e-b-1}\varepsilon^{\mu vx}dv,$$
$$+Bx^b\int_0^1 v^{b-e}(1-v)^{e-a-1}\varepsilon^{\mu vx}dv.$$

I must here remark, that the difference of (a) and (b) is here supposed less than unity. If not, one of these series will require to be modified in a way that will be easily seen. The same remark will apply to several of the equations we are about to consider.

Let the equation be

$$x^3\frac{d^3u}{dx^3}-\{(a+b+c-3)x^2+\mu x^3\}\frac{d^2u}{dx^2}$$
$$+\{(ab+ac+bc-a-b-c+1)x+\mu(e+q-3)x^2\}\frac{du}{dx}$$
$$-\{abc+\mu(eq-c-q+1)x\}u=0.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)u-\mu(D-e)(D-q)\varepsilon^{\omega}u=0;$$

therefore

$$u=Ax^a\left\{1+\frac{(a-e+1)(a-q+1)}{(a-b+1)(a-c+1)}\mu x\right.$$
$$\left.+\frac{(a-e+1)(a-e+2)(a-q+1)(a-q+2)}{(a-b+1)(a-b+2)(a-c+1)(a-c+2)}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\}$$
$$+Bx^b\left\{1+\frac{(b-e+1)(b-q+1)}{(b-c+1)(b-a+1)}\mu x\right.$$
$$\left.+\frac{(b-e+1)(b-e+2)(b-q+1)(b-q+2)}{(b-c+1)(b-c+2)(b-a+1)(b-a+2)}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\}$$
$$+Cx^c\left\{1+\frac{(c-e+1)(c-q+1)}{(c-a+1)(c-b+1)}\mu x\right.$$
$$\left.+\frac{(c-e+1)(c-e+2)(c-q+1)(c-q+2)}{(c-a+1)(c-a+2)(c-b+1)(c-b+2)}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\}.$$

Hence we shall find

$$u=Ax^a\int_0^1\int_0^1 v^{a-e}z^{a-q}(1-v)^{e-b-1}(1-z)^{q-c-1}\,\varepsilon^{\mu vzx}\,dv\,dz$$
$$+Bx^b\int_0^1\int_0^1 v^{b-e}z^{b-q}(1-v)^{e-c-1}(1-z)^{q-a-1}\,\varepsilon^{\mu vzx}\,dv\,dz$$
$$+Cx^c\int_0^1\int_0^1 v^{c-e}z^{c-q}(1-v)^{e-a-1}(1-z)^{q-b-1}\,\varepsilon^{\mu vzx}\,dv\,dz.$$

Let $x^3\dfrac{d^3u}{dx^3}-(a+b+c-3)\,x^2\dfrac{d^2u}{dx^2}$

$$+\{(ab+ac+bc-a-b-c+1)\,x-\mu x^2\}\frac{du}{dx}-\{abc-\mu(e-1)\,x\}\,u=0.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)-\mu(D-e)\,\varepsilon^{\omega}u=0;$$

therefore

$$u=Ax^a\left\{1+\frac{a-e+1}{(a-b+1)(a-c+1)}\mu x\right.$$
$$\left.+\frac{(a-e+1)(a-e+2)}{(a-b+1)(a-b+2)(a-c+1)(a-c+2)}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\},$$
$$+Bx^b\left\{1+\frac{b-e+1}{(b-c+1)(b-a+1)}\mu x\right.$$
$$\left.+\frac{(b-e+1)(b-e+2)}{(b-c+1)(b-c+2)(b-a+1)(b-a+2)}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\},$$

$$+\, Cx^e \left\{1 + \frac{c-e+1}{(c-a+1)(c-b+1)}\,\mu x \right.$$

$$\left. + \frac{(c-e+1)(c-e+2)}{(c-a+1)(c-a+2)(c-b+1)(c-b+2)} \cdot \frac{\mu x^2}{1.2} + \ldots \right\}.$$

Consequently

$$u = Ax^a \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{e-b-1}\, v^{a-e}\, \varepsilon^{\mu vx \cos^2\theta} \cos^{a-c-1}\theta$$

$$\cos\{\mu vx \cos\theta \sin\theta + (a-c+1)\theta - \tan\theta\}$$

$$+\, Bx^b \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{e-c-1}\, v^{b-e}\, \varepsilon^{\mu vx \cos^2\theta} \cos^{b-a-1}\theta$$

$$\cos\{\mu vx \cos\theta \sin\theta + (b-a+1)\theta - \tan\theta\}$$

$$+\, Cx^c \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{e-a-1}\, v^{c-e}\, \varepsilon^{\mu vx \cos^2\theta} \cos^{c-b-1}\theta$$

$$\cos\{\mu vx \cos\theta \sin\theta + (c-b+1)\theta - \tan\theta\}.$$

Let $x^3 \dfrac{d^3u}{dx^3} - (a+b+c-3)\, x^2 \dfrac{d^2u}{dx^3}$

$$+ (ab + ac + bc - a - b - c + 1)\, x \frac{du}{dx} - (abc + \mu x)\, u = 0.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)\, u - \mu\varepsilon^{\omega} u = 0;$$

therefore

$$u = Ax^a \left\{1 + \frac{\mu x}{(a-b+1)(a-c+1).1} \right.$$

$$\left. + \frac{\mu^2 x^2}{(a-b+1)(a-b+2)(a-c+1)(a-c+2).1.2} + \&c.\right\}$$

$$+\, Bx^b \left\{1 + \frac{\mu x}{(b-c+1)(b-a+1).1} \right.$$

$$\left. + \frac{\mu^2 x^2}{(b-c+1)(b-c+2)(b-a+1)(b-a+2).1.2} + \&c.\right\}$$

$$+\, Cx^c \left\{1 + \frac{\mu x}{(c-a+1)(c-b+1).1} \right.$$

$$\left. + \frac{\mu^2 x^2}{(c-a+1)(c-a+2)(c-b+1)(c-b+2).1.2} + \&c.\right\};$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)u-\mu(D-e)(D-q)\,\varepsilon^{\theta}u=0;$$

therefore

$$\begin{aligned}
u = {} & Ax^a\left\{1+\frac{(a-e+1)(a-q+1)}{(a-b+1)(a-c+1)}\mu x\right.\\
&\left.+\frac{(a-e+1)(a-e+2)(a-q+1)(a-q+2)}{(a-b+1)(a-b+2)(a-c+1)(a-c+2)}\cdot\frac{\mu^2x^2}{1.2}+\&\text{c.}\right\}\\
&+Bx^b\left\{1+\frac{(b-e+1)(b-q+1)}{(b-c+1)(b-a+1)}\mu x\right.\\
&\left.+\frac{(b-e+1)(b-e+2)(b-q+1)(b-q+2)}{(b-c+1)(b-c+2)(b-a+1)(b-a+2)}\cdot\frac{\mu^2x^2}{1.2}+\&\text{c.}\right\}\\
&+Cx^c\left\{1+\frac{(c-e+1)(c-q+1)}{(c-a+1)(c-b+1)}\mu x\right.\\
&\left.+\frac{(c-e+1)(c-e+2)(c-q+1)(c-q+2)}{(c-a+1)(c-a+2)(c-b+1)(c-b+2)}\cdot\frac{\mu^2x^2}{1.2}+\&\text{c.}\right\}.
\end{aligned}$$

Hence we shall find

$$\begin{aligned}
u = {} & Ax^a\int_0^1\!\!\int_0^1 v^{a-e}z^{a-q}(1-v)^{e-b-1}(1-z)^{q-c-1}\,\varepsilon^{\mu vzx}\,dv\,dz\\
&+Bx^b\int_0^1\!\!\int_0^1 v^{b-e}z^{b-q}(1-v)^{e-c-1}(1-z)^{q-a-1}\,\varepsilon^{\mu vzx}\,dv\,dz\\
&+Cx^c\int_0^1\!\!\int_0^1 v^{c-e}z^{c-q}(1-v)^{e-a-1}(1-z)^{q-b-1}\,\varepsilon^{\mu vzx}\,dv\,dz.
\end{aligned}$$

$$+ Cx^c\left\{1 + \frac{c-e+1}{(c-a+1)(c-b+1)}\,\mu x\right.$$

$$\left. + \frac{(c-e+1)(c-e+2)}{(c-a+1)(c-a+2)(c-b+1)(c-b+2)} \cdot \frac{\mu x^2}{1.2} + \ldots\right\}.$$

Consequently

$$u = Ax^a \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{e-b-1}\, v^{a-e}\, \varepsilon^{\mu v x \cos^2\theta} \cos^{a-c-1}\theta$$

$$\cos\{\mu v x \cos\theta \sin\theta + (a-c+1)\theta - \tan\theta\}$$

$$+ Bx^b \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{e-c-1}\, v^{b-e}\, \varepsilon^{\mu v x \cos^2\theta} \cos^{b-a-1}\theta$$

$$\cdot \cos\{\mu v x \cos\theta \sin\theta + (b-a+1)\theta - \tan\theta\}$$

$$+ Cx^c \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{e-a-1}\, v^{c-e}\, \varepsilon^{\mu v x \cos^2\theta} \cos^{c-b-1}\theta$$

$$\cos\{\mu v x \cos\theta \sin\theta + (c-b+1)\theta - \tan\theta\}.$$

Let
$$x^3 \frac{d^3u}{dx^3} - (a+b+c-3)\, x^2 \frac{d^2u}{dx^2}$$

$$+ (ab+ac+bc-a-b-c+1)\, x \frac{du}{dx} - (abc + \mu x)\, u = 0.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)\, u - \mu\varepsilon^{\omega} u = 0;$$

therefore

$$u = Ax^a\left\{1 + \frac{\mu x}{(a-b+1)(a-c+1).1}\right.$$

$$\left. + \frac{\mu^2 x^2}{(a-b+1)(a-b+2)(a-c+1)(a-c+2).1.2} + \&c.\right\}$$

$$+ Bx^b\left\{1 + \frac{\mu x}{(b-c+1)(b-a+1).1}\right.$$

$$\left. + \frac{\mu^2 x^2}{(b-c+1)(b-c+2)(b-a+1)(b-a+2).1.2} + \&c.\right\}$$

$$+ Cx^c\left\{1 + \frac{\mu x}{(c-a+1)(c-b+1).1}\right.$$

$$+ \frac{\mu^2 x^2}{(c-a+1)(c-a+2)(c-b+1)(c-b+2).1.2}$$

Now $\dfrac{3}{1+a-b} + \dfrac{3}{2+a-b} + \dfrac{3}{3+a-b} + \ldots + \dfrac{3}{n+a-b}$

$$= 3\int_0^1 \frac{v^{a-b}dv(v^n - 1)}{v-1}$$

$$\frac{2(a-b)}{1(1+a-b)} + \frac{2(a-b)}{2(2+a-b)} + \frac{2(a-b)}{3(3+a-b)} \cdots + \frac{2(a-b)}{n(n+a-b)}$$

$$= 2(a-b)\int_0^1\int_0^1 dv\,dz\,\frac{(v^n z^n - 1)\,z^{a-b}}{vz-1},$$

whence the rest is obvious.

The preceding equations are all included under the general symbolical form

$$u + \phi(D)\,\varepsilon^{r\varpi}u = 0,$$

to which the still more general symbolical form

$$u + a_1\phi(D)\,\varepsilon^{\varpi}u + a_2\phi(D)\,\phi(D-1)\,\varepsilon^{2\varpi}u + \&c. = 0$$

can always be reduced.

I shall conclude this paper with two examples, which are well calculated to exhibit the power of the method I have been investigating.

Let $\dfrac{d^n y}{dx^n} = x^m y$.

$$y = A_0\left\{1 + \frac{x^{m+n}}{(m+1)\ldots(n+m)} + \frac{x^{2(n+m)}}{(m+1)\ldots(n+m)(n+2m+1)\ldots(2n+2m)}\right.$$

$$\left. + \frac{x^{3(n+m)}}{(m+1)\ldots(n+m)(n+2m+1)\ldots(2n+2m)(2n+3m+1)\ldots3(n+m)} + \&c.\right\}$$

$$+ A_1 x\left\{1 + \frac{x^{m+n}}{(m+2)\ldots(n+m+1)} + \frac{x^{2(m+n)}}{(m+2)\ldots(n+m+1)(n+2m+2)\ldots(2n+2m+1)}\right.$$

$$\left. + \frac{x^{3(m+n)}}{(m+2)\ldots(n+m+1)(n+2m+2)\ldots(2n+2m+1)(2n+3m+2)\ldots(3n+3m+1)} + \&c.\right\}$$

$$+ A_2 x^2\left\{1 + \frac{x^{m+n}}{(m+3)\ldots(n+m+2)} + \frac{x^{2(m+n)}}{(m+3)\ldots(n+m+2)(n+2m+3)\ldots(2n+2m+2)}\right.$$

$$\left. + \frac{x^{3(n+m)}}{(m+3)\ldots(n+m+2)(n+2m+3)\ldots(2n+2m+2)(2n+3m+3)\ldots(3n+3m+2)} + \&c.\right\}$$

$+$ &c.

$$= A_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\varepsilon^{i(z+z'+\,..\,z^{(n-2)})}\,dz\,dz'\dots dz^{(n-2)}}{(1+iz)^{\frac{m+1}{n+m}}(1+iz')^{\frac{m+2}{n+m}}\dots(1+iz^{(n-2)})^{\frac{n+m-1}{n+m}}}\,\varepsilon^{\frac{x^{n+m}}{(n+m)^n(1+iz)(1+iz')\dots}}$$

$$+ A_1 x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \frac{\varepsilon^{i(z+z'+\,..\,z^{(n-2)})}\,dz\,dz'\dots dz^{(n-2)}}{(1+iz)^{\frac{m+2}{n+m}}(1+iz')^{\frac{m+3}{n+m}}\dots(1+iz^{(n-2)})^{\frac{n+m+1}{n+m}}}\,\varepsilon^{\frac{x^{n+m}}{(n+m)^n(1+iz)(1+iz')\dots}}$$

$$+ A_2 x^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \frac{\varepsilon^{i(z+z'+\,..\,z^{(n-2)})}\,dz\,dz'\dots dz^{(n-2)}}{(1+iz)^{\frac{m+3}{n+m}}(1+iz')^{\frac{m+4}{n+m}}\dots(1+iz^{(n-2)})^{\frac{n+m+2}{n+m}}}\,\varepsilon^{\frac{x^{n+m}}{(n+m)^n(1+iz)(1+iz')\dots}}$$

+ &c.,

where the integral may be transformed by putting

$$z = \tan\theta, \quad z' = \tan\theta'.$$

Let $x\dfrac{d^n y}{dx^n} + \dfrac{d^{n-1}y}{dx^{n-1}} - y = 0.$

$$y = y_0\left\{1 + \frac{x^{n-1}}{1.2.3\dots(n-1).1} + \frac{x^{2n-2}}{1.2.3\dots(2n-2).1.n} + \frac{x^{3n-3}}{1.2.3\dots(3n-3).1.n.(2n-1)} + \dots\right\}$$

$$+ y_0'\left\{x + \frac{x^n}{1.2.3\dots n.2} + \frac{x^{2n-1}}{1.2.3\dots(2n-1).2.(n+1)} + \frac{x^{3n-2}}{1.2.3\dots(3n-2).2(n+1).2n} + \dots\right\}$$

$$+ y_0''\left\{\frac{x^2}{1.2} + \frac{x^{n+1}}{1.2.3\dots(n+1).3} + \frac{x^{2n}}{1.2.3\dots2n.3.(n+2)} + \frac{x^{3n-1}}{1.2.3\dots(3n-1).3.(n+2)(2n+1)} + \dots\right\}$$

+ &c. to $(n-1)$ terms,

$$= E_0 \int_{-\infty}^{\infty} \frac{\varepsilon^{iz} dz}{(1+iz)^{\frac{1}{n-1}}} \left\{ \varepsilon^{\frac{x}{\rho}} + \alpha^{n-1} \varepsilon^{\frac{\alpha x}{\rho}} + \alpha^{2(n-1)} \varepsilon^{\frac{\alpha^2 x}{\rho}} + \&c. \right\}$$

$$+ E_1 \int_{-\infty}^{\infty} \frac{\varepsilon^{iz} dz}{(1+iz)^{\frac{1}{n-1}}} \left\{ \varepsilon^{\frac{x}{\rho}} + \alpha^{n-2} \varepsilon^{\frac{\alpha x}{\rho}} + \alpha^{2(n-2)} \varepsilon^{\frac{\alpha^2 x}{\rho}} + \&c. \right\}$$

$$+ E_2 \int_{-\infty}^{\infty} \frac{\varepsilon^{iz} dz}{(1+iz)^{\frac{1}{n-1}}} \left\{ \varepsilon^{\frac{x}{\rho}} + \alpha^{n-3} \varepsilon^{\frac{\alpha x}{\rho}} + \alpha^{2(n-3)} \varepsilon^{\frac{\alpha^2 x}{\rho}} + \&c. \right\}$$

$+ \&c.,$

where $\rho = (n-1)^{\frac{1}{n-1}} (1+iz)^{\frac{1}{n-1}}$; $1, \alpha, \alpha^2, \ldots$ are the $(n-1)$ roots of the equation $\xi^{n-1} - 1 = 0$, and each of the series contained within the brackets is continued to $(n-1)$ terms. We may transform this expression by putting $z = \tan\theta$. By the above process we shall obtain $(n-1)$ particular integrals only; but when $(n-1)$ particular integrals of a linear equation of the n^{th} order are known, the remaining integral can always be found.

and always by the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon^{i\alpha(x-x')} F(x') \frac{d\alpha . dx'}{2\pi};$$

consequently, in both cases, the quantities under the integral sign in the successive differential coefficients of these integrals with respect to (x) are in geometrical progression. Hence the substitution of these integrals reduces the proposed series to forms already considered. In this paper I intend to use the first of these expressions, as it is the simpler of the two; the application of the second is to be conducted in a similar manner. The method of summation I have employed in this paper, will be understood on comparing it with the paper to which I have already referred.

Let $$x_1^3 \frac{d^3u}{dx_1^3} - (a+b+c-3)\, x_1^2 \frac{d^2u}{dx_1^2}$$
$$+ (bc + ca + ab - a - b - c + 1)\, x_1 \frac{du}{dx_1} - abcu = x_1 \frac{du}{dx_2}.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)\, u - \frac{d}{dx_2}\, \varepsilon^{\omega} u = 0.$$

Hence,

$$u = x_1^a \left\{ 1 + \frac{x_1}{(a-b+1)(a-c+1).1} \frac{d}{dx_2} \right.$$
$$\left. + \frac{x_1^2}{(a-b+1)(a-b+2)(a-c+1)(a-c+2).1.2} \frac{d^2}{dx_2^2} + \&c. \right\} F_a x_2,$$
$$+ x_1^b \left\{ 1 + \frac{x_1}{(b-c+1)(b-a+1).1} \frac{d}{dx_2} \right.$$
$$\left. + \frac{x_1^2}{(b-c+1)(b-c+2)(b-a+1)(b-a+2).1.2} \frac{d^2}{dx_2^2} + \&c. \right\} F_b x_2,$$
$$+ x_1^c \left\{ 1 + \frac{x_1}{(c-a+1)(c-b+1).1} \frac{d}{dx_2} \right.$$
$$\left. + \frac{x_1^2}{(c-a+1)(c-a+2)(c-b+1)(c-b+2).1.2} \frac{d^2}{dx_2^2} + \&c. \right\} F_c x_2.$$

Let $F_a x_2 = \int_0^\rho \varpi_a(\rho)\, \varepsilon^{\rho x_2} d\rho$, and similarly for $F_b x_2$, $F_c x_2$,

$$u = x^a \int_0^\rho \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \varepsilon^{\rho(x_1 \cos\theta \cos\phi \cos(\theta+\phi) + x_2)} \cos^{a-b-1}\theta \cos^{a-c-1}\phi\, \varpi_a($$

$$\cos\{\rho x_1 \cos\theta \cos\phi \sin(\theta+\phi) + (a-b+1)\theta + (a-c+1)\phi - (\tan$$

$$+x^b\int_0^\rho\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\varepsilon^{\rho(x_1\cos\theta\cos\phi\cos(\theta+\phi)+x_2)}\cos^{b-c-1}\theta\cos^{b-a-1}\phi\,\varpi_b(\rho)\,d\theta\,d\phi\,d\rho$$

$$\cos\{\rho x_1\cos\theta\cos\phi\sin(\theta+\phi)+(b-c+1)\theta+(b-a+1)\phi-(\tan\theta+\tan\phi)\},$$

$$+x^c\int_0^\rho\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\varepsilon^{\rho(x_1\cos\theta\cos\phi\cos(\theta+\phi)+x_2)}\cos^{c-a-1}\theta\cos^{c-b-1}\phi\,\varpi_c(\rho)\,d\theta\,d\phi\,d\rho$$

$$\cos\{\rho x_1\cos\theta\cos\phi\sin(\theta+\phi)+(c-a+1)\theta+(c-b+1)\phi-(\tan\theta+\tan\phi)\}.$$

Let $$x_1\frac{d^2u}{dx_1^2}+\frac{du}{dx_1}-\frac{du}{dx_2}=0.$$

The symbolical form of this equation is

$$D^2u-\frac{d}{dx_2}\varepsilon^{\varpi}u=0,$$

$$u=F_0x_2+x_1F_1x_2+x_1^2F_2x_2+\&c.+\log_\varepsilon x_1(f_0x_2+x_1f_1x_2+x_1^2f_2x_2+\&c.)$$

where $F_nx_2=\dfrac{d}{dx_2}\cdot\dfrac{F_{n-1}x_2}{n^2}-2\dfrac{d}{dx_2}\cdot\dfrac{f_{n-1}x_2}{n^3}$, $\quad f_nx_2=\dfrac{d}{dx_2}\dfrac{f_{n-1}x_2}{n^2}$.

I shall discuss the first only of these series,

$$F_nx_2=\frac{d^n}{dx_2^n}\frac{F_0x_2}{1^2.2^2.3^2\ldots n^2}-2\frac{d^n}{dx_2^n}\frac{f_0x_2}{1^2.2^2.3^2\ldots n^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right).$$

Let $F_0x_2=\int_0^\rho\Pi(\rho)\,\varepsilon^{\rho x_2}d\rho$, and similarly for f_0x_2; then

$$F_nx_2=\int_0^\rho\frac{\Pi(\rho)\,\varepsilon^{\rho x_2}\rho^n\,d\rho}{1^2.2^2.3^2\ldots n^2}$$

$$-2\int_0^\rho\frac{\varpi(\rho)\,\varepsilon^{\rho x_2}\rho^n\,d\rho}{1^2.2^2.3^2\ldots n^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right),$$

a form which I have considered in the paper I have before mentioned.

Let $$\frac{d^vu}{dx_1\,dx_2\,dx_3\ldots dx_v}=u.$$

$$u=\int dx_vF_vx_v+\frac{x_1x_2x_3\ldots x_{v-1}}{1^{v-1}}\iint dx_v^2F_vx_v$$

$$+\frac{x_1^2x_2^2x_3^2\ldots x_{v-1}^2}{1^{v-1}.2^{v-1}}\iiint dx_v^3F_vx_v+\&c.,$$

$$+\int dx_{v-1}F_{v-1}x_{v-1}+\frac{x_1x_2x_3\ldots x_v}{1^{v-1}}\iint dx_{v-1}^2F_{v-1}x_{v-1}$$

$$+\frac{x_1^2x_2^2x_3^2\ldots x_v^2}{1^{v-1}.2^{v-1}}\iiint dx_{v-1}^3F_{v-1}x_{v-1}+\&c.,$$

$$+ \int dx_{v-2} F_{v-2} x_{v-2} + \frac{x_1 x_2 x_3 \ldots x_v}{1^{v-1}} \iint dx^2_{v-2} F_{v-2} x_{v-2}$$

$$+ \frac{x_1^2 x_2^2 x_3^2 \ldots x_v^2}{1^{v-1}.2^{v-1}} \iiint dx^3_{v-2} F_{v-2} x_{v-2} + \text{\&c.},$$

+ &c.

Now $\int_0 \int_0 \int_0 \ldots dx^n Fx = \dfrac{1}{1.2.3\ldots(n-1)} \int_0^x (x-\rho)^{n-1} F(\rho) d\rho$;

therefore

$$u = \int_0^{x_v} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots F_v \rho \, \frac{\varepsilon^{i(z+z'+\ldots z^{(v-2)})} dz\, dz' \ldots dz^{(v-2)} d\rho}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})} \, \varepsilon^{\frac{x_1 x_2 x_3 \ldots x_{v-1}(x_v - \rho)}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}}$$

$$+ \int_0^{x_{v-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots F_{v-1} \rho \, \frac{\varepsilon^{i(z+z'+\ldots z^{(v-2)})} dz\, dz' \ldots dz^{(v-2)} d\rho}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})} \, \varepsilon^{\frac{x_1 x_2 x_3 \ldots (x_{v-1} - \rho)\, x_v}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}}$$

$$+ \int_0^{x_{v-2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots F_{v-2} \rho \, \frac{\varepsilon^{i(z+z'+\ldots z^{(v-2)})} dz\, dz' \ldots dz^{(v-2)} d\rho}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})} \, \varepsilon^{\frac{x_1 x_2 x_3 \ldots (x_{v-2} - \rho) \ldots x_v}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}}$$

+ &c.,

where the integral may be transformed by putting $z = \tan\theta$, $z' = \tan\theta'$, &c.

SOME THEOREMS ON CONFOCAL SURFACES OF THE SECOND ORDER.

By the Rev. J. Booth, F.R.S., &c.

The following demonstrations of some general theorems on Confocal Surfaces of the Second Order may be acceptable to such readers of the Journal as take an interest in researches of this kind.

Theorem I. *Three confocal surfaces of the second order intersect in a common point Q the vertex of a c envelopes a fourth confocal surface; to deter*

of this cone referred to the normals of the three surfaces, at the common point Q, as axes of coordinates.

It is very generally known that confocal surfaces mutually intersect at right angles;* the normals of the three surfaces will therefore constitute a system of rectangular coordinates.

Let the equation of the fourth surface, an ellipsoid suppose, be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ ....(1)},$$

and let α, β, γ be the coordinates of the point Q, the vertex of the enveloping cone.

Let $\quad x-\alpha=m(z-\gamma), \quad y-\beta=n(z-\gamma)$ ....(2),

be the equations of a right line passing through the point $(\alpha\beta\gamma)$, and piercing the surface (1) in two points. If we substitute the values of x and y derived from (2) in (1), we shall have a resulting equation of the form

$$Uz^2+2Vz+W=0 \text{ ....(3)};$$

of which the two roots will express the values of the vertical ordinates of the two points in which the line pierces the surface. When the line becomes a side of the enveloping cone, the two roots become equal, and we get the well-known condition

the equation of the enveloping cone becomes

$$\left.\begin{array}{r}\left[\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}+\frac{(z-\gamma)^2}{c^2}\right]\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right] \\ =\left[\frac{\alpha(x-\alpha)}{a^2}+\frac{\beta(y-\beta)}{b^2}+\frac{\gamma(z-\gamma)}{c^2}\right]^2\end{array}\right\}\ldots(8).$$

We shall leave the equation of the enveloping cone in this form, to facilitate its future transformation.

Let the tangent planes at the common intersection Q of the three confocal surfaces be taken for the coordinate planes, to which the equation of the enveloping cone is to be referred, and let $a_{,}b_{,}c_{,}$, $a_{,,}b_{,,}c_{,,}$, $a_{,,,}b_{,,,}c_{,,,}$ be the semi-axes of the three confocal surfaces. Moreover, let p, $p_{,}$, $p_{,,}$ be the perpendiculars from the common centre of the surfaces on the three tangent planes at the point Q. Now the normals, at the point Q, to the three surfaces being parallel to the perpendiculars p, $p_{,}$, $p_{,,}$, make the same angles with the original axes of coordinates.

Let $\lambda\mu\nu$, $\lambda'\mu'\nu'$, $\lambda''\mu''\nu''$, be the angles which the perpendiculars p, $p_{,}$, $p_{,,}$ respectively make with the axes of coordinates. Then as each abscissa is equal to the projections of the three new ones upon it,

$$\left.\begin{array}{l}x-\alpha=x'\cos\lambda+y'\cos\lambda'+z'\cos\lambda'', \\ y-\beta=x'\cos\mu+y'\cos\mu'+z'\cos\mu'', \\ z-\gamma=x'\cos\nu+y'\cos\nu'+z'\cos\nu'',\end{array}\right\}\ldots\ldots(9),$$

but $$\cos\lambda=\frac{p\alpha}{a_{,}^2},\quad \cos\lambda'=\frac{p_{,}\alpha}{a_{,,}^2},\quad \cos\lambda''=\frac{p_{,,}\alpha}{a_{,,,}^2}\ldots\ldots(10);$$

finding similar values for μ, μ', μ'' and ν, ν', ν'', then substituting the resulting values in (9), we shall have

$$\left.\begin{array}{l}x-\alpha=\alpha\left[\frac{px'}{a_{,}^2}+\frac{p_{,}y'}{a_{,,}^2}+\frac{p_{,,}z'}{a_{,,,}^2}\right] \\ y-\beta=\beta\left[\frac{px'}{b_{,}^2}+\frac{p_{,}y'}{b_{,,}^2}+\frac{p_{,,}z'}{b_{,,,}^2}\right] \\ z-\gamma=\gamma\left[\frac{px'}{c_{,}^2}+\frac{p_{,}y'}{c_{,,}^2}+\frac{p_{,,}z'}{c_{,,,}^2}\right]\end{array}\right\}\ldots\ldots\ldots(11).$$

Substituting these values in (8), the equation veloping cone, and eliminating the equivalent shall have, after some reductions,

$$\left.\begin{aligned}
&\left[\left(\frac{\alpha\beta}{ab}\right)^2+\left(\frac{\alpha\gamma}{ac}\right)^2-\left(\frac{\alpha}{a}\right)^2\right]\left[\frac{px}{a_{,}^2}+\frac{p_{,}y}{a_{,,}^2}+\frac{p_{,,}z}{a_{,,,}^2}\right]^2\\
&+\left[\left(\frac{\beta\gamma}{bc}\right)^2+\left(\frac{\beta\alpha}{ba}\right)^2-\left(\frac{\beta}{b}\right)^2\right]\left[\frac{px}{b_{,}^2}+\frac{p_{,}y}{b_{,,}^2}+\frac{p_{,,}z}{b_{,,,}^2}\right]^2\\
&+\left[\left(\frac{\gamma\alpha}{ca}\right)^2+\left(\frac{\gamma\beta}{cb}\right)^2-\left(\frac{\gamma}{c}\right)^2\right]\left[\frac{px}{c_{,}^2}+\frac{p_{,}y}{c_{,,}^2}+\frac{p_{,,}z}{c_{,,,}^2}\right]^2\\
&+2\left(\frac{\beta\gamma}{bc}\right)^2\left[\frac{px}{b_{,}^2}+\frac{p_{,}y}{b_{,,}^2}+\frac{p_{,,}z}{b_{,,,}^2}\right]\left[\frac{px}{c_{,}^2}+\frac{p_{,}y}{c_{,,}^2}+\frac{p_{,,}z}{c_{,,,}^2}\right]\\
&+2\left(\frac{\gamma\alpha}{ca}\right)^2\left[\frac{px}{c_{,}^2}+\frac{p_{,}y}{c_{,,}^2}+\frac{p_{,,}z}{c_{,,,}^2}\right]\left[\frac{px}{a_{,}^2}+\frac{p_{,}y}{a_{,,}^2}+\frac{p_{,,}z}{a_{,,,}^2}\right]\\
&=2\left(\frac{\alpha\beta}{ab}\right)^2\left[\frac{px}{a_{,}^2}+\frac{p_{,}y}{a_{,,}^2}+\frac{p_{,,}z}{a_{,,,}^2}\right]\left[\frac{px}{b_{,}^2}+\frac{p_{,}y}{b_{,,}^2}+\frac{p_{,,}z}{b_{,,,}^2}\right]
\end{aligned}\right\}\ldots(12),$$

omitting the traits over the xyz as no longer necessary.

Now, as the surfaces are confocal, let

$$\left.\begin{aligned}
a_{,}^2&=a^2+k^2, & b_{,}^2&=b^2+k^2, & c_{,}^2&=c^2+k^2\\
a_{,,}^2&=a^2+k_{,}^2, & b_{,,}^2&=b^2+k_{,}^2, & c_{,,}^2&=c^2+k_{,}^2\\
a_{,,,}^2&=a^2+k_{,,}^2, & b_{,,,}^2&=b^2+k_{,,}^2, & c_{,,,}^2&=c^2+k_{,,}^2
\end{aligned}\right\}\ldots(13);$$

find that the coefficient of the term xy will be as follows:

$$2pp_{,}\left[\left(\frac{\beta\gamma(b^2-c^2)}{bc\,b_{,}c_{,}\,b_{,,}c_{,,}}\right)^2+\left(\frac{\gamma\alpha(c^2-a^2)}{ca\,c_{,}a_{,}\,c_{,,}a_{,,}}\right)^2+\left(\frac{\alpha\beta(a^2-b^2)}{ab\,a_{,}b_{,}\,a_{,,}b_{,,}}\right)^2\right]$$

$$-\,2pp_{,}\left[\frac{\alpha^2}{a^2a_{,}^2a_{,,}^2}+\frac{\beta^2}{b^2b_{,}^2b_{,,}^2}+\frac{\gamma^2}{c^2c_{,}^2c_{,,}^2}\right]\ \ldots\ldots(16).$$

Now if we eliminate from this expression the quantities α, β, γ, by the help of (15), the first term of the preceding expression will become

$$\left[\frac{a^2(b^2-c^2)(b^2+k_{,,}^2)(c^2+k_{,,}^2)+b^2(c^2-a^2)(a^2+k_{,,}^2)(c^2+k_{,,}^2)+c^2(a^2-b^2)(a^2+k_{,,}^2)(b^2+k_{,,}^2)}{a^2b^2c^2(b^2-c^2)(c^2-a^2)(a^2-b^2)}\right];$$

and this expression may be reduced to

$$\left[\frac{a^2(b^4-c^4)+b^2(c^4-a^4)+c^2(a^4-b^4)}{a^2b^2c^2(b^2-c^2)(c^2-a^2)(a^2-b^2)}\right]k_{,,}^2.$$

Now the numerator of this fraction may be written in the form $(b^2-c^2)(c^2-a^2)(a^2-b^2)$. Hence the first member of the coefficient of xy in (16) may be reduced to $\dfrac{2pp_{,}k_{,,}^2}{a^2b^2c^2}$.

If in the same manner we calculate the second member of (16), we shall find it to be $\dfrac{-2pp_{,}k_{,,}^2}{a^2b^2c^2}$. Hence the coefficient of xy is 0. In the same way it may be shown that the coefficients of xz and yz are each $= 0$.

We have now to determine the coefficient of x^2 in the equation (12) of the cone.

Collecting the terms by which x^2 is multiplied in that equation, we shall find their sum to be

$$p^2\left[\left(\frac{\beta\gamma}{bc}\right)^2\left(\frac{1}{b_{,}^2}-\frac{1}{c_{,}^2}\right)^2+\left(\frac{\gamma\alpha}{ca}\right)^2\left(\frac{1}{c_{,}^2}-\frac{1}{a_{,}^2}\right)^2+\left(\frac{\alpha\beta}{ab}\right)^2\left(\frac{1}{a_{,}^2}-\frac{1}{b_{,}^2}\right)^2\right]$$

$$-\,p^2\left[\frac{\alpha^2}{a^2a_{,}^4}+\frac{\beta^2}{b^2b_{,}^4}+\frac{\gamma^2}{c^2c_{,}^4}\right]\ \ldots\ldots(17).$$

If, in the first member of this expression, we substitute for $\alpha\beta\gamma$ their values derived from (15), the transformed result will become

$$p^2\left\{\begin{array}{l}+a^2(b^2-c^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)(c^2+k_{,}^2)(c^2+k_{,,}^2)(a^2+\\+b^2(c^2-a^2)(c^2+k_{,}^2)(c^2+k_{,,}^2)(a^2+k_{,}^2)(a^2+k_{,,}^2)(b^2\\+c^2(a^2-b^2)(a^2+k_{,}^2)(a^2+k_{,,}^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)(c^2\end{array}\right.$$

divided by the denominator

$$a^2b^2c^2 a_{,}^2b_{,}^2c_{,}^2(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

Making like substitutions in the second member of (17), the resulting expression will become

$$-p^2\left\{\begin{array}{l} b^2c^2(b^2-c^2)(a^2+k_{,}^2)(a^2+k_{,,}^2)(b^2+k^2)(c^2+k^2) \\ +c^2a^2(c^2-a^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)(a^2+k^2)(c^2+k^2) \\ +a^2b^2(a^2-b^2)(c^2+k_{,}^2)(c^2+k_{,,}^2)(a^2+k^2)(b^2+k^2)\end{array}\right\}\ldots(19),$$

divided by the same denominator as the preceding expression.

If we add together the terms in (18) that are multiplied by a^2c^2, we shall find the result to be

$$-a^2c^2(a^2-c^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)\left[a^2c^2+(a^2+c^2)k^2+k^2(k_{,}^2+k_{,,}^2)-k_{,}^2k_{,,}^2\right],$$

and the corresponding term in (19) is

$$a^2c^2(a^2-c^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)\left[a^2c^2+k^2(a^2+c^2)+k^4\right];$$

adding these expressions together, the resulting expression will become

$$a^2c^2(a^2-c^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)(k^2-k_{,}^2)(k^2-k_{,,}^2)\ldots(19);$$

or, developing this expression,

$$\left[a^4b^4c^2+a^4b^2c^2(k_{,}^2+k_{,,}^2)+a^4c^2k_{,}^2k_{,,}^2\right](k^2-k_{,}^2)(k^2-k_{,,}^2)$$
$$-\left[a^2c^4b^4+a^2b^2c^4(k_{,}^2+k_{,,}^2)+a^2c^4k_{,}^2k_{,,}^2\right](k^2-k_{,}^2)(k^2-k_{,,}^2).$$

Hence, if in like manner we develope and combine the remaining terms in (18) and (19), the whole coefficient of x^2 will be the sum of the following twenty-four terms,

$$\left\{\begin{array}{l} a^4b^4c^2+a^4b^2c^2(k_{,}^2+k_{,,}^2)+a^4c^2k_{,}^2k_{,,}^2 \\ -a^2b^4c^4-a^2b^2c^4(k_{,}^2+k_{,,}^2)-a^2c^4k_{,}^2k_{,,}^2 \\ +b^4c^4a^2+b^4c^2a^2(k_{,}^2+k_{,,}^2)+b^4a^2k_{,}^2k_{,,}^2 \\ -b^2c^4a^4-b^2c^2a^4(k_{,}^2+k_{,,}^2)-b^2a^4k_{,}^2k_{,,}^2 \\ +c^4a^4b^2+c^4a^2b^2(k_{,}^2+k_{,,}^2)+c^4b^2k_{,}^2k_{,,}^2 \\ -c^2a^4b^4-c^2a^2b^4(k_{,}^2+k_{,,}^2)-c^2b^4k_{,}^2k_{,,}^2\end{array}\right\}p^2(k^2-k_{,}^2)(k^2-k_{,,}^2)\ldots(20),$$

divided by the denominator

$$-a^2b^2c^2a_{,}^2b_{,}^2c_{,}^2(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

Now if we add the terms of this coefficient vertically, the sum of the first column $=0$, the sum of the second is also $=0$,

and the sum of the terms in the third column may be reduced to

$$(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)k_{,}^2 k_{,,}^2.$$

Hence the coefficient of x^2 becomes

$$\frac{p^2 k^2 k_{,}^2 k_{,,}^2 (k^2 - k_{,}^2)(k^2 - k_{,,}^2)}{k^2 a_{,}^2 b_{,}^2 c_{,}^2 a^2 b^2 c^2} \text{.............} (21).$$

The symmetrical factor of this expression $\frac{k^2 k_{,}^2 k_{,,}^2}{a^2 b^2 c^2}$ will appear in the coefficients of y^2 and z^2, and may therefore be eliminated by division from the equation of the cone. Hence the coefficient of x^2 may thus be reduced to

$$\frac{p^2 (k^2 - k_{,}^2)(k^2 - k_{,,}^2)}{k^2 a_{,}^2 b_{,}^2 c_{,}^2} \text{..................} (22).$$

Let A, B be the semiaxes of the section of the ellipsoid $(a_{,} b_{,} c_{,})$ conjugate to the diameter passing through the vertex of the cone, and we shall have, by a well-known relation,

$$p.A.B = a_{,} b_{,} c_{,} \text{.....................} (23).$$

Hence (22) may now be reduced to

$$\frac{(k^2 - k_{,}^2)(k^2 - k_{,,}^2)}{k^2 A^2 B^2} \text{..................} (24).$$

From (15) it follows that

$$\alpha^2 + \beta^2 + \gamma^2 = a^2 + b^2 + c^2 + k^2 + k_{,}^2 + k_{,,}^2 \text{......} (25),$$

and by a common theorem

$$\alpha^2 + \beta^2 + \gamma^2 + A^2 + B^2 = a_{,}^2 + b_{,}^2 + c_{,}^2 = a^2 + b^2 + c^2 + 3k^2 \text{...} (26).$$

Hence, combining (25) and (26),

$$A^2 + B^2 = (k^2 - k_{,}^2) + (k^2 - k_{,,}^2) \text{...........} (27).$$

Now the confocal surfaces $(a_{,} b_{,} c_{,})$ and $(a_{,,} b_{,,} c_{,,})$ intersect in a line of curvature, and for the whole of this line A is constant, or when k and $k_{,}$ are constant A is constant; hence

$$A^2 = (k^2 - k_{,}^2), \quad B^2 = (k^2 - k_{,,}^2) \text{...........} (28),$$

and the coefficient of x^2 in the equation of the cone, namely

$$\frac{(k^2 - k_{,}^2)(k^2 - k_{,,}^2)}{k^2 A^2 B^2},$$

becomes simply

$$\frac{1}{k^2} \text{...........................} (29).$$

By a similar process we shall find the coefficients of y^2 and z^2 to be $\frac{1}{k_{,}^2}$ and $\frac{1}{k_{,,}^2}$. Hence we derive the following very simple equation for the enveloping cone,

$$\frac{x^2}{k^2} + \frac{y^2}{k_{,}^2} + \frac{z^2}{k_{,,}^2} = 0 \dots\dots\dots\dots\dots\dots (30).$$

If we turn to (15) we shall easily perceive that *one* of the factors in the numerator of the value of β^2 must be negative. Let $k_{,}^2$ therefore be taken with a negative sign, and let it be greater than b^2. In order that the value of γ^2 may be real, since one of the factors of the numerator is negative, *two* must be negative, hence $k_{,,}^2$ must be taken with a negative sign, and that there may *not* be *two* negative factors in the value of β^2, $k_{,,}^2$ must be less than b^2. Now if $a > b > c$ in the order of magnitude, we shall have

$$b_{,,}^2 = b^2 - k_{,}^2, \quad c_{,,}^2 = c^2 - k_{,}^2.$$

Hence $b_{,,}^2$ and $c_{,,}^2$ must be taken with negative signs. Since $b_{,,,}^2 = b^2 - k_{,,}^2$, $c_{,,,}^2 = c^2 - k_{,,}^2$, $b_{,,,}^2$ must be taken with a positive sign and $c_{,,,}$ with a negative sign. Therefore the surface $(a_{,}b_{,}c_{,})$ is an ellipsoid, the surface $(a_{,,}b_{,,}c_{,,})$ is an hyperboloid of two sheets, and $(a_{,,,}b_{,,,}c_{,,,})$ is an hyperboloid of one sheet. (30) may now be written

respectively, these surfaces will intersect in the origin O of the first system of coordinates, and the fourth confocal surface $(kk_,k_{,,})$ may be enveloped by a cone whose vertex is at O and whose equation referred to the original axes of coordinates is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0 \ldots\ldots\ldots\ldots\ldots\ldots (32).$$

The original axes of coordinates Ox, Oy, Oz are normals to the second group of confocal surfaces, as Qx', Qy', Qz' are normals to the first, and the sums of the squares of the nine semiaxes in each group will obviously be equal to each other, as also an axis in each pair of corresponding surfaces. It is also obvious, from an inspection of (15), that α, β, γ, the coordinates of the point Q in the first system, become the perpendiculars from the point Q, the origin of the second system, on the tangent planes to the second group of surfaces having their common point of intersection at O.

III. *Let two cones having their common vertex on a surface of the second order, an ellipsoid suppose, $(a_,b_,c_,)$ envelop two confocal surfaces. The diametral plane of the surface conjugate to the diameter passing through the common vertex of the two cones will cut off from their common side a constant length, independent of the position of the common vertex of the two cones on the surface $(a_,b_,c_,)$.*

Let a, b, c; α, β, γ be the semiaxes of the confocal surfaces. And, as in the preceding theorem, let $(a_,b_,c_,)$, $(a_{,,}b_{,,}c_{,,})$, $(a_{,,,}b_{,,,}c_{,,,})$ be the axes of three confocal surfaces passing through the vertex of the cones. Hence we shall have, as in (13),

$$\left.\begin{matrix} a_,^2=a^2+k^2, & a_{,,}^2=a^2+k_,^2, & a_{,,,}^2=a^2+k_{,,}^2 \\ a_,^2=\alpha^2+h^2, & a_{,,}^2=\alpha^2+h_,^2, & a_{,,,}^2=\alpha^2+h_{,,}^2 \end{matrix}\right\} \ldots(33).$$

The equations of the cones will be, as in (30),

$$\frac{x^2}{k^2}+\frac{y^2}{k_,^2}+\frac{z^2}{k_{,,}^2}=0\,;\quad \frac{x^2}{h^2}+\frac{y^2}{h_,^2}+\frac{z^2}{h_{,,}^2}=0\,;$$

or as these equations may manifestly be written,

$$\left.\begin{matrix} \dfrac{x^2}{a_,^2-a^2}+\dfrac{y^2}{a_{,,}^2-a^2}+\dfrac{z^2}{a_{,,,}^2-a^2}=0 \\ \dfrac{x^2}{a_,^2-\alpha^2}+\dfrac{y^2}{a_{,,}^2-\alpha^2}+\dfrac{z^2}{a_{,,,}^2-\alpha^2}=0 \end{matrix}\right\}\ldots\ldots$$

Now the distance between the vertex and the diametral plane is p, and as p coincides with the new axis of x, we shall have

$$x = p \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

Let D be the length of the common side of the cones, then

$$D^2 = x^2 + y^2 + z^2.$$

And if we find the values of x, y, z from the equations (34) and (35), we shall have

$$\frac{D^2}{p^2} = \frac{a_{,}^2 a_{,,}^2 (a_{,}^2 - a_{,,}^2) + a_{,,}^2 a_{,,,}^2 (a_{,,}^2 - a_{,,,}^2) + a_{,,,}^2 a_{,}^2 (a_{,,,}^2 - a_{,}^2)}{(a_{,,}^2 - a_{,,,}^2)(a_{,}^2 - a^2)(a_{,}^2 - \alpha^2)} \;\ldots(36).$$

In (23) it was shewn that

$$p^2 = \frac{a_{,}^2 b_{,}^2 c_{,}^2}{(k^2 - k_{,}^2)(k^2 - k_{,,}^2)};$$

or as $\qquad k^2 - k_{,}^2 = a_{,} - a_{,,}^2, \quad k^2 - k_{,,}^2 = a_{,}^2 - a_{,,,}^2,$

$$p^2 = \frac{a_{,}^2 b_{,}^2 c_{,}^2}{(a_{,}^2 - a_{,,}^2)(a_{,}^2 - a_{,,,}^2)}.$$

Now the numerator of (36) may be resolved into the product of the three factors

$-(a_{,,}^2 - a_{,,,}^2)(a_{,,,}^2 - a_{,}^2)(a_{,}^2 - a_{,,}^2)$, and $a_{,}^2 - a^2 = k^2$, $a_{,}^2 - \alpha^2 = h^2$.

Hence, making the substitutions indicated,

$(x_i y_i z_i)$ being the vertex of the cone.

Let $$\frac{x^2}{a^2 + k^2} + \frac{y^2}{b^2 + k^2} + \frac{z^2}{c^2 + k^2} = 1 \ldots\ldots\ldots (39),$$

be the equation of the confocal surface.

The equation of the polar plane of $(x_i y_i z_i)$ with reference to this last surface is

$$\frac{xx_i}{a^2 + k^2} + \frac{yy_i}{b^2 + k^2} + \frac{zz_i}{c^2 + k^2} = 1 \ldots\ldots\ldots (40).$$

The equations to the normal at the point $(x_i y_i z_i)$ in (38) are

$$x - x_i = \frac{\cos\lambda}{\cos\nu}(z - z_i), \quad y - y_i = \frac{\cos\mu}{\cos\nu}(z - z_i) \ldots (41),$$

but $$\frac{\cos\lambda}{\cos\nu} = \frac{c^2 x_i}{a^2 z_i}, \quad \frac{\cos\mu}{\cos\nu} = \frac{c^2 y_i}{b^2 z_i}.$$

Now $$\Delta^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2,$$

or substituting in this expression the values derived from the preceding equations,

$$\Delta^2 = \left[\frac{x_i^2}{a^4} + \frac{y_i^2}{b^4} + \frac{z_i^2}{c^4}\right] \frac{c^4 (z - z_i)^2}{z_i^2} \ldots\ldots (42).$$

We must now determine the value of z for the point in which the axis of the cone meets the polar plane. For this purpose, from the equation of the polar plane

$$\frac{xx_i}{a^2 + k^2} + \frac{yy_i}{b^2 + k^2} + \frac{zz_i}{c^2 + k^2} = 1,$$

subtract the identity

$$\frac{x_i^2}{a^2 + k^2} + \frac{y_i^2}{b^2 + k^2} + \frac{z_i^2}{c^2 + k^2} = \frac{x_i^2}{a^2 + k^2} + \frac{y_i^2}{b^2 + k^2} + \frac{z_i^2}{c^2 + k^2};$$

replacing 1 by its value

$$\frac{x_i^2}{a^2} + \frac{y_i^2}{b^2} + \frac{z_i^2}{c^2},$$

the result will be found

$$\frac{x_i(x - x_i)}{a^2 + k^2} + \frac{y_i(y - y_i)}{b^2 + k^2} + \frac{z_i(z - z_i)}{c^2 + k^2}$$
$$= k^2 \left[\frac{x_i^2}{a^2(a^2 + k^2)} + \frac{y_i^2}{b^2(b^2 + k^2)} + \frac{z_i^2}{c^2(c^2 + k^2)}\right];$$

or putting for $(x - x_,)$ and $(y - y_,)$ their values derived from the equations of the normal (41), we find

$$c^2\left(\frac{z - z_,}{z_,}\right) = k^2.$$

Whence combining this expression with (42), we find

$$\Delta^2 = k^4\left[\frac{x_,^2}{a^4} + \frac{y_,^2}{b^4} + \frac{z_,^2}{c^4}\right],$$

or as this latter expression is equal to $\frac{1}{p^2}$, finally

$$\Delta = \frac{k^2}{p} \text{ ....(43)};$$

for any other confocal surface, the vertex of the cone remaining unchanged,

$$\Delta' = \frac{k_,^2}{p},$$

or

$$\Delta : \Delta' :: k^2 : k_,^2.$$

V. *To transform the equation of a surface of the second order, so that the axes of coordinates shall be the normal to the surface at a given point, and the two right lines in the tangent plane at this point, which are tangents to the lines of greatest and least curvature.*

Adding these terms vertically, the sum of the first column is manifestly = 1. The sum of the terms in the second column is $\frac{2x_i}{p}$. The sum of the terms in the third column is

$$p\,y_i \left\{\frac{\alpha^2}{a_i^2 a_{ii}^2} + \frac{\beta^2}{b_i^2 b_{ii}^2} + \frac{\gamma^2}{c_i^2 c_{ii}^2}\right\}.$$

Now the cosines of the angles λ, μ, ν, which the axes of coordinates make with the perpendicular p (let fall on the tangent plane through the point $(\alpha\beta\gamma)$ on the surface $(a_i b_i c_i)$,) are $\frac{p\alpha}{a_i^2}$, $\frac{p\beta}{b_i^2}$, $\frac{p\gamma}{c_i^2}$; and the cosines of the angles λ', μ', ν' which p_i makes, the perpendicular on the tangent plane through the point $(\alpha\beta\gamma)$ on the surface $(a_{ii} b_{ii} c_{ii})$ are $\frac{p_i\alpha}{a_{ii}^2}$, $\frac{p_i\beta}{b_{ii}^2}$, $\frac{p_i\gamma}{c_{ii}^2}$; and as these planes are at right angles,

$$\cos\lambda \cos\lambda' + \cos\mu \cos\mu' + \cos\nu \cos\nu' = 0.$$

Hence the third column in the coefficient of $y_i = 0$. In like manner the fourth column on the coefficient of $z_i = 0$. The fifth column is

$$p^2 x_i^2 \left\{\frac{\alpha^2}{a_i^6} + \frac{\beta^2}{b_i^6} + \frac{\gamma^2}{c_i^6}\right\}.$$

Now as

$$\cos^2\lambda = \frac{p^2\alpha^2}{a_i^4}, \quad \cos^2\mu = \frac{p^2\beta^2}{b_i^4}, \quad \cos^2\gamma = \frac{p^2\gamma^2}{c_i^4},$$

the coefficient of x_i^2 may be written

$$\frac{\cos^2\lambda}{a_i^2} + \frac{\cos^2\mu}{b_i^2} + \frac{\cos^2\nu}{c_i^2}.$$

This expression is $= \frac{1}{r^2}$, if we denote by r the semi-diameter of the surface parallel to p.

In like manner the coefficients of y_i^2 and z_i^2 are $\frac{1}{r_i}$ and $\frac{1}{r_{ii}}$ respectively; r_i and r_{ii} being parallel to p_i and p_{ii}.

The coefficient of $x_i y_i$ is

$$2pp_i \left[\frac{\alpha^2}{a_i^4 a_{ii}^2} + \frac{\beta^2}{b_i^4 b_{ii}^2} + \frac{\gamma^2}{c_i^4 c_{ii}^2}\right];$$

multiply the terms of this expression by the equivalent factors $a_{ii}^2 - a_i^2 = b_{ii}^2 - b_i^2 = c_{ii}^2 - c_i^2 = k_{ii}^2 - k^2$, dividing by

this latter, and the expression will be transformed into

$$\frac{2pp_{,}}{k_{,}^2 - k^2}\left[\frac{\alpha^2}{a_{,}^4} + \frac{\beta^2}{b_{,}^4} + \frac{\gamma^2}{c_{,}^4} - \left(\frac{\alpha^2}{a_{,}^2 a_{,,}^2} + \frac{\beta^2}{b_{,}^2 b_{,,}^2} + \frac{\gamma^2}{c_{,}^2 c_{,,}^2}\right)\right].$$

Now the first of these groups is equal to $\frac{1}{p^2}$, and the second, as we have already shewn, is $= 0$; hence, the coefficient of xy is $\frac{2p_{,}}{p(k_{,}^2 - k^2)}$.

In the same manner it may be shewn that the coefficient of $x_{,}z_{,}$ is $\frac{2p_{,,}}{p(k_{,,}^2 - k^2)}$.

Let $r_{,}$ and $r_{,,}$ be the axes of the section parallel to the tangent plane at the point $(\alpha\beta\gamma)$, then, as we have found in (28),

$$r_{,,}^2 = k^2 - k_{,,}^2, \quad r_{,}^2 = k^2 - k_{,}^2.$$

Introducing into the equation (44) the resulting expressions thus found, the equation of the surface will at length become

$$\frac{x^2}{r^2} + \frac{y^2}{r_{,}^2} + \frac{z^2}{r_{,,}^2} - \frac{2p_{,}}{pr_{,}^2}xy - \frac{2p_{,,}}{pr_{,,}^2}xz + \frac{2x}{p} = 0 \ \dots \ (45).$$

In this equation the coefficients are the perpendiculars p, $p_{,}$, $p_{,,}$ from the centre on the coordinate planes, and the three diameters of the surface which coincide with these perpendiculars.

whose focal lines coincide with the optic axes of the surface or with the perpendiculars to its circular sections.

Since a line of curvature is generated by the intersection of two confocal surfaces, let the equations of these surfaces be

$$\frac{x_i^2}{a^2}+\frac{y_i^2}{b^2}+\frac{z_i^2}{c^2}=1, \quad \text{and} \quad \frac{x_i^2}{a^2+k^2}+\frac{y_i^2}{b^2+k^2}+\frac{z_i^2}{c^2+k^2}=1 \ldots (46),$$

Let λ, μ, ν be the angles which p makes with the axes, then

$$\cos\lambda=\frac{px_i}{a^2}, \quad \cos\mu=\frac{py_i}{b^2}, \quad \cos\nu=\frac{pz_i}{c^2};$$

from these five equations eliminating $x_i y_i z_i$ and p, and putting for $\frac{\cos\lambda}{\cos\nu}$, $\frac{\cos\mu}{\cos\nu}$, their values $\frac{x}{z}$, $\frac{y}{z}$, the resulting equation will become

$$\frac{a^2x^2}{a^2+k^2}+\frac{b^2y^2}{b^2+k^2}+\frac{c^2z^2}{c^2+k^2}=0 \ldots\ldots\ldots\ldots (47).$$

Now the angles which the focals of this cone make with the internal axe are given by the equation

$$\tan^2\varepsilon=\frac{c^2(a^2-b^2)}{b^2(a^2-c^2)},$$

a result independent of k; hence, all the cones so generated are confocal. Hence, the supplemental cones to these have their circular sections parallel to those of the given surface, and therefore these supplemental cones are the intersections of spheres with the given surface.

The angles which the optic axes make with the vertical axes are given by the same equation

$$\tan^2\varepsilon_i=\frac{c^2(a^2-b^2)}{b^2(a^2-c^2)}, \quad \text{or } \varepsilon_i=\varepsilon.$$

NOTES ON MOLECULAR MECHANICS.

By the Rev. Samuel Haughton.

No. III. Normal and Transverse Vibrations.

In my last Note, Vol. VIII. p. 159, I have given the general equations of motion of an elastic system, derived from the function of the second order of the differential coefficients of the displacements; together with the integral

corresponding to plane waves. I shall now restrict the results there obtained, so that the vibrations shall be normal and transversal.

If, in equations (10), we substitute (l, m, n) for $(\cos\alpha, \cos\beta, \cos\gamma)$, and eliminate v^2, we shall have the conditions necessary to express that one of the vibrations is normal to the wave plane and the other two transverse; the resulting equations are

$$\left.\begin{array}{l}(Q' - R')mn + F'^2(n^2 - m^2) + H'ln - G'lm = 0 \\ (R' - P')nl + G'^2(l^2 - n^2) + F'ml - H'mn = 0 \\ (P' - Q')lm + H'^2(m^2 - l^2) + G'nm - F'nl = 0\end{array}\right\} \ldots(24).$$

Introducing into these equations the values of P', Q', &c., we find the following twenty-four conditions:

$$\left.\begin{array}{c}(a_1b_1) = (a_1c_1) = (b_2c_2) = 0 \\ (a_2b_2) = (a_3c_3) = (b_3c_3) = 0 \\ (b_1c_1) + (a_1c_2) + (a_2c_1) = (b_1c_1) + (a_1b_3) + (a_3b_1) = 0 \\ (c_2a_2) + (b_2a_3) + (b_3a_2) = (c_2a_2) + (b_2c_1) + (b_1c_2) = 0 \\ (a_3b_3) + (c_3b_1) + (c_1b_3) = (a_3b_3) + (c_3a_2) + (c_2a_3) = 0 \\ (a_1a_2) = (b_1b_2) = (c_1c_2) - (a_3b_3)\end{array}\right\} \ldots (25).$$

tain only thirty-six coefficients, the coefficients of $\Sigma(\alpha_1\beta_2)$ *being equal in pairs.*"

As such a supposition appears to be, *a priori*, very probable, I shall adopt it in the investigations which follow in this note, and consider the consequences which follow from it.

It supplies us with the following nine equations:

$$\left.\begin{array}{lll} (b_2c_3)=(b_3c_2), & c_2a_3=c_3a_2, & a_2b_3=a_3b_2 \\ (b_3c_1)=(b_1c_3), & c_3a_1=c_1a_3, & a_3b_1=a_1b_3 \\ (b_1c_2)=(b_2c_1), & c_1a_2=c_2a_1, & a_1b_2=a_2b_1 \end{array}\right\} \ldots (27).$$

Combining these conditions with (25), and introducing them into (26), I find

$$\left.\begin{array}{l} 2V = A\lambda + B\mu + C\nu + 2D\phi + 2E\chi + 2F\psi \\ \quad + P(X^2-u_1) + Q(Y^2-v_2) + R(Z^2-w_3) \\ \quad + L\{2YZ-(v_3+w_2)\} + M\{2XZ-(w_1+u_3)\} \\ \qquad\qquad + N\{2XY-(u_2+v_1)\} \end{array}\right\} \ldots (28),$$

where

$$\left.\begin{array}{l} A=(a_1^2), \quad D=(a_1a_3), \\ B=(b_2)^2, \quad E=(b_2b_3), \\ C=(c_3)^2, \quad F=(c_1c_3), \\ P=-2(b_2c_3), \quad L=-(b_1c_1), \\ Q=-2(c_3a_1), \quad M=-(c_2a_2), \\ R=-2(a_1b_2), \quad N=-(a_3b_3), \\ \lambda=\alpha_1^2+\beta_1^2+\gamma_1^2, \quad \phi=\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2, \\ \mu=\alpha_2^2+\beta_2^2+\gamma_2^2, \quad \chi=\alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3, \\ \nu=\alpha_3^2+\beta_3^2+\gamma_3^2, \quad \psi=\alpha_1\alpha_3+\beta_1\beta_3+\gamma_1\gamma_3, \\ X=\beta_3-\gamma_2, \\ Y=\gamma_1-\alpha_3, \\ Z=\alpha_2-\beta_1, \\ u_1=\beta_2\gamma_3-\beta_3\gamma_2, \quad v_1=\gamma_2\alpha_3-\gamma_3\alpha_2, \quad w_1=\alpha_2\beta_3-\alpha_3\beta_2 \\ u_2=\beta_3\gamma_1-\beta_1\gamma_3, \quad v_2=\gamma_3\alpha_1-\gamma_1\alpha_3, \quad w_2=\alpha_3\beta_1-\alpha_1\beta_3 \\ u_3=\beta_1\gamma_2-\beta_2\gamma_1, \quad v_3=\gamma_1\alpha_2-\gamma_2\alpha_1, \quad w_3=\alpha_1\beta_2-\alpha_2\beta_1 \end{array}\right\} \ldots (29).$$

It is not difficult to prove that if the coordinate axes be changed in direction, the origin remaining the same, that the functions $\lambda, \mu, \nu, \phi, \chi, \psi$ are transformed by the same equations as $x^2, y^2, z^2, yz, xz, xy$; and that $u_1, v_2, w_3, v_3+w_2, w_1+u_3, u_2+v_1$, are transformed by the same equations as

$x^2, y^2, z^2, 2yz, 2xz, 2xy$; and it is well known that **X, Y, Z,** may be transformed by the same equations as x, y, z.

Hence the function V may be reduced to a function of nine coefficients in either of two ways, 1st by assuming for coordinate axes the axes of the ellipsoid

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy = 1\,;$$

or, 2nd, of the ellipsoid

$$Px^2 + Qy^2 + Rz^2 + 2Lyz + 2Mxz + 2Nxy = 1.$$

The position of the first set of axes of elasticity depends on the external conditions to which the body is subject, and the position of the second set of axes appears to depend on the inherent structure of the body.

If we suppose the axes chosen to be the axes of the second ellipsoid, so as to destroy the coefficients L, M, N, and seek the equation of the surface of *Wave Slowness* (12), we shall find, after some reductions,

$$(S-1)\left\{\begin{aligned}&(x^2+y^2+z^2)\,(QRx^2+PRy^2+PQz^2)\\ &\quad+(S-1)\,\{(Q+R)\,x^2+(P+R)\,y^2+(P+Q)\,z^2\}\\ &\quad+(S-1)^2\end{aligned}\right\}=0..(30),$$

where

$$S = Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy.$$

Equation (30) is composed of two factors; one of the second

Hence we find

$$\left.\begin{aligned}
-\frac{dV}{d\alpha_1} &= A\alpha_1 - \tfrac{1}{2}P(\beta_2 + \gamma_3) \\
-\frac{dV}{d\beta_2} &= A\beta_2 - \tfrac{1}{2}P(\gamma_3 + \alpha_1) \\
-\frac{dV}{d\gamma_3} &= A\gamma_3 - \tfrac{1}{2}P(\alpha_1 + \beta_2) \\
-\frac{dV}{d\alpha_2} &= (A + P)\,\alpha_2 - \tfrac{1}{2}P\beta_1 \\
-\frac{dV}{d\beta_1} &= (A + P)\,\beta_1 - \tfrac{1}{2}P\alpha_2 \\
-\frac{dV}{d\beta_3} &= (A + P)\,\beta_3 - \tfrac{1}{2}P\gamma_2 \\
-\frac{dV}{d\gamma_2} &= (A + P)\,\gamma_2 - \tfrac{1}{2}P\beta_3 \\
-\frac{dV}{d\gamma_1} &= (A + P)\,\gamma_1 - \tfrac{1}{2}P\alpha_3 \\
-\frac{dV}{d\alpha_3} &= (A + P)\,\alpha_3 - \tfrac{1}{2}P\gamma_1
\end{aligned}\right\} \quad \text{..........(33).}$$

As these coefficients are the forces acting on the faces of the elementary parallelepiped (vol. v. p. 176), it is necessary that they should have opposite signs to those of the displacements and strains, by which they are produced, and consequently the right-hand members of these equations must be essentially positive.

Hence, adding the first three together, we find

$$A - P > 0,$$

and adding the last six together in pairs, we find

$$A + \tfrac{1}{2}P > 0.$$

These equations, taken together, prove that A must be positive and cannot be zero or negative; and that P may have any value lying between $+A$ and $-2A$, which limits, positive and negative, it cannot exceed. From this investigation it appears that A cannot vanish without P vanishing at the same time in an uncrystalline body, and similar reasoning would apply to the case of a crystallized body

The equations of motion deducible from (32) may be written thus:

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= T^2\left(\frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}+\frac{d^2\xi}{dz^2}\right)+(N^2-T^2)\frac{d\omega}{dx}\\ \frac{d^2\eta}{dt^2} &= \text{..............}\\ \frac{d^2\zeta}{dt^2} &= \text{..............}\end{aligned}\right\}\text{...(34)},$$

where $\omega = \frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}$, $\frac{A}{\varepsilon}=N^2$, and $\frac{P}{\varepsilon}=T^2-N^2$.

Equations (34) may be transformed into the following:

$$\left.\begin{aligned}\frac{d^2\omega}{dt^2} &= N^2\left(\frac{d^2\omega}{dx^2}+\frac{d^2\omega}{dy^2}+\frac{d^2\omega}{dz^2}\right)\\ \frac{d^2X}{dt^2} &= T^2\left(\frac{d^2X}{dx^2}+\frac{d^2X}{dy^2}+\frac{d^2X}{dz^2}\right)\\ \frac{d^2Y}{dt^2} &= T^2\left(\frac{d^2Y}{dx^2}+\frac{d^2Y}{dy^2}+\frac{d^2Y}{dz^2}\right)\\ \frac{d^2Z}{dt^2} &= T^2\left(\frac{d^2Z}{dx^2}+\frac{d^2Z}{dy^2}+\frac{d^2Z}{dz^2}\right)\end{aligned}\right\}\text{...... (35).}$$

transverse vibration; and secondly, by supposing the normal vibration to be evanescent. I have just shewn, that in a body constituted as I have supposed, it is possible for the normal velocity to become equal to the transverse velocity, without endangering the stability of the system; and I shall now inquire whether the normal velocity may be considered equal to the transverse velocity in the problem of refraction.

Let the axis of z be perpendicular to the surface, the plane (x, z) being the plane of incidence, and let the incident vibration be confined to the plane of incidence. By (34), since $N = T$, we have

$$\left\{\frac{d^2}{dt^2} - T^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\right\}\{\xi, \eta, \zeta\} = 0$$

for the equations of wave propagation, and by comparing (33) and (5), (6), we find for the conditions at the limits

$$\xi_0 = \xi_0', \quad \zeta_0 = \zeta_0', \quad A\gamma_3 = A'\gamma_3', \quad A\alpha_3 = A'\alpha_3' \ldots (37).$$

Let $\tau, \tau_{,}, \tau_{,,}$ denote the incident, reflected transverse, and reflected normal vibration; and τ', τ'' the refracted transverse and refracted normal vibration; then (37) may be written thus:

$$\left.\begin{aligned}
&\tau\cos\alpha + \tau_{,}\cos\alpha_{,} + \tau_{,,}\cos\alpha_{,,} = \tau'\cos\alpha' + \tau''\cos\alpha'' \\
&\tau\cos\gamma + \tau_{,}\cos\gamma_{,} + \tau_{,,}\cos\gamma_{,,} = \tau'\cos\gamma' + \tau''\cos\gamma'' \\
&A\left(\frac{\tau}{\lambda}n\cos\gamma + \frac{\tau_{,}}{\lambda_{,}}n_{,}\cos\gamma_{,} + \frac{\tau_{,,}}{\lambda_{,,}}n_{,,}\cos\gamma_{,,}\right) \\
&\qquad\qquad = A'\left(\frac{\tau'}{\lambda'}n'\cos\gamma' + \frac{\tau''}{\lambda''}n''\cos\gamma''\right) \\
&A\left(\frac{\tau}{\lambda}n\cos\alpha + \frac{\tau_{,}}{\lambda_{,}}n_{,}\cos\alpha_{,} + \frac{\tau_{,,}}{\lambda_{,,}}n_{,,}\cos\alpha_{,,}\right) \\
&\qquad\qquad = A'\left(\frac{\tau'}{\lambda'}n'\cos\alpha' + \frac{\tau''}{\lambda''}n''\cos\alpha''\right)
\end{aligned}\right\}\ldots(38),$$

and the angles contained in these equations are connected with the angles of incidence and refraction of the respective rays in the following way:

$$\left.\begin{array}{llll}
\cos\alpha = \cos i, & n = -\cos i, & l_{,,} = \cos\alpha_{,,} = & \sin i \\
\cos\gamma = \sin i, & l = \sin i, & n_{,,} = \cos\gamma_{,,} = & \cos i \\
\cos\alpha_{,} = \cos i, & n_{,} = \cos i, & & \\
\cos\gamma_{,} = -\sin i, & l_{,} = \sin i, & l'' = \cos\alpha'' = & \sin r \\
\cos\alpha' = \cos r, & n' = -\cos r, & n'' = \cos\gamma'' = & -\cos r \\
\cos\gamma' = \sin r, & l' = \sin r & &
\end{array}\right\}\ldots(39),$$

Introducing these values into equations (38), and remembering that the normal and transverse velocities are equal, we find

$$\left.\begin{aligned}
(\tau+\tau_{,})\cos i + \tau_{,,}\sin i &= \tau'\cos r + \tau''\sin r \\
(\tau-\tau_{,})\sin i + \tau_{,,}\cos i &= \tau'\sin r - \tau''\cos r \\
(\tau+\tau_{,})\sin i\cos i - \tau_{,,}\cos^2 i & \\
&= k\mu\left\{\tau'\sin r\cos r - \tau''\cos^2 r\right\} \\
(\tau-\tau_{,})\cos^2 i - \tau_{,,}\sin i\cos i & \\
&= k\mu\left\{\tau'\cos^2 r + \tau''\sin r\cos r\right\}
\end{aligned}\right\} \ldots\ldots (40),$$

where $k = \frac{A'}{A}$, and μ is the refractive index $= \frac{\sin i}{\sin r}$.

If, to simplify the calculations, we make $k = 1$,* equations (40) may be reduced to the following:

$$(\tau+\tau_{,})\sin 2i - \tau_{,,}\cos 2i = 2\tau'\sin i\cos r - \mu\tau''\cos 2r,$$

$$(\tau-\tau_{,})\cos 2i - \tau_{,,}\sin 2i = \tau'\mu\cos 2r + 2\tau''\sin i\cos r,$$

$$\tau - \tau_{,} = \mu\tau',$$

$$\tau_{,,} = \mu\tau'',$$

which may be converted into the following:

second part of the expression (41), which depends on the first power of the tangent; and therefore when the refractive index is small, the numerical values of (41) will be nearly represented by the formula

$$\frac{\tau_1}{\tau} = -\frac{\tan(i-r)}{\tan(i+r)} \quad \text{..................... (42)},$$

which coincides with the well-known law of Brewster and Fresnel.

If the incident vibration be perpendicular to the plane of incidence, the conditions at the limits will be

$$\eta_0 = \eta_0', \quad (A\beta_3)_0 = (A'\beta'_3)_0 \quad \text{............ (43)}.$$

These equations give, assuming as before $A = A'$,

$$\left.\begin{aligned} \tau + \tau_1 &= \tau' \\ (\tau - \tau_1)\cos i &= \mu\tau' \cos r \end{aligned}\right\} \quad \text{............... (44)}.$$

From which it is easy to find

$$2\tau \cos i = \tau'(\cos i + \mu \cos r),$$

$$2\tau_1 \cos i = \tau'(\cos i - \mu \cos r).$$

And finally

$$\frac{\tau_1}{\tau} = -\frac{\sin(i-r)}{\sin(i+r)} \quad \text{.................... (45)},$$

which is the value given by Brewster and Fresnel's law.

Trinity College, Dublin,
Nov. 12, 1853.

SUR UNE PROPRIÉTÉ D'UN PRODUIT DE FACTEURS LINÉAIRES.

Par Professeur François Brioschi.

Un mémoire de M. Cayley* publié le dernier cahier de ce *Journal* m' à suggéré une remarquable propriété pour le produit d'un nombre déterminé de facteurs linéaires.

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* On the Rationalization of certain Algebraical Equations, May 1853.

En considérant deux éléments x, y, on voit tout de suite, que les seuls résultats essentiellement différents qu'on obtient en les ajoutant l'un à l'autre, sont donnés par les expressions

$$x + y, \quad x - y.$$

Or si l'on suppose

$$x + y = 0, \quad x^2 = a, \quad y^2 = b,$$

et on multiplie les termes de la première par 1 et par xy; on obtient en observant les deux autres équations

$$x + y = 0,$$
$$bx + ay = 0,$$

et par conséquent

$$\Delta = \begin{vmatrix} 1 & 1 \\ b & a \end{vmatrix} = 0.$$

Si l'on considère l'équation

$$x - y = 0,$$

et on fait sur elle des opérations analogues aux supérieures on a

$$\nabla = \begin{vmatrix} 1, & -1 \\ -b, & a \end{vmatrix} = \Delta = 0.$$

J'observe que l'équation $\Delta = 0$ a son origine dans la $x+y=0$,

et Δ sera divisible par $x+y+z$, $x+y-z$, &c., et on aura

$$(x+y+z)(x+y-z)(x-y+z)(-x+y+z)=-\Delta.$$

Si on représente avec x, y, z les trois côtés d'un triangle, et avec A sa surface on aura

$$\Delta = (4A)^2,$$

et aussi
$$\Delta = \frac{(2A)^4}{r\,r_1 r_2 r_3},$$

r, r_1, r_2, r_3, désignant les rayons des cercles inscrits et exinscrits aux triangle. Observons qu'on a évidemment

$$\Delta = \frac{1}{x^2y^2z^2}\begin{vmatrix} \cdot & x & y & z \\ x & \cdot & cxy & bxz \\ y & cxy & \cdot & ayz \\ z & bxz & ayz & \cdot \end{vmatrix} = \begin{vmatrix} \cdot & x & y & z \\ x & \cdot & z & y \\ y & z & \cdot & x \\ z & y & x & \cdot \end{vmatrix}$$

Pour quatre éléments le nombre des expressions différentes étant huit, on aura les équations

$$x+y+z+t=0,\quad -x+y+z+t=0,\quad x-y+z+t=0,\quad x+y-z+t=0,$$
$$x+y+z-t=0,\quad -x+y+z-t=0,\quad -x+y-z+t=0,\quad -x-y+z+t=0,$$

desquelles en posant

$$x^2=a,\quad y^2=b,\quad z^2=c,\quad t^2=d,$$

et en multipliant avec ordre par

$$1,\ zt,\ yt,\ xt,\ zy,\ zx,\ yx,\ xyzt,$$

on aura

$$\Delta = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & d & c \\ \cdot & 1 & \cdot & 1 & \cdot & d & \cdot & b \\ \cdot & 1 & 1 & \cdot & d & \cdot & \cdot & a \\ 1 & \cdot & \cdot & 1 & \cdot & c & b & \cdot \\ 1 & \cdot & 1 & \cdot & c & \cdot & a & \cdot \\ 1 & 1 & \cdot & \cdot & b & a & \cdot & \cdot \\ d & c & b & a & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0,$$

et Δ sera égal au produit des premiers membres des équations supérieures.

Si x, y, z, t désignent les aires des quatre faces d'un tétrahèdre, V le volume, et $r, r_1, \ldots r_7$ les rayons des sphères

inscrites et ex-inscrites, on aura

$$\Delta = \frac{(3V)^8}{r\,r_1 \ldots r_7},$$

et Δ *réprésente* la norme de $\pm x \pm y \pm z \pm t$. (Voir un memoire de Mr. Sylvester dans le même cahier).* On a aussi

$$\Delta = \begin{vmatrix} . & . & . & . & x & y & z & t \\ . & . & x & y & . & . & t & z \\ . & z & . & x & . & t & . & y \\ . & y & z & . & t & . & . & x \\ x & . & . & t & . & z & y & . \\ y & . & t & . & z & . & x & . \\ z & t & . & . & y & x & . & . \\ t & z & y & x & . & . & . & . \end{vmatrix}$$

Si en général on considère n éléments

$$x_1, x_2 \ldots x_n,$$

en posant
$$X_n = x_1 + x_2 \ldots + x_n,$$

et en désignant par

$$| X_n(1, 2, \ldots m) |,$$

le produit des facteurs linéaires qu'on deduit de X_n en

Il est évident que le nombre des facteurs linéaires du produit (1) sera

$$1 + n + \frac{n(n-1)}{2} + \ldots + \frac{n(n-1)\ldots\frac{(n-3)}{2}}{1.2\ldots\frac{n-1}{2}} = 2^{n-1},$$

et le nombre des facteurs du produit (2) sera

$$1+n+\frac{n(n-1)}{2}+\ldots+\frac{n(n-1)\ldots\left(\frac{n}{2}+2\right)}{1.2\ldots\left(\frac{n}{2}-1\right)}+\frac{1}{2}\,\frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1.2\ldots\frac{n}{2}} = 2^{n-1},$$

et que le nombre des équations qui découlent de l'équation $X_n = 0$ sera, soit pour n impair, soit pour n pair

$$1 + \frac{n(n-1)}{2} + \ldots + \frac{n(n-1)\ldots 3.2.1}{1.2.3\ldots(n-1)} = 2^{n-1}.$$

Il importe d'observer que, en supposant que les éléments $x_1, x_2 \ldots$ soient des radicaux quadratiques, le produit Δ des fonctions linéaires de ces radicaux est rationnel.

Si l'on désigne par $1, \alpha, \beta$ les trois racines de l'équation $w^3 - 1 = 0$, les expressions

$$x + y, \quad x + \alpha y, \quad x + \beta y,$$

sont les seules essentiellement différentes qu'on peut former en combinant deux éléments x, y. Supposons

$$x + y = 0, \quad x + \alpha y = 0, \quad x + \beta y = 0,$$

$$x^3 = a, \quad y^3 = b.$$

En multipliant chacune des équations supérieures par $1, x^2y, xy^2$, on aura

$$\Delta = \begin{vmatrix} . & 1 & 1 \\ 1 & . & a \\ 1 & b & . \end{vmatrix} = 0, \quad \Delta_1 = \begin{vmatrix} . & 1 & \alpha \\ \alpha & . & a \\ 1 & ab & . \end{vmatrix} = 0, \quad \Delta_2 = \begin{vmatrix} . & 1 & \beta \\ \beta & . & a \\ 1 & \beta b & . \end{vmatrix} = 0,$$

et puisque Δ_1 se réduit identique à Δ en multipliant la dernière signe, et la seconde colonne par α, et en divisant la première signe et la première colonne par α; et analoguement en réduit $\Delta_2 = \Delta$ on aura

$$(x + y)(x + \alpha y)(x + \beta y) = \Delta.$$

Trois éléments x, y, z donnent neuf expressions différentes,

$$\left|\begin{array}{c|c|c} x+y+z & x+y+\alpha z & x+y+\beta z \\ x+\alpha y+\beta z & x+\alpha y+z & x+\beta y+z \\ x+\beta y+\alpha z & \alpha x+y+z & \beta x+y+z \end{array}\right| \ldots\ldots (3),$$

chacune desquelles égalée à zéro, et multipliée par 1, z^2y, z^2x, y^2z, y^2x, x^2z, x^2y, $x^2y^2z^2$, xyz donnera

$$\Delta = \left|\begin{array}{ccccccccc} . & . & . & . & . & . & 1 & 1 & 1 \\ . & . & 1 & . & . & 1 & . & c & . \\ . & . & 1 & . & 1 & . & c & . & . \\ . & 1 & . & . & . & 1 & . & . & b \\ . & 1 & . & 1 & . & . & b & . & . \\ 1 & . & . & . & 1 & . & . & . & a \\ 1 & . & . & 1 & . & . & . & a & . \\ . & . & . & c & b & a & . & . & . \\ 1 & 1 & 1 & . & . & . & . & . & . \end{array}\right| = 0,$$

ayant posé

$$x^3 = a, \quad y^3 = b, \quad z^3 = c\,;$$

et le produit des expressions (3) egalera $\pm\Delta$. La valeur de Δ est remarquable pour sa forme; en effet on a

par les vingt-sept quantités

$$1,\ t^2z\ldots,\ y^2z^2t^2\ldots,\ yzt\ldots,\ xy^2z^2t^2\ldots,\ x^2y^2z^2t^2,\ xyzt,$$

donneront par l' élimination un déterminant qui sera égal au produit des vingt-sept expressions supérieures.

En général le nombre des expressions sera 3^{n-1} pour n éléments ; et l' on aura 3^{n-1} quantités par lesquelles on doit multiplier les équations qui en découlent ; et par conséquent le déterminant sera de l' ordre 3^{n-1}. Et si les éléments seront des radicaux cubiques leur produit sera rationnel. Cette loi s' étend évidemment aux produit d' un nombre déterminé de radicaux de l' ennième ordre. Nous ajouterons seulement le cas du produit des quatre expressions qu' on obtient en combinant deux éléments avec les quatre racines de l' equation $w^4 - 1 = 0$. En désignant ces racines avec $1, \alpha, \beta, \gamma$ on aura les équations,

$$x + y = 0, \quad x + \alpha y = 0, \quad x + \beta y = 0, \quad x + \gamma y = 0\ ;$$

chacune desquelles multipliée par $1, xy^3, x^2y, x^3y^2$ donne

$$\Delta = \begin{vmatrix} . & . & 1 & 1 \\ 1 & . & b & . \\ . & 1 & . & a \\ 1 & 1 & . & . \end{vmatrix} = 0,$$

en posant $x^4 = a$, $y^4 = b$; et par conséquent

$$(x + y)(x + \alpha y)(x + \beta y)(x + \gamma y) = - \Delta,$$

et on soit que si x, y étaient des radicaux biquadratiques, le produit serait rationnel.

Nous ajouterons une dernière remarque sur une propriété des sommes des puissances égales de expressions supérieures. Par exemple le trinome

$$(x + y)^2 + \beta(x + \alpha y)^2 + \alpha(x + \beta y)^2 = H,$$

qui evidémment s' annule pour $x = 0$, $y = 0$ sera divisible par le produit xy ; et le quotient sera 3, 2 ; parceque le produit xy est répété trois fois, et chaque fois avec le coefficient 2. Par conséquent sera

$$H = 3.2xy.$$

Analoguement on peut démontrer l' équation

$$(x + y + z)^3 + (x + \alpha y + \beta z)^3 + (x + \beta y + \alpha z)^3$$
$$+ \beta\{(x + y + \alpha z)^3 + (x + \alpha y + z)^3 + (\alpha x + y + z)^3\}$$
$$+ \alpha\{(x + y + \beta z)^3 + (x + \beta y + z)^3 + (\beta x + y + z)^3\} = 3^2.2.3xyz,$$

et ainsi de suite. Ces relations sont analogues à quelques-unes de celles que M. Cauchy* a trouvées en considérant seulement les deux racines de l'équation $w^2 - 1 = 0$.

NOTE.—PROB. 7.† Si $\alpha = 0$, $\beta = 0$, $\gamma = 0$ sont les équations des trois côtés d'un triangle l, m, n, trois indéterminées on a

$$\begin{vmatrix} . & \alpha & \beta & \gamma \\ \alpha & . & n & m \\ \beta & n & . & l \\ \gamma & m & l & . \end{vmatrix} = 0,$$

pour l'équation d'une conique inscrite. En égard à ce qu'on a démontré supérieurment cette équation pourra s'écrire

$$\pm (l\alpha)^{\frac{1}{2}} \pm (m\beta)^{\frac{1}{2}} \pm (n\gamma)^{\frac{1}{2}} = 0.$$

Pour déterminer l, m, n, j'observe que l'ellipse *maximum* inscrite dans un triangle touche les milieux de ces côtés. Or les équations des droites qui unissant deux à deux les points de contact étant en général

$$m\beta + n\gamma - l\alpha = 0, \quad n\gamma + l\alpha - m\beta = 0, \quad l\alpha + m\beta - n\gamma = 0\ ;$$

et dans ce cas ces droites étant respectivement parallèles aux côtés du triangle, on aura les équations

$$m \sin(\alpha - \beta) + n \sin(\alpha - \gamma) = 0,$$

ON A PARTICULAR CASE OF THE DESCENT OF A HEAVY BODY IN A RESISTING MEDIUM.

By J. C. Maxwell.

Every one must have observed that when a slip of paper falls through the air, its motion, though undecided and wavering at first, sometimes becomes regular. Its general path is not in the vertical direction, but inclined to it at an angle which remains nearly constant, and its fluttering appearance will be found to be due to a rapid rotation round a horizontal axis. The direction of deviation from the vertical depends on the direction of rotation.

If the positive directions of an axis be toward the right hand and upwards, and the positive angular direction opposite to the direction of motion of the hands of a watch, then, if the rotation is in the positive direction, the horizontal part of the mean motion will be positive.

These effects are commonly attributed to some accidental peculiarity in the form of the paper, but a few experiments with a rectangular slip of paper (about two inches long and one broad), will shew that the direction of rotation is determined, not by the irregularities of the paper, but by the initial circumstances of projection, and that the symmetry of the form of the paper greatly increases the distinctness of the phenomena. We may therefore assume that if the form of the body were accurately that of a plane rectangle, the same effects would be produced.

The following investigation is intended as a general explanation of the true cause of the phenomenon.

I suppose the resistance of the air caused by the motion of the plane to be in the direction of the normal and to vary as the square of the velocity estimated in that direction.

Now though this may be taken as a sufficiently near approximation to the magnitude of the resisting force on the plane taken as a whole, the pressure on any given element of the surface will vary with its position so that the resultant force will not generally pass through the centre of gravity.

It is found by experiment that the position of the centre of pressure depends on the tangential part of the motion, that it lies on that side of the centre of gravity towards which the tangential motion of the plane is directed, and that its distance from that point increases as the tangenti[al] velocity increases.

I am not aware of any mathematical investigation of this effect. The explanation may be deduced from experiment.

Place a body similar in shape to the slip of paper obliquely in a current of some visible fluid. Call the edge where the fluid first meets the plane the first edge, and the edge where it leaves the plane, the second edge, then we may observe that

(1) On the anterior side of the plane the velocity of the fluid increases as it moves along the surface from the first to the second edge, and therefore by a known law in hydrodynamics, the pressure must diminish from the first to the second edge.

(2) The motion of the fluid behind the plane is very unsteady, but may be observed to consist of a series of eddies diminishing in rapidity as they pass behind the plane from the first to the second edge, and therefore relieving the posterior pressure most at the first edge.

Both these causes tend to make the total resistance greatest at the first edge, and therefore to bring the centre of pressure nearest to that edge.

Hence the moment of the resistance about the centre of gravity will always tend to turn the plane towards a position perpendicular to the direction of the current, or, in the case of the slip of paper, to the path of the body

The motion, speaking roughly, is one of descent, that is, in the negative direction along the axis of y.

The resolved part of the resistance in the vertical direction will always act upwards, being greatest when the plane of the paper is horizontal, and vanishing when it is vertical.

When the motion has become regular, the effect of this force during a whole revolution will be equal and opposite to that of gravity during the same time.

Since the resisting force increases while the normal is in its first and third quadrants, and diminishes when it is in its second and fourth, the maxima of velocity will occur when the normal is in its first and third quadrants, and the minima when it is in the second and fourth.

The resolved part of the resistance in the horizontal direction will act in the positive direction along the axis of x in the first and third quadrants, and in the negative direction during the second and fourth; but since the resistance increases with the velocity, the whole effect during the first and third quadrants will be greater than the whole effect during the second and fourth. Hence the horizontal part of the resistance will act on the whole in the positive direction, and will therefore cause the general path of the body to incline in that direction, that is, toward the right.

That part of the moment of the resistance about the centre of gravity which depends on the angular velocity will vary in magnitude, but will always act in the negative direction. The other part, which depends on the obliquity of the plane of the paper to the direction of motion, will be positive in the first and third quadrants and negative in the second and fourth; but as its magnitude increases with the velocity, the positive effect will be greater than the negative.

When the motion has become regular, the effect of this excess in the positive direction will be equal and opposite to the negative effect due to the angular velocity during a whole revolution.

The motion will then consist of a succession of equal and similar parts performed in the same manner, each part corresponding to half a revolution of the paper.

These considerations will serve to explain the lateral motion of the paper, and the maintenance of the rotatory motion.

Similar reasoning will shew that whatever be the initial motion of the paper, it cannot remain uniform.

Any accidental oscillations will increase till their amplitude exceeds half a revolution. The motion will then be

one of rotation, and will continually approximate to that which we have just considered.

It may be also shewn that this motion will be unstable unless it take place about the longer axis of the rectangle.

If this axis is inclined to the horizon, or if one end of the slip of paper be different from the other, the path will not be straight, but in the form of a helix. There will be no other essential difference between this case and that of the symmetrical arrangement.

Trinity College, April 5, 1853.

ON THE RELATION BETWEEN DIFFERENT CURVES AND CONES CONNECTED WITH A SERIES OF CONFOCAL ELLIPSOIDS.

By John Y. Rutledge, A.M., Trinity College, Dublin.

No. II.

In a preceding number of this *Journal* we established a natural connection between a very interesting and important system of curves traced upon the surface of an ellipsoid; or, more generally, traced upon any central surface of the second order. The present article seeks to complete that investigation, and by some additional ex-

curvature. Its asymptotic cone will also intersect the ellipsoid in a curve, which, for convenience sake, we purpose calling the 'asymptotic correlative' of the line of curvature. This asymptotic cone will intersect the surface of elasticity in an analogous curve: now any curve being given on the surface of elasticity, we can readily obtain its conjugate on the ellipsoid; this conjugate, in the present instance, we propose calling the 'correlative' of the line of curvature. We consequently see that, any line of curvature being given, we are given at the same time two correlative curves, which we may always discuss in connection with it. The equations of the respective curves are

$$\left.\begin{aligned} \frac{x^2}{a^2-\rho^2} + \frac{y^2}{b^2-\rho^2} + \frac{z^2}{c^2-\rho^2} &= 0, \\ \frac{x^2}{a^2(a^2-\rho^2)} + \frac{y^2}{b^2(b^2-\rho^2)} + \frac{z^2}{c^2(c^2-\rho^2)} &= 0, \\ \frac{x^2}{a^4(a^2-\rho^2)} + \frac{y^2}{b^4(b^2-\rho^2)} + \frac{z^2}{c^4(c^2-\rho^2)} &= 0, \end{aligned}\right\} \ldots\ldots(1).$$

The first being the equation of the 'asymptotic correlative' line; the second of the line of curvature; and the third of the line 'correlative' of the line of curvature.

Now let the complete differential equation of any surface referred to ordinary rectangular coordinates, be

$$Pdx + Qdy + Rdz = 0,$$

making $$X = \frac{P}{R} \text{ and } Y = \frac{Q}{R}.$$

Geometers have called X and Y the normal coordinates of any point upon the surface; and it has been remarked, that X and Y being given, the point upon the surface may be determined by the equations X, Y, and the equation of the surface $F(x, y, z) = 0$.

It is easy to see that the equations of the preceding curves, expressed in normal coordinates, are

$$\left.\begin{aligned} (\xi)\quad \frac{a^4X^2}{a^2-\rho^2} + \frac{b^4Y^2}{b^2-\rho^2} + \frac{c^4}{c^2-\rho^2} &= 0, \\ (\eta)\quad \frac{a^2X^2}{a^2-\rho^2} + \frac{b^2Y^2}{b^2-\rho^2} + \frac{c^2}{c^2-\rho^2} &= 0, \\ (\zeta)\quad \frac{X^2}{a^2-\rho^2} + \frac{Y^2}{b^2-\rho^2} + \frac{1}{c^2-\rho^2} &= 0, \end{aligned}\right\} \ldots\ldots\ldots(2).$$

These are now the equations of sphero-conics traced upon a sphere whose radius is unity, by tangent-planes parallel to the planes tangent to the ellipsoid along the respective curves under consideration; (ξ) is therefore the sphero-conic derived from the asymptotic correlative line, (η) the sphero-conic derived from the line of curvature, and (ζ) the sphero-conic derived from the correlative line of the line of curvature. From the mere fact of these curves being of the second order, it follows that a tangent plane at any point makes angles whose sum or difference is constant with two fixed planes, viz. the planes tangent to the sphere at their respective foci. It is well known that the equation of a sphero-conic, referred to its principal diametral arcs for arcs of reference, is

$$\frac{X^2}{\tan^2\alpha} + \frac{Y^2}{\tan^2\beta} = 1,$$

where α and β denote the distances from the origin of the points in which the curve meets the X and Y arcs of reference. The foci are of course determined by the equation

$$\tan^2\gamma = \frac{\tan^2\alpha - \tan^2\beta}{1 + \tan^2\beta},$$

and the cyclic arcs by the equations

$$\cos^2\phi = \frac{\tan^2\gamma}{\tan^2\alpha}, \quad \sin^2\phi = \frac{\sin^2\beta}{\sin^2\alpha},$$

The ratios, therefore, of the tangents of $(\alpha, \alpha', \&c.)$ are independent of the semi-diameter of the ellipsoid (ρ), and are identical for all the derived curves of the form (2) traced upon the surface of the sphere. Again, for (ξ), we have

$$\tan^2\gamma = \frac{c^4}{a^4} \cdot \frac{a^2 - b^2}{b^2 - c^2} \cdot \left\{\frac{a^2b^2 - \rho^2(a^2 + b^2)}{b^2c^2 - \rho^2(b^2 + c^2)}\right\},$$

for (η)

$$\tan^2\gamma' = \frac{c^2}{a^2} \cdot \frac{a^2 - b^2}{b^2 - c^2},$$

and for (ζ)

$$\tan^2\gamma'' = \frac{a^2 - b^2}{b^2 - c^2};$$

we consequently have

$$\frac{\tan\gamma'}{\tan\gamma''} = \frac{\tan\alpha'}{\tan\alpha''};$$

and since $\tan\gamma'$ and $\tan\gamma''$ are independent of (ρ), it follows "that all the curves of the sphere corresponding to the lines of curvature on the ellipsoid are biconfocal," the same theorem holding also for the spherical curves corresponding to the lines 'correlative' of the line of curvature. We also perceive that the two last-mentioned conics of the sphere are, for any given value of (ρ), biconcyclic. From the values found for $\tan\gamma$, $\tan\gamma'$, and $\tan\gamma''$ it follows, that the foci of the conic (ξ) correspond to the points upon the ellipsoid, at which the tangent planes are parallel to the circular sections (of the cone) of the line of constant curvature; *i.e.* the curve-locus of the points at which the measure of curvature of the ellipsoid has a constant value. The equation of this curve, as has been shewn in the paper to which we have already referred, is

$$\frac{a^2 - \rho^2}{a^4} \cdot x^2 + \frac{b^2 - \rho^2}{b^4} \cdot y^2 + \frac{c^2 - \rho^2}{c^4} \cdot z^2 = 0,$$

where (ρ) denotes the constant perpendicular from the centre upon the tangent plane to the ellipsoid at any point upon the curve. The tangent planes to the ellipsoid, therefore, along the asymptotic correlative line of the line of curvature, make angles whose sum or difference is constant with the circular sections of the corresponding line of constant curvature. The foci of the curve (η) evidently correspond to the umbilics of the ellipsoid; we have therefor
of the well-known theorem, "*The tangent-pla*
ellipsoid along a line of curvature make angles

or difference is constant with the planes of circular section." This pleasing proof was first given by Professor Graves. The foci of the conic (ζ) correspond to the points upon the ellipsoid at which the tangent-planes are normal to the asymptotes of the focal hyperbola, and consequently the tangent-planes to the ellipsoid along the line correlative of the line of curvature make angles whose sum or difference is constant with the planes normal to the asymptotes of the focal hyperbola. We also evidently have the interesting theorem, "*The curves upon the sphere which correspond to the lines correlative of the line of curvature, when considered upon an infinite series of confocal ellipsoids, are all biconfocal.*" From what has been stated it is easy to see that the tangent-planes to the ellipsoid along the asymptotic correlative of the line of curvature form by their ultimate intersections a developable surface, of which the 'arêtes' are parallel to the sides of a central cone (*i.e.* which has its vertex at the centre) intersecting the surface of the ellipsoid in the line of constant curvature. Also, as is well known, the tangent-planes to the ellipsoid, along the line of curvature, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central cone intersecting the surface of the ellipsoid in a sphero-conic. In like manner, the tangent-planes to the ellipsoid, along the line correlative of the line of curvature, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central cone that intersects the surface of the ellipsoid in a curve, which in our former paper we have named the curve of 'reference' of the sphero-conic. The last-mentioned cone we have also shewn to be the reciprocal of the asymptotic cone of the fixed hyperboloid. Hence follows the curious theorem— "*The tangent-plane to the ellipsoid at any point upon the line correlative to the line of curvature is normal to a side of the corresponding asymptotic cone;*" or, in other words, that the different normals to the ellipsoid along the line correlative of the line of curvature are parallel to the sides of the corresponding asymptotic cone. The equations of this new system of curves are

$$\left.\begin{aligned} \frac{a^2-\rho^2}{a^4}x^2+\frac{b^2-\rho^2}{b^4}y^2+\frac{c^2-\rho^2}{c^4}z^2=0 \\ \frac{a^2-\rho^2}{a^2}x^2+\frac{b^2-\rho^2}{b^2}y^2+\frac{c^2-\rho^2}{c^2}z^2=0 \\ (a^2-\rho^2)\,x^2+(b^2-\rho^2)\,y^2+(c^2-\rho^2)\,z^2=0 \end{aligned}\right\} \quad \text{........(3).}$$

The first being the equation of the line of constant curvature, the second of the sphero-conic, and the third of the sphero-conic's curve of reference. The equations of the corresponding curves upon the sphere whose radius is unity are

$$\left.\begin{array}{l}(\xi_1)\ (a^2-\rho^2)\,X^2+(b^2-\rho^2)\,Y^2+(c^2-\rho^2)=0\\ (\eta_1)\ a^2(a^2-\rho^2)\,X^2+b^2(b^2-\rho^2)\,Y^2+c^2(c^2-\rho^2)=0\\ (\zeta_1)\ a^4(a^2-\rho^2)\,X^2+b^4(b^2-\rho^2)\,Y^2+c^4(c^2-\rho^2)=0\end{array}\right\}\ \ldots (4).$$

The first is the equation of the conic corresponding to the line of constant curvature, the second of the conic corresponding to the sphero-conic, and the third of the conic corresponding to the sphero-conic's curve of reference.

For the equation (ξ_1) we have

$$\tan^2\alpha_1=\frac{\rho^2-c^2}{a^2-\rho^2},\qquad \tan^2\beta_1=\frac{\rho^2-c^2}{b^2-\rho^2};$$

for (η_1) we have

$$\tan^2\alpha_1'=\frac{c^2}{a^2}\cdot\frac{\rho^2-c^2}{a^2-\rho^2},\qquad \tan^2\beta_1'=\frac{c^2}{b^2}\cdot\frac{\rho^2-c^2}{b^2-\rho^2};$$

and for (ζ_1) we have

$$\tan^2\alpha_1''=\frac{c^4}{a^4}\cdot\frac{\rho^2-c^2}{a^2-\rho^2},\qquad \tan^2\beta_1''=\frac{c^4}{b^4}\cdot\frac{\rho^2-c^2}{b^2-\rho^2}.$$

From which we obtain the following relations,

$$\tan\alpha_1\,\tan\alpha_1''=\tan\alpha_1'^2,\quad \tan\beta_1\,\tan\beta_1''=\tan\beta_1'^2.$$

So that the tangents of the arcs (α_1, α_1', &c.) are in continued proportion. Also

$$\tan\alpha\,.\,\tan\alpha_1=\tan\alpha'\,.\,\tan\alpha_1'=\tan\alpha''\,.\,\tan\alpha_1''=\frac{c^2}{a^2},$$

$$\tan\beta\,.\,\tan\beta_1=\tan\beta'\,.\,\tan\beta_1'=\tan\beta''\,.\,\tan\beta_1''=\frac{c^2}{b^2},$$

$$\tan^2\alpha_1\,\tan^2\alpha''=\tan^2\beta_1\,\tan^2\beta''=1.$$

We in like manner perceive that the ratios

$$\frac{\tan\alpha_1}{\tan\alpha_1''}=\frac{a^2}{c^2},\quad \frac{\tan\alpha_1'}{\tan\alpha_1''}=\frac{a}{c},\quad \frac{\tan\beta_1}{\tan\beta_1''}=\frac{b^2}{c^2},\quad \frac{\tan\beta_1'}{\tan\beta_1''}=\frac{b}{c},$$

are the reciprocals of the analogous ratios which we already found for the equations (2). The foci of the

(ξ_1, η_1, ζ_1) are determined by the equations

$$\tan^2\gamma_1 = \frac{c^2-\rho^2}{a^2-\rho^2} \cdot \frac{a^2-b^2}{b^2-c^2}, \quad \tan^2\gamma_1' = \frac{c^2}{a^2} \cdot \frac{c^2-\rho^2}{a^2-\rho^2} \cdot \frac{a^2-b^2}{b^2-c^2} \cdot \frac{a^2+b^2-\rho^2}{b^2+c^2-\rho^2},$$

$$\tan^2\gamma_1'' = \frac{c^4}{a^4} \cdot \frac{c^2-\rho^2}{a^2-\rho^2} \cdot \frac{a^2-b^2}{b^2-c^2} \cdot \frac{a^4+a^2b^2+b^4-\rho^2(a^2+b^2)}{b^4+b^2c^2+c^4-\rho^2(b^2+c^2)};$$

we also have $$\cos^2\phi_1 = -\tan^2\gamma''.$$

The preceding relations between the system of six conics upon the auxiliary sphere of radius unity are sufficiently interesting to repay attention. Since the preceding values for $\tan\gamma_1$, $\tan\gamma_1'$, &c. all include (ρ), we perceive that no one of the conics represented by equations (4) is biconfocal for the system of conics derived from the corresponding system of curves upon the surface of the ellipsoid obtained by the variation of (ρ). We at the same time, however, perceive that the sphero-conics (ζ, ξ_1) are identical for their corresponding curves derived upon an infinite series of confocal ellipsoids from one and the same fixed hyperboloid. The conics (ζ) and (ξ_1) are in fact the reciprocals one of the other; *i.e.* the equation of one being

$$\frac{X^2}{\tan^2\alpha} + \frac{Y^2}{\tan^2\beta} = 1,$$

conic, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central cone intersecting the ellipsoid in the line correlative of the line of curvature. If we look for the focal lines of the last-mentioned cone, we can easily see that the tangent planes to the ellipsoid at the points in which the focal lines pierce the surface, make with the axis of (x) angles, the tangents of which bear a constant ratio to the tangents of the angles $+\gamma_1''$ or $-\gamma_1''$ for the infinite series of confocal ellipsoids; the curve on each ellipsoid being of course considered as derived from one and the same fixed hyperboloid. An analogous theorem holds for the tangents of the angles made with the axis of (x) by the planes tangent to the ellipsoid at the points in which the focal lines of the cone of constant curvature pierce its surface and the tangents of the angles $+\gamma$ or $-\gamma$. Let the tangent planes in the first instance make with the axis of (x) the angle $(\pm\theta')$, in the second instance the angle $(\pm\theta)$; we shall then have

$$\frac{\tan^2\theta}{\tan^2\gamma} = \frac{\tan^2\gamma_1''}{\tan^2\theta'} = \frac{c^2-\rho^2}{a^2-\rho^2}.$$

Let $(x'y'z')$ denote any point upon the sphero-conic, traced upon the surface of the ellipsoid by the sphere-cone of radius (ρ); then if we write

$$\frac{x'}{a} = \cos i, \quad \frac{y'}{b} = \cos i_{,}, \quad \frac{z'}{c} = \cos i_{,,},$$

we shall have determined a corresponding side of the cone of reference, such that if a plane be drawn normal to their side and tangent to the ellipsoid, the perpendicular intercept measured from the centre will equal (ρ), and the point of contact of the tangent plane with the ellipsoid will of course be a point upon the line of constant curvature. Any side of any one of the three preceding cones being given, we have indicated in our former paper a geometrical construction, by means of the ellipsoid of reference, to determine the two corresponding sides upon the two remaining cones; the ellipsoid of reference being such that the original given ellipsoid is with respect to it the reciprocal polar of the auxiliary sphere of radius unity. Similarly, let (x_1, y_1, z_1) be the coordinates of a point upon the line of curv
if we then refer the radius vector drawn to this
its line of reference, we shall obtain a side of
asymptotic cone; and a tangent plane drawn norm
side will touch the ellipsoid in a point upon

correlative to the line of curvature. The same geometrical construction to which we have alluded, by means of the ellipsoid of reference, will enable us, any side of any one of the three preceding cones being given, to determine the two corresponding sides upon the two remaining cones.

So far we perceive that, resulting from one and the same fixed hyperboloid, *there are upon the surface of the ellipsoid six curves and six cones which have a natural and intimate connection;* while at the same time there are six derived sphero-conics upon the surface of the auxiliary sphere of radius unity. We next perceive that, since when viewed through the medium of corresponding points, the asymptotic cone is fixed with reference to an infinite series of confocal ellipsoids, as also its reciprocal cone, the cone of reference of the equiradial cone of radius (ρ), *there are two systems of parallel developable surfaces which envelope the entire series of confocal ellipsoids;* viz. *the developable surface along the line of constant curvature, and the developable along the line correlative of the line of curvature.* In fact, when the ellipsoid degenerates into a point, the first-named developable surface degenerates into the asymptotic cone, and the second into the asymptotic cone's reciprocal or supplementary cone. Let the ellipsoid be one of revolution round its mean axis suppose, and we shall have for the developable surface, circumscribed to the ellipsoid along the line of con-

The equation of this locus surface will be of the form

$$\frac{a^2 - \rho^2}{a^4} x^2 + \frac{b^2 - \rho^2}{b^4} y^2 + \frac{c^2 - \rho^2}{c^4} z^2 = 1,$$

the asymptotic cone of which we perceive to be the central cone which intersects the ellipsoid in the line of constant curvature. If we now consider the two semi-diameters of the ellipsoid conjugate with (ρ), we shall have two more fixed hyperboloids, the consideration of which will give us the previously enumerated curves and cones in sets of three, as well as the derived sphero-conics upon the auxiliary sphere of radius unity. In our former paper we have stated the curious relations which connect any two of these three fixed hyperboloids with the third; it is needless therefore to repeat them here.

Professor Chasles, in the notes to his admirable history of Geometry, has demonstrated the very beautiful theorem, "*Given in magnitude and position any three conjugate semi-diameters of an ellipsoid, construct the surface.*" This theorem in a remarkable manner completes our theory; for we see that if any three conjugate semi-diameters of an ellipsoid be given in magnitude and position, we can construct the surface; while at the same time we have three hyperboloids fixed with reference to an infinite series of confocal ellipsoids, the consideration of which gives on each ellipsoid of the series the remarkable and interesting system of curves that we have enumerated in the present paper. To avoid needless repetition, we have hitherto restricted ourselves to the consideration of the ellipsoid; the reader can, however, without difficulty extend the preceding theorems to all central surfaces of the second order. The modifications which they undergo in the case of the paraboloids may be also had, by remembering that instead of the semi-diameters (ρ), we must consider the parallel bifocal chords. Let us now consider the line of curvature

$$\frac{x^2}{a^2(a^2 - \rho^2)} + \frac{y^2}{b^2(b^2 - \rho^2)} + \frac{z^2}{c^2(c^2 - \rho^2)} = 0$$

upon the surface of the fixed hyperboloid. The equation of its derived curve upon the auxiliary sphere of radius unity then is

$$\frac{a^2 - \rho^2}{a^2} X^2 + \frac{b^2 - \rho^2}{b^2} Y^2 + \frac{c^2 - \rho^2}{c^2} = 0;$$

we consequently have

$$\tan^2\alpha = \frac{a^2}{c^2}\cdot\frac{(\rho^2 - c^2)}{a^2 - \rho^2}, \qquad \tan^2\beta = \frac{b^2}{c^2}\cdot\frac{\rho^2 - c^2}{b^2 - \rho^2},$$

$$\tan^2\gamma = \frac{c^2 - \rho^2}{a^2 - \rho^2}\cdot\frac{a^2 - b^2}{b^2 - c^2}, \qquad \cos^2\phi = -\frac{c^2}{a^2}\cdot\frac{a^2 - b^2}{b^2 - c^2};$$

from which we perceive that this curve is the reciprocal of the conic (η), and that it is biconfocal with the conic (ξ_1). From the value of $\cos\phi$ it is manifest that the conics derived from the lines of curvature upon the fixed hyperboloids, formed by the intersection of any one ellipsoid, are all bicon-cyclic; we have already seen that the conics derived from the lines of curvature, when considered upon the ellipsoid, are all biconfocal. The tangent planes to the fixed hyper-boloid, along the line of curvature, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central equiradial cone, the equation of which is

$$\frac{a^2x^2}{a^2 - \rho^2} + \frac{b^2y^2}{b^2 - \rho^2} + \frac{c^2z^2}{c^2 - \rho^2} = 0,$$

the reciprocal of the equiradial cone already found for the ellipsoid. The equation of its derived conic is

$$a^2(a^2 - \rho^2)\,X^2 + b^2(b^2 - \rho^2)\,Y^2 + c^2(c^2 - \rho^2) = 0,$$

which is identical with the equation (η_1), from which we perceive that the tangent planes to the ellipsoid and the fixed hyperboloid along their respective related sphero-conics are parallel, and make angles whose sum or difference is constant with the same pair of fixed planes.

Let two confocal hyperboloids, the equations of which are

$$(\phi)\ \frac{x^2}{\mu^2} + \frac{y^2}{\mu'^2} + \frac{z^2}{\mu''^2} = 1, \quad (\omega)\ \frac{x^2}{\nu^2} + \frac{y^2}{\nu'^2} + \frac{z^2}{\nu''^2} = 1,$$

intersect the surface of the ellipsoid in the point $(x'y'z')$, and in each surface let the right line of reference corresponding to the radius vector (ρ') be found; then, if we draw perpen-dicular to each a plane tangent to the surface to which it belongs, the points of contact will be corresponding points, *i.e.*

$$\frac{x_{,}}{a} = \frac{x_{,,}}{\mu} = \frac{x_{,,,}}{\nu},$$

where $(x_{,}, y_{,}, \&c., x_{,,}, y_{,,}, \&c.)$ represent the points of contact on the respective surfaces. Let us next consider the cor-

responding points to (x', y', z') on the series of confocal ellipsoids. It is well known that the curve of intersection of ϕ and ω will be the locus of these corresponding points; consequently, if we repeat the preceding construction for each corresponding point on the several ellipsoids of the series, the points of contact of the several tangent planes to the surfaces ϕ and ω, will give on each a locus of corresponding points. The respective equations of these loci on the surfaces ϕ and ω, are

$$\frac{x^2}{\mu^4\nu^2} + \frac{y^2}{\mu'^4\nu'^2} + \frac{z^2}{\mu''^4\nu''^2} = 0, \quad \frac{x^2}{\nu^4\mu^2} + \frac{y^2}{\nu'^4\mu'^2} + \frac{z^2}{\nu''^4\mu''^2} = 0.$$

It is easy to see that these equations represent on each surface the lines correlative to the lines of curvature indicated by the equation

$$\frac{x^2}{\mu^2\nu^2} + \frac{y^2}{\mu'^2\nu'^2} + \frac{z^2}{\mu''^2\nu''^2} = 0.$$

This results from the fact that the asymptotic cone of one of two intersecting confocal surfaces of the second order is the cone of reference of the central cone, which passes through the line of curvature on the other formed by their intersection. In the ellipsoid, since the right line of reference of (ρ') lies at once on the asymptotic cones of ϕ and ω, it is evident that this right line must be one of the four sides of intersection of the two asymptotic cones; if then, perpendicular to either of these sides, we draw a plane tangent to the ellipsoid at the point $(x_, y_, z_,)$, and if A denote the double of the triangle formed by the perpendicular intercept from the centre, the radius vector $(\rho_,)$ and its projection on the tangent plane, we shall have

$$A^2 = \mu^2\nu^2 - (\mu^2 + \nu'^2)(\mu^2 + \nu''^2).$$

So that this area is constant for the infinite series of confocal surfaces. This expression for an area, which in the theory of rotation has an important physical signification, will not be found in the solution of many questions devoid of utility.

Let $(r, r_,, r_{,,})$ be the angles formed by a side of intersection of the asymptotic cones with the axes of (x, y, z), and we shall have

$$\cos^2 r = \frac{\mu^2\nu^2}{(a^2 - b^2)\,(a^2 - c^2)}, \qquad \cos^2 r_, = \frac{\mu'^2\nu'^2}{(b^2 - a^2)\,(b^2 - c^2)},$$

Since however, if with the point $(x'y'z')$ as centre we construct the three confocal surfaces intersecting in the centre of the original ellipsoid, the equations of which referred to the three normals at the point $(x'y'z')$ as axes of (ξ, η, ζ), are

$$(\varpi)\ \frac{\xi^2}{a^2}+\frac{\eta^2}{\mu^2}+\frac{\zeta^2}{\nu^2}=1,\quad (\varpi')\ \frac{\xi^2}{b^2}+\frac{\eta^2}{\mu'^2}+\frac{\zeta^2}{\nu'^2}=1,$$

$$(\varpi'')\ \frac{\xi^2}{c^2}+\frac{\eta^2}{\mu''^2}+\frac{\zeta^2}{\nu''^2}=1,$$

then the four sides of intersection of the two asymptotic cones of ϕ and ω are the four bifocal chords, which can be drawn from the centre of the original ellipsoid through the focal curves of the newly constructed system, we shall have, as I have elsewhere demonstrated,

$$\cos^2 r=\frac{x'^2}{a^2},\quad \cos^2 r_{\prime}=\frac{y'^2}{b^2},\quad \cos^2 r_{\prime\prime}=\frac{z'^2}{c^2};$$

we therefore have the known equations for the coordinates of the point of intersection of three confocal surfaces

$$x'^2=\frac{a^2\mu^2\nu^2}{(a^2-b^2)(a^2-c^2)},\quad y'^2=\frac{b^2\mu'^2\nu'^2}{(b^2-a^2)(b^2-c^2)},\quad z'^2=\frac{c^2\mu''^2\nu''^2}{(a^2-c^2)(b^2-c^2)}.$$

Now, if we suppose the point $(x'y'z')$ fixed in space and the surface ϖ given, it is manifest that this latter surface

plane, and (a) the semi-major axis of the surface, we have

$$\cos r = \frac{P}{a},$$

where (r) denotes the angle which the bifocal chord drawn from the point of contact of the tangent plane makes with the normal to the surface at that point; it follows, that along the line of constant curvature, *the bifocal chord makes a constant angle with the normal to the surface.*

DEMONSTRATION OF A THEOREM OF JACOBI, RELATIVE TO FUNCTIONAL DETERMINANTS.

By Professor W. F. Donkin.

The following proposition is of fundamental importance in Jacobi's general theory of multipliers of a system of differential equations.

Let $u_1, u_2, \ldots u_n$ be n functions of the n variables $x_1, x_2, \ldots x_n$; and let Δ be the determinant (of n^2 terms) formed with the differential coefficients

$$\frac{du_1}{dx_1},\ \frac{du_1}{dx_2}, \ldots \frac{du_1}{dx_n};\quad \frac{du_2}{dx_1}, \ldots \frac{du_2}{dx_n};\ \&\text{c.}$$

If we put

$$\Delta = \frac{du_1}{dx_1}\nabla_1 + \frac{du_1}{dx_2}\nabla_2 + \ldots + \frac{du_1}{dx_n}\nabla_n,$$

$\nabla_1, \nabla_2, \ldots \nabla_n$ will be the n minor determinants obtained by leaving out the first horizontal row in Δ, and then, successively, each vertical row in the remaining terms. The proposition in question is expressed by the equation

$$\frac{d\nabla_1}{dx_1} + \frac{d\nabla_2}{dx_2} + \ldots + \frac{d\nabla_n}{dx_n} = 0 \ldots\ldots\ldots\ldots\ldots (\text{I.})$$

(It will be observed that u_1 does not appear in this equation at all; it has been introduced merely for the sake of s[illegible] - metry and clearness.)

This theorem may be very simply demonstrat[illegible] help of the following property of determinants.

If the first term in each vertical (or in eac[illegible] row be equal to the sum of the remaining terms

the determinant vanishes.* For it is plain that the determinant (of n^2 terms)

$$\begin{vmatrix} a_2 + a_3 + \dots + a_n, & b_2 + b_3 + \dots + b_n, \dots\dots & k_2 + k_3 + \dots + k_n \\ a_2 \quad , & b_2 \quad ,\dots\dots & k_2 \\ a_3 \quad , & b_3 \quad ,\dots\dots & k_3 \\ \vdots \quad \vdots & \vdots \quad \vdots\dots\dots & \vdots \\ a_n \quad , & b_n \quad ,\dots\dots & k_n \end{vmatrix}$$

is the product of two determinants, namely,

$$\begin{vmatrix} 0, & 1, & 1, \dots 1 \\ 0, & 1, & 0, \dots 0 \\ 0, & 0, & 1, \dots 0 \\ \vdots & \vdots & \vdots \dots \vdots \\ 0, & 0, & 0, \dots 1 \end{vmatrix} \begin{vmatrix} a_1, & b_1, \dots k_1, \\ a_2, & b_2, \dots k_2, \\ a_3, & b_3, \dots k_3, \\ \vdots & \vdots \dots \vdots \\ a_n, & b_n, \dots k_n \end{vmatrix}$$

of which the first vanishes, since its first vertical row is composed wholly of zeros.

Returning now to the determinant Δ, it is evident that it may be represented in the form

$$D.u_1 u_2 \dots u_n,$$

where D is an operation performed upon the product $u_1 u_2 \dots u_n$, and defined by the equation

by the determinant thus modified E, then it is also plain that

$$E.u_2 u_3 \dots u_n,$$

will represent precisely the expression on the left of the equation (I.)

Now it follows from the form of the subject of operation $(u_2 u_3 \dots u_n)$ that we may substitute for the unrestricted symbol $\frac{d}{dx_i}$, the expression

$$\frac{d_2}{dx_i} + \frac{d_3}{dx_i} + \dots + \frac{d_n}{dx_i}.$$

But by this substitution the determinant E is transformed into one in which the first term in each vertical row is the sum of the remaining terms in that row. Consequently E vanishes identically, and the equation (I) is established.

Oxford, Feb. 27, 1854.

ON A THEOREM OF M. LEJEUNE DIRICHLET'S.

By Arthur Cayley.

The following formula,

$$\Sigma q^{ax^2+2bxy+cy^2} + \Sigma q^{a'x^2+2b'xy+c'y^2} = 2\Sigma \delta^{\frac{n-1}{2}} \varepsilon^{\frac{n^2-1}{8}} \left(\frac{n}{P}\right) q^{nn'} \dots (3),$$

is given in Lejeune Dirichlet's well-known memoir "Recherches sur diverses applications, &c. (*Crelle*, tom. XXI. p. 8). The notation is as follows:—On the left-hand side (a, b, c), (a', b', c') ... are a system of properly primitive forms to the *negative* determinant D (*i.e.* a system of positive forms); x, y are positive or negative integers including zero, such that in the sum $\Sigma q^{ax^2+2bxy+cy^2}$, $ax^2 + 2bxy + cy^2$ is prime to $2D$, and similarly in the other sums; q is indeterminate and the summations extend to the values first mentioned, of x and y. On the right-hand side we have to consider the form of D, viz. we have $D = PS^2$ or else $D = 2PS^2$, where S^2 is the greatest square factor in D and where P is odd: this obviously defines P, and the values of δ, ε, which are always ± 1 (or, as I prefer to express it, are always $\pm$) are given as follows, viz.

$$
\begin{array}{lll}
D = PS^2, & P \equiv 1 \pmod 4, & \delta, \varepsilon = +\,+ \\
D = PS^2, & P \equiv 3 \pmod 4, & \delta, \varepsilon = -\,+ \\
D = 2PS^2, & P \equiv 1 \pmod 4, & \delta, \varepsilon = +\,- \\
D = 2PS^2, & P \equiv 3 \pmod 4, & \delta, \varepsilon = -\,-
\end{array}
$$

n, n' are any positive numbers prime to $2D$, $\left(\dfrac{n}{P}\right)$ is Legendre's symbol as generalized by Jacobi, viz. in general if p be a positive or negative prime not a factor of n, then $\left(\dfrac{n}{p}\right) = +$ or $-$ according as n is or is not a quadratic residue of p (or, what is the same thing, p being positive, $\left(\dfrac{n}{\pm p}\right) \equiv n^{\frac{p-1}{2}} \pmod p$), and for $P = pp'p''\ldots$

$$\left(\frac{n}{P}\right) = \left(\frac{n}{p}\right)\left(\frac{n}{p'}\right)\left(\frac{n}{p''}\right),$$

and the summation extends to all the values of n, n' of the form above mentioned. In the particular case $D = -1$, it is necessary that the second side should be doubled. The method of reducing the equation is indicated in the memoir. The following are a few particular cases.

$D = -1,\qquad \Sigma q^{x^2+y^2} = 4\Sigma(-)^{\frac{n-1}{2}}\, q^{nn'},$

or $\qquad (1 + 2q^4 + 2q^{16} + 2q^{36} + \ldots)(q + q^9 + q^{25} + q^{49} + \ldots)$

$$= \frac{q}{1-q^2} - \frac{q^3}{1-q^6} + \frac{q^5}{1-q^{10}} - \frac{q^7}{1-q^{14}} + \ldots$$

$D = -2,\qquad \Sigma q^{x^2+2y^2} = 2\Sigma(-)^{\frac{n-1}{2} + \frac{n^2-1}{8}}\, q^{nn'},$

or $\qquad (1 + 2q^2 + 2q^8 + 2q^{18}\ldots)(q + q^9 + q^{25} + q^{49} + \ldots)$

$$= \frac{q}{1-q^2} + \frac{q^3}{1-q^6} - \frac{q^5}{1-q^{10}} - \frac{q^7}{1-q^{14}} + \&c.$$

an example given in the memoir.

$D = -3,\qquad \Sigma q^{x^2+3y^2} = 2\Sigma(\tfrac{1}{3}n)\, q^{nn'},$

or $\qquad (q^1 + q^{25} + q^{49} + q^{121} + q^{169}\ldots)(1 + 2q^{12} + 2q^{48} + 2q^{108}\ldots)$

$$+ 2(q^3 + q^{27} + q^{75} + q^{147} + \ldots)(q^4 + q^{16} + q^{64} + q^{100}\ldots)$$

$$= \frac{q+q^5}{1-q^6} - \frac{q^5+q^{25}}{1-q^{30}} + \frac{q^7+q^{35}}{1-q^{42}} - \frac{q^{11}+q^{55}}{1-q^{66}} + \ldots$$

I am not aware that the above theorem is quoted or referred to in any subsequent memoir on Elliptic Functions, or on the class of series to which it relates, and the theorem is so distinct in its origin and form from all other theorems relating to the same class of series, and, independently of the researches in which it originates, so remarkable as a result, that I have thought it desirable to give a detached statement of it in this paper.

2, *Stone Buildings, Lincoln's Inn,*
March 8, 1854.

MATHEMATICAL NOTES.

To the Editor of the Mathematical Journal.

In a note appended to a paper which appears in the recently published number of the *Journal* (No. xxxiv. p. 76) the late Mr. Weddle has kindly pointed out slight errors in the form in which two geometrical theorems are given by me in a paper published in the seventh volume of the *Journal.* The modifications in their enunciations necessary to rectify these errors, which Mr. Weddle has suggested, are at once seen to be proper; and I can only attribute to the haste with which these two special applications of a general theorem were written, my oversight of the correct method of deducing them. My object in addressing this note to you is that of briefly showing how these two theorems, as corrected by Mr. Weddle, are to be deduced agreeably to the method employed in my paper alluded to.

I conceive a surface S of the second order to be cut in two small conics by the planes L, L', and a cone to be described passing through the two conics. By the general theorem on which the method of the paper is founded, the rectangle of segments of a secant of the cone parallel to a circular section, drawn from any point in S, will bear a constant ratio to the rectangle of perpendiculars let fall from the same point on the planes L, L'. Then, supposing the planes L, L' moving parallel to themselves to become tangent planes, and the cone to degenerate into one of infinitesimal aperture, the first of the two corrected theorems will be true; the fixed pla to which the line drawn to meet the chord of cont parallel, being one parallel to the ultimate direction circular section of the cone. Again, if a second surf likewise of the second order, be conceived to tou

planes L, L' in the same two points as S, the rectangle of the perpendiculars let fall from any point in S upon the planes will bear a constant ratio to the segments of any chord (or secant) of S' drawn from the same point parallel to a fixed line; and by compounding ratios it will follow that the rectangle of the segments of a chord (or secant) of S,' drawn from any point in S parallel to a fixed line, will bear a constant ratio to the square of a line drawn from the same point to the chord of contact parallel to a certain fixed plane.

Writing on a geometrical subject, I am induced to send you the following remarks on two geometrical interpretations of which the equation to the circle is capable in its general form. They occurred to me some time back, and perhaps you might think them not out of place among your Mathematical Notes, as the first of the two ought, I conceive, to find a place in all elementary treatises on analytic geometry.

The particular values of the coefficients of the terms x^2, xy, y^2 in the general equation of the second order when it represents a circle express two well-known *metrical* properties of that curve, the opposite sides of an inscribed tetragon being chosen as axes of coordinates. To show the former of these it is necessary to premise the following:

LEMMA. If the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (a)$$

But from comparing the product of their numerators with the expression

$$(Ax^2 + Bxy + Cy^2 + Dx + Ey + F)\sin^2\phi,$$

it appears that

$$\alpha\alpha' = A, \quad \beta\beta' = C; \quad \alpha^2\beta'^2 + \alpha'^2\beta^2 = B^2 - 2AC;$$

$$\alpha\beta(\alpha'^2 + \beta'^2) + \alpha'\beta'(\alpha^2 + \beta^2) = (\alpha\alpha' + \beta\beta')(\alpha\beta' + \alpha'\beta) = (A + C)B,$$

whence the product of the two fractions may be easily identified with the formula (*b*).

If in the equation of the two right lines we suppose $A = C = 1$, (*i.e.* that the two right lines are subcontrary one to the other with respect to the axes,) the expression for the rectangle of perpendiculars from $x'y'$ becomes

$$\frac{(x'^2 + Bx'y' + y'^2 + Dx' + Ey' + F)\sin^2\phi}{B - 2\cos\phi} \;\ldots\ldots (c).$$

Suppose now the axes of coordinates to be the opposite sides of a tetragon inscribed in a circle, and B to be a certain constant quantity; the equation of the circle may easily be thrown into the form

$$\frac{(x^2 + Bxy + y^2 + Dx + Ey + F)\sin^2\phi}{B - 2\cos\phi} = xy\sin^2\phi \ldots (d).$$

Let B now be assumed equal to either of the expressions

$$\frac{DE \pm \{(D^2 - 4F)(E^2 - 4F)\}^{\frac{1}{2}}}{2F},$$

and the equation (*d*) will at once be seen to express the equality of the rectangles of the three pairs of perpendiculars let fall from any point in a circle on the three pairs of opposite sides of an inscribed tetragon.

In the same case, supposing $x'y'$ to be the point of intersection of a pair of opposite sides of the tetragon, it is easily shown that $Dx' + Ey' = -2F$. Let l be the length of the line drawn from the origin to $x'y'$, and t, t' the lengths of tangents drawn from the same points; then

$$\pm t'^2 = x'^2 + 2\cos\phi x'y' + y'^2 + Dx' + Ey' + F$$
$$= x'^2 + 2\cos\phi x'y' + y'^2 - F = l^2 - t^2,$$

or

$$l^2 = t^2 \pm t'^2,$$

the upper or lower sign being used according as $x'y'$ is without or within the circle (the origin being without), in which

latter case t' must be considered as half the chord drawn through $x'y'$ so as to be bisected at that point. This is another well-known metrical property of the circle.

By pursuing a very similar method of investigation, this theorem may be slightly generalized thus: Suppose a circle and any central conic to circumscribe the same tetragon; the sum of the squares of tangents drawn to the circle from the point of intersection of a pair of opposite sides of the tetragon, and from the centre of the conic is equal to the difference between the square of the line joining those points and the sum of squares of the semi-diameters of the conic which are parallel to that pair of opposite sides of the tetragon.

JOHN WALKER.

Dollymount, near Dublin,
Feb. 27, 1854.

SOME FORMULÆ IN FINITE DIFFERENCES.

Since $\log(1+\Delta) = \frac{d}{dx}$; therefore,

$$\psi\{x+\log(1+\Delta)\}\, y^n = \psi\left(x+\frac{d}{dy}\right) y^n,$$

and clearly, when $y = 0$, the only term in the development of $\psi\left(x+\frac{d}{dy}\right) y^n$ which does not vanish, is

$$\frac{1}{1.2\ldots n}\,\psi^{(n)}(x)\,\frac{d^n}{dy^n}\,(y^n) \text{ or } \psi^{(n)}(x);$$

therefore $$\frac{d^n}{dx^n}\,\psi(x) = \psi\{x+\log(1+\Delta)\}\,0^n.$$

Putting $x = 0$ and using Maclaurin's series, we obtain Herschel's theorem. Again, let

$$\psi(x) = B_0 + B_1 x + \ldots + B_r x^r + \ldots;$$

therefore $\psi\left(\frac{d}{dx}\right)\phi(x) = B_0\phi(x) + \ldots + B_r\phi\{x+\log(1+\Delta)\}\,0^r + \ldots;$

therefore $\psi\left(\frac{d}{dx}\right)\phi(x) = \phi\{x+\log(1+\Delta)\}\,\psi(0).$

Whence it easily follows that

$$F(\Delta)\,\psi(0) = \psi\left(\frac{d}{dx}\right) F(\varepsilon^x - 1) \text{ when } x = 0.$$

Let $$F(\varepsilon^x - 1) = C_0 + C_1 x + \ldots + C_p x^p + \ldots;$$

therefore $F(\Delta)\,\psi(0) =$ value of

$$\left\{\psi(0) + \ldots + \psi^{(p)}(0)\,\frac{d^p}{dx^p} + \ldots\right\}(C_0 + \ldots + C_p x^p \ldots) \text{ when } x = 0,$$

$$= \psi(0)\,C_0 + \psi'(0)\,C_1 + \ldots + \psi^{(p)}\,C_p + \ldots.$$

The following results are consequences of this formula.

$$S_n = \psi(0) + \psi(1) + \ldots + \psi(n-1) = \frac{(1+\Delta)^n - 1}{\Delta}\,\psi(0)$$

$$= \psi\left(\frac{d}{dx}\right)\left(\frac{\varepsilon^{nx} - 1}{\varepsilon^x - 1}\right) \text{ when } n = 0,$$

$$= n\psi(0) + \psi'(0)\,S_1 + \ldots + \psi^{(p)}(0)\,\frac{S_p}{\lfloor p} + \ldots,$$

where $$S_p = 1^p + \ldots + (n-1)^p.$$

Also $$\frac{\log(1+\Delta)}{\Delta}\,\psi(0) = \psi(0) - \frac{\psi'(0)}{2} + \ldots + (-1)^{n+1} B_{2n-1}\,\frac{\psi^{2n}(0)}{\lfloor 2n} + \ldots,$$

B_{2n-1} being one of Bernoulli's numbers.

ARTHUR COHEN.

THE following proof of Legendre's theorem (here very briefly given) may be worth attention, if it be not already known. The notation is as usual; accented letters being the angles of the plane triangle. By common substitution for $\sin A$, $\cos A$, &c.

$$\sin(A-B) = \frac{2\sqrt{(\sin s . \sin s - a . \sin s - b . \sin s - c)}}{\sin a \,\sin b\, \sin^2 c}(\cos b - \cos a)(1 + \cos c),$$

$$\sin(A'-B') = \frac{2\sqrt{(s . s - a . s - b . s - c)}}{abc^2} \cdot \frac{a^2 - b^2}{2}(1+1).$$

Hence, using $\dfrac{\sin x}{x} = 1 - \dfrac{x^2}{6} + \ldots$ &c., and expanding

$$\sin(A-B) : \sin(A'-B'),$$

we find for this ratio the product of nine series of the form 1 + term of second order + term of fourth order + &c., the nine terms of the second order being

$$-\frac{s^2}{12}, -\frac{(s-a)^2}{12}, -\frac{(s-b)^2}{12}, -\frac{(s-c)^2}{12}, +\frac{a^2}{6}, +\frac{b^2}{6}, +\frac{c^2}{3}, -\frac{a^2+b^2}{12}, -\frac{c^2}{4},$$

the sum of which remains. Hence, as far as excess of the whole over the ratio of the areas, and that of the angles, is a ratio of equality, or $A - A' = B - B' =$ (for similar reasoning) $C - C'$; whence Legendre's theorem immediately follows.

A. DE MORGAN.

Lagrange, in a paper on spherical triangles (Jour. Éc. Polytechn., vol. ii., translated in Leybourn's Repository, vol. i.), notes that the dependence of the area on the spherical excess, first announced by Albert Girard in 1629, was not demonstrated by him, but was first demonstrated by Cavalieri, in his Directorium, &c., published in 1632. Lagrange therefore considers that the theorem belongs rather to Cavalieri than to Albert Girard. This transfer, however, cannot be made. It is a matter of fact that Girard discovered the theorem, and Cavalieri the demonstration; it is also a matter of fact, that for one who can discover, there are hundreds who can demonstrate, when they know what is to be demonstrated. At the same time, looking at dates and countries, it is most probable that Cavalieri was an independent discoverer.

As the *Invention Nouvelle*, &c. of Albert Girard is the rarest of all the works in which mathematical discovery has been announced, a short statement of what is there found may be acceptable. Girard considers the surface of the sphere as composed of 720 equal *superficial degrees*; and then announces and exemplifies the theorem, or science, "incogneue jusques à present, si ce n'est devant le deluge," that every spherical polygon has as many superficial degrees as the sum of all the angles contains degrees more than the sum of all the angles of a corresponding rectilinear figure. This he first proves, easily of course, of a semilune, or *fibulle*, as he calls it. He then proceeds to the case of a right-angled triangle having each side less than a quadrant; and this case he does not attempt to prove except *en conclusion probable*. The triangle being ABC, right-angled at C, he shows that if a small circle be drawn having A for its pole, and cutting the circles AB, AC in M and N, so that the area AMN may be equal to the semilune which has the same spherical excess as ABC, then M falls within AB, and N on AC produced, so that MN cuts BC, say in P. If he show MPB, CPN to be equal, the theorem wo proved; and here he stops, calling the theorem m

et probable. In the steps which follow this assumption there is nothing to notice. It should be remarked, that what Girard does prove is very much more difficult than the general theorem. Had he thought of the manner in which three circles divide the sphere into eight triangles, the demonstration given by Cavalieri would not have been concealed for one minute from the geometer who could demonstrate that MN cuts BC.

A. DE MORGAN.

To eliminate x, y, λ from

$$ax + by + \lambda(\alpha x + \beta y) = 0,$$
$$a'x + b'y + \lambda(\alpha' x + \beta' y) = 0,$$
$$a''x + b''y + \lambda(\alpha'' x + \beta'' y) = 0.$$

Assume $1 + \lambda\mu = 0$, then ξ, η being arbitrary,

$$\xi x + \eta y + \lambda(\mu\xi x + \mu\eta y) = 0.$$

And eliminating x, y, λx, λy from the four equations, therefore

$$\begin{vmatrix} \xi & \eta & \mu\xi & \mu\eta \\ a & b & \alpha & \beta \\ a' & b' & \alpha' & \beta' \\ a'' & b'' & \alpha'' & \beta'' \end{vmatrix} = 0,$$

an equation which may be written

$$A\xi + B\eta + C\mu\xi + D\mu\eta = 0;$$

and the equation being true, independently of the values of ξ, η, we have

$$A + C\mu = 0,$$
$$B + D\mu = 0.$$

Or, eliminating μ,

$$AD - BC = 0,$$

which is the result of the elimination of x, y, λ from the given equations, or, what is the same thing, the result of the elimination of x, y from the equations

$$\frac{ax + by}{\alpha x + \beta y} = \frac{a'x + b'y}{\alpha' x + \beta' y} = \frac{a''x + b''y}{\alpha'' x + \beta'' y}.$$

A. C.

To the Editors of the Cambridge and Dublin Mathematical Journal.

GENTLEMEN,

I request the insertion of the following statement in the next number of your *Journal.*

Having read a paper by Mr. Matthew Collins "On Clairaut's Theorem," which appeared in the February number of the *Journal,* I think it right to state, that the substance of that communication has been taken, without acknowledgment, from a series of Lectures delivered by the late Professor MacCullagh in the University of Dublin, in Hilary and Michaelmas Terms, 1846.

I attended the Lectures alluded to, in company with Mr. Collins amongst others; and a memoir containing an account of them has been prepared by me for the Royal Irish Academy, and was published some months since in its *Transactions.*

In the course of last year Mr. Collins published a pamphlet absolutely identical with the paper which appeared in the *Journal.* He sent several copies of this pamphlet to distinguished members of the University of Dublin. I have now before me one of these copies, on the title-page of which Mr. Collins has *written* the following acknowledgment—"Extracted from manuscript notes taken at MacCullagh's Lectures."

I have the honor to be, Gentlemen,

Your obedient Servant,

GEORGE J. ALLMAN, LL.D.,

Professor of Mathematics, Queen's College, Galway.

April 10, 1854.

SUR LA THEORIE DES FONCTIONS HOMOGENES À DEUX INDETERMINEES.

Par M. HERMITE.

MES premières recherches sur la théorie des formes à deux indéterminées, ont pour objet la démonstration de cette proposition arithmétique élémentaire, *que les formes à coefficients entiers et en nombre infini, qui ont les mêmes invariants, ne donnent qu'un nombre essentiellement limité de classes distinctes.*

Une notion générale sur les invariants s'est offerte dans ces recherches, amenée par une considération purement arithmétique, la réduction des simples formes quadratiques définies, et l'application très facile que j'ai pu faire pour les formes cubiques et biquadratiques, m'a donné leurs invariants obtenus déja par M. Cayley, en suivant une toute autre voie. Mais à partir du cinquiême degré l'application de cette méthode devenait si pénible, que j'ai du renoncer à l'espoir d'en tirer *explicitement* les expressions de leurs invariants, et à plus forte raison celle des invariants des formes des degrés plus élevés. Ramené demièrement à ces questions, j'ai été conduit à les envisager sous un point de vue nouveau, et j'ai pu enfin aborder les formes du cinquième degré, qui n'avaient pu être traitées par ma première méthode. Les circonstances singulières, que j'ai rencontrées dans cette recherche, me semblent ajouter encore à l'interet de la grande théorie que MM. Cayley et Sylvester ont déja enrichie de tant de découvertes. Mais j'ai eu surtout en vue la théorie arithmétique, dont j'ai ainsi trouvé les véritables éléments, comme l'on verra par la suite de mes recherches: dès à présent néanmoins on pourra reconnâitre que la théorie des formes binaires, dans toute sa généralité, est étroitement liée à la *composition des classes quadratiques*, résultat singulier et qui ouvrera des nouvelles perspectives dans l'étude des propriétés les plus cachées des nombres. La loi de réciprocité dont M. Sylvester a bien voulu déja annoncer la découverte, étant le point de départ de mon analyse, je dira d'abord en peu de mots en quoi elle consiste.

Section I.—Loi de Réciprocité.

Elle est contenue dans le théoréme: *A tout covariant d'une forme de degré m, et qui par rapport aux coefficients de cette forme est du degré p, correspond un covariant du degré m par rapport aux coefficients, d'une forme du degré p.*

Soit,

$$f(x, y) = a(x+\alpha y)(x+\alpha' y)\ldots(x+\alpha^{(m-1)}y) = a \text{ Norme } (x+\alpha y),$$

une forme du degré m décomposée en facteurs linéaires, et $\phi(x, y)$, un covariant de cette forme du degré p quant aux coefficients, et d'un degré quelconque en x et y.

Si nous faisons

$$F = a^p \text{ Norme } (X+\alpha Y+\alpha^2 Z+\ldots+\alpha^p T),$$

les coefficients de F, seront des fonctions entières de

des coefficients de f, et on démontrera facilement ces deux lemmes.

1°. *Toute fonction entière et du degré p des coefficients de f, s'exprime linéairement par ceux de la forme F.*

2°. *Les coefficients de F ne sauraient être liés par aucune relation du premier degré dont les coefficients seraient numériques,* c. à. d. *indépendents des coefficients de f. D'où résulte qu'une fonction du degré p de ces coefficients n'est absolument susceptible que d'une seule expression linéaire par ceux de F.*

Cela étant, voici comment du covariant $\phi(x, y)$ qui se rapporte à la forme f, du degré m, se déduit un covariant se rapportant à une forme du degré p.

Soit,

$$f(x, y) = ax^m + mbx^{m-1}y + \frac{m.m-1}{1.2}\, cx^{m-2}y^2 + \text{etc.}$$

de sorte que les constantes a, b, c.... soient ce que nous avons appelé les coefficients de f; nous leurs donnerons une désignation plus expressive, en les représentant de cette manière

$$a = (x_0^m), \quad b = (x_0^{m-1}y_0), \quad c = (x_0^{m-2}y_0^2), \ldots$$

ainsi l'expression de $f(x, y)$ deviendra par la suppression des parenthèses la puissance

$$(xx_0 + yy_0)^m.$$

Faisons de même

$$\begin{aligned} F = a^p \text{ Norme } (X + \alpha Y + \alpha^2 Z + \ldots + \alpha^p T) \\ = (XX_0 + YY_0 + \ldots + TT_0)^m, \end{aligned}$$

en convenant après le developpement de la puissance d'écrire

$$p.\ ex.\ (X_0^m)\, X^m, \quad (X_0^{m-1}\, Y_0)\, X^{m-1}\, Y, \text{ etc.,}$$

respectivement au lieu de

$$X_0^m X^m, \quad X_0^{m-1}\, Y_0, \quad X^{m-1}\, Y, \text{ etc.,}$$

ce qui sera une désignation commode des coefficients de F. Cela posé, d'après le premier des lemmes ci-dessus, on pourra, et d'une manière seulement, exprimer linéairement les coefficients du covariant $\phi(x, y)$, par les quantités

$$(X_0^m), \ (X_0^{m-1}\, Y), \text{ etc.;}$$

or il se présente cette conséquence remarquable,

Ayant exprimé $\phi(x, y)$ par les quantités

$$(X_0^m),\ (X_0^{m-1}Y),\ \text{etc.},$$

concevons que l'on supprime les parenthèses, on arrivera par la à une fonction du m^e degré par rapport aux quantités

$$X_0,\ Y_0,\ Z_0,\ \ldots\ T_0.$$

Or cette fonction sera un covariant de la forme suivante de degré p,

$$X_0x^p + pY_0x^{p-2} + \frac{p.p-1}{1.2}Z_0x^{p-2}y^2 + \ldots + T_0y^p.$$

Rien d'ailleurs ne vient içi changer le degré des indéterminées x et y, dans cette métamorphose que subit la fonction $\phi(x, y)$, ainsi ce sont des covariants de même degré par rapport aux indéterminées, qui se trouvent liés l'un à l'autre par la loi de réciprocité. Mais il y a une seconde manière de passer ainsi d'un covariant se rapportant à une forme d'un certain degré, à un covariant se rapportant à une forme d'un autre degré. L'analyse précédente conduit en effet, et très aisément, a ce second théorème :

Etant donné un covariant quelconque du m^e degré par rapport aux coefficients de la forme

$$X_0x^p + pY_0x^{p-1}y + \frac{p.p-1}{1.2}Z_0x^{p-2}y^2 + \ldots + T_0y^p.$$

Si l'on transforme en symboles, dans l'expression de ce covariant, les quantités X_0^m, $X_0^{m-1}Y$, etc., ... en les remplacant respectivement par (X_0^m), $(X_0^{m-1}Y)$, etc., coefficients de la forme F, ce covariant se transformera en un autre se rapportant à la forme $f(x, y)$, et du degré p, relativement aux coefficients de cette forme.

Section II.—Conséquences de la Loi de Réciprocité.

Nous considérons en premier lieu les invariants qui sont un cas particulier des covariants, lorsqu'on suppose leur degré nul par rapport aux indéterminées x et y. La connaissance complète que nous avons des invariants des formes du second, troisième, et quatrième degré, nous donnera alors immediatement pour des formes de degré quelconque, les invariants qui sont du second, troisième, et quatrième degré par rapport aux coefficients de ces formes. Ainsi les formes quadratiques

$$f = ax^2 + 2bxy + cy^2 = a(x + \alpha y)(x + \alpha' y),$$

ont pour expression générale de leurs invariants, la fonction de degré 2μ,

$$\Delta = (b^2 - ac)^\mu = a^{2\mu}(\alpha - \alpha')^{2\mu},$$

donc toutes les formes de degré 2μ possèdent un invariant quadratique, que nous allons calculer. Soit pour cela

$$F = a^{2\mu}(X + \alpha X' + \alpha^2 X'' + \ldots + \alpha^{2\mu} X^{(2\mu)})(X + \alpha' X' + \alpha'^2 X'' + \ldots + \alpha'^{2\mu} X^{(2\mu)});$$

en représentant symboliquement cette forme, suivant notre convention par

$$F = (XX_0 + X'X_0' + \ldots + X^{(2\mu)}X_0^{(2\mu)})^2,$$

on trouvera bien aisément

$$(X_0^{(i)})^2 = a^{2\mu}\alpha^i\alpha'^i,$$

$$2(X_0^{(i)}X_0^{(j)}) = a^{2\mu}(\alpha^i\alpha'^j + \alpha^j\alpha'^i).$$

Maintenant il viendra par le developpement de la puissance

$$\Delta = a^{2\mu}(\alpha - \alpha')^{2\mu}$$

$$= a^{2\mu}[(\alpha^{2\mu} + \alpha'^{2\mu}) - \mu_1(\alpha^{2\mu-1}\alpha' + \alpha'^{2\mu-1}\alpha) + \mu_2(\alpha^{2\mu-2}\alpha'^2 + \alpha'^{2\mu-2}\alpha^2) - \text{etc.}],$$

en rapprochant les termes équidistants des extrêmes, et nommant pour abreger $\mu_1, \mu_2, \ldots$ les coefficients binomiaux. Cela fait on peut immediatement introduire les coefficients de F, et il viendra

Nous avons d'ailleurs, l'invariant quadratique

$$\Delta = ae - 4bd + 3c^2,$$

et si nous posons

$$\Delta' = ace + 2bcd - ad^2 - c^3 - b^2e,$$

l'expression générale des invariants des formes biquadratiques, sera la fonction de degré $2m + 3n$,

$$\Delta^m \Delta'^n,$$

comme l'a démontré M. Sylvester. Donc toutes les formes de degré $\mu = 2m + 3n$, possédent des invariants du quatrième degré en nombre égal à celui des solutions entières et positives de cette équation $\mu = 2m + 3n$. C'est là encore un des beaux résultats obtenus par M. Cayley dans son mémoire sur les hyperdéterminants. Mais les conséquences de la loi de réciprocité, dont j'aurai besoin principalement dans la suite, se rapportant aux covariants, j'y arrive immediatement en omettant beaucoup de remarques auxquelles les résultats précédents donneraient lieu.

Considérant d'abord les formes quadratiques

$$f = ax^2 + 2bxy + cy^2,$$

nous avons cette expression générale de leurs covariants savoir,

$$\phi = (b^2 - ac)^\mu (ax^2 + 2bxy + cy^2)^\nu = a^{2\mu+\nu}(\alpha - \alpha')^{2\mu}(x + \alpha y)^\nu (\alpha x + \alpha' y)^\nu,$$

de degré $2\mu + \nu$ par rapport aux coefficients de f. Donc faisant

$$2\mu + \nu = m,$$

nous aurons autant de covariants du second degré par rapport aux formes de degré m, qu'il y a de solutions entières et positives de cette équation. D'ailleurs le nombre 2ν représente le degré de chacun de ces covariants en x et y. Dans le cas ou m est impair, et dans ce cas seulement, on peut faire $\nu = 1$, on est alors conduit à un covariant du second degré en x et y, dont nous allons donner l'expression générale a cause de son importance. A cet effet posons

$$m = 2\mu + 1, \quad \phi = Ax^2 + Bxy + Cy^2,$$

de sorte que

$$A = a^{2\mu+1}(\alpha - \alpha')^{2\mu}, \quad B = a^{2\mu+1}(\alpha - \alpha')^{2\mu}(\alpha + \alpha'),$$
$$C = a^{2\mu+1}(\alpha - \alpha')^{2\mu}\alpha\alpha',$$

il s'agira d'exprimer ces diverses quantités au moyen des coefficients de la forme

$$F = a^m(X + \alpha X' + \alpha^2 X'' + \ldots + \alpha^m X^{(m)})(X + \alpha' X' + \alpha'^2 X'' + \ldots + \alpha'^m X^{(m)}),$$

coefficients dont nous avons précédemment employé les valeurs savoir

$$2(X_0^{(i)}X_0^{(j)}) = a^m(\alpha^i\alpha'^j + \alpha^j\alpha'^i).$$

Or on trouve immediatement A et C, par le même calcul qui nous a donné l'invariant quadratique des formes de degré pair, savoir

$$A = 2(X_0^{(m-1)}X_0) - 2\mu_1(X_0^{(m-2)}X_0') + 2\mu_2(X_0^{(m-3)}X_0'') - \text{etc.},$$

$$C = 2(X_0^{(m)}X_0') - 2\mu_1(X_0^{(m-1)}X_0') + 2\mu_2(X_0^{(m-2)}X_0''') - \text{etc.},$$

quant a B, après quelques réductions très faciles on obtiendra

$$B = 2(X_0^{(m)}X_0) - 2(\mu_1 - 1)(X_0^{(m-1)}X_0')$$
$$+ 2(\mu_2 - \mu_1)(X_0^{(m-2)}X_0'') - \text{etc.},$$

la dernier terme étant

$$2(\mu_\mu - \mu_{\mu-1})(X_0^{(\mu+1)}X_0^{(\mu)}).$$

Pour les formes cubiques,

$$f = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

nous avons le covariant quadratique

$$(b^2 - ac)\,x^2 + (bc - ad)\,xy + (c^2 - bd)\,y^2,$$

que donneraient les formules précédentes, mais en le multi-

de substitutions, au déterminant *un*, propres à faire disparaitre les coefficients des carrés des indéterminées, et à réduire ϕ à l'expression suivante

$$XY.\sqrt{\Delta}.$$

Soit

$$\left.\begin{aligned} x &= \alpha X + \beta Y \\ y &= \gamma X + \delta Y \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

l'une quelconque de ces substitutions, toutes les autres s'en déduiront comme on sait, en la faisant suivre de celle-ci

$$X = \omega\eta, \quad Y = -\frac{1}{\omega}\xi \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

on ω est une quantité arbitraire. Cela posé, nous définirons comme forme canonique de $f(x, y)$, la transformée qui en résulte par la substitution (1). Cette forme canonique contiendra essentiellement dans les coefficients une quantité arbitraire qu'on mettra en évidence si l'on veut, en y faisant la substitution (2).

Mais posons d'abord

$$f(\alpha X + \beta Y, \gamma X + \delta Y) = F(X, Y),$$

nous aurons cette proposition fondamentale : *Toute fonction entière des coefficients de F, qui se reproduit identiquement dans la transformée obtenue par la substitution* (2), *est une fonction rationelle des coefficients de la proposée $f(x, y)$, le dénominateur de cette fonction étant une puissance de Δ, et la numérateur un invariant de f. En second lieu, toute fonction qui se reproduit au signe près, redonne si on la multiplie par $\sqrt{\Delta}$, le même expression que les précédentes.*

Voici donc la principe d'une nouvelle méthode pour la recherche des invariants, puisque tout invariant de la forme $f(x, y)$, s'exprime par une fonction semblable des coefficients de la transformée F, qui possédera evidemment la propriété mentionnée dans notre proposition. Nous allons en faire l'application aux formes du cinquième degré.

Section IV.

Recherche des Invariants des Formes du Cinquième Degré.

Nous représenterons la forme proposée par

$$f(x, y) = ax^5 + 5bx^4y + 10cx^3y^2 + 10c'x^2y^3 + 5b'xy^4 + a'y^5.$$

Le covariant quadratique, par

$$\phi = (ab' - 4bc' + 3c^2)\,x^2 + (aa' - 3bb' + 2cc')\,xy + (a'b - 4b'c + 3c'^2)\,y^2,$$

et enfin la transformée canonique par

$$F = AX^5 + 5BX^4Y + 10CX^3Y^2 + 10C'X^2Y^3 + 5B'XY^4 + A'Y^5.$$

Cela posé, puisque le covariant quadratique de F, se reduit par hypothése à l'expression $XY\sqrt{\Delta}$, nous aurons entre les coefficients de F, les relations suivantes,

$$AB' - 4BC + 3C^2 = 0, \quad AA' - 3BB' + 2CC' = \sqrt{\Delta},$$
$$A'B - 4B'C + 3C'^2 = 0,$$

et c'est sous ces conditions, qu'il nous faut obtenir l'expression la plus générale d'une fonction entière des coefficients de F, qui ne change pas en y faisant la substitution

$$X = \omega\eta, \quad Y = -\frac{1}{\omega}\xi.$$

Une analyse plus longue que difficile, et que je n'ai pas encore assez simplifiée pour l'exposer ici, m'a donné les propositions suivantes.

1°. *Toute fonction entière des coefficients A, B, C, etc., qui ne change pas quand on transforme F, par la substitution*

$$X = \omega\eta, \quad Y = -\frac{1}{\omega}\xi,$$

$f(x, y)$, on a donc

$$I = \Delta^k \sqrt{\Delta}(ACB'^2 - A'C'B^2)$$
$$= (AA' - 3BB' + 2CC')^{2k+1}(ACB'^2 - A'C'B^2),$$

ce qui est une fonction de degré impairement pair, quel que soit l'entier k, par rapport aux coefficients de F, et par suite par rapport à ceux de f, comme on la reconnait avec une legère attention. Mais il nous faut encore approfondir la nature de ces quatre quantités,

$$AA', \ BB', \ CC', \ ACB'^2 - A'C'B^2,$$

qui viennent s' offrir comme éléments simples dans l'expression générale des invariants des formes du cinquième degré. C' est l'objet des considérations qui vont suivre.

Section V.—Des Covariants Similaires.

Revenant au cas général des formes de degré quelconque $f(x, y)$ qui ont un covariant quadratique, soit comme plus haut

$$F = AX^m + mBX^{m-1}Y + \ldots + mB'X^{m-1}Y + A'Y^m,$$

la transformée à laquelle nous avons donné le nom de forme canonique. Par définition même, le covariant quadratique de F sera simplement $XY\sqrt{\Delta}$, cela posé, nous réunirons par la dénomination commune de covariants similaires de f ceux qui jouissent de cette propriété, qu'en y faisant la substitution par laquelle f devient F, leurs coefficients reproduisent toujours à un facteur numérique près, les quantités $A, B, \ldots B', A'$, multipliées par une puissance de $\sqrt{\Delta}$. Cette définition dépend essentiellement du covariant quadratique en x et y, qu'on prend pour base de la réduction à la forme canonique, de sorte qu'on parviendra à un groupe différent de covariants similaires, en employant pour la réduction à la forme canonique, un covariant quadratique en x et y, mais d'un autre degré par rapport aux coefficients. Pour fixer les idées, nous ne considerons que les groupes se rapportant aux covariants quadratiques dont nous avons en commençant établi l'existence par la loi de réciprocité, et nous en donnerons une première série, en nous fondant sur ce théorème :

Soient $\phi(x, y)$ et $\psi(x, y)$ deux covariants quelconques de f, le degré du second étant supposé non-inférieur à celui du premier, en faisant

$$\phi(y, -x) = \alpha x^p + p\beta x^{p-1}y + \ldots + p\beta' xy^{p-1} + \alpha' y^p,$$

le forme

$$\chi = \alpha \frac{d^p\psi}{dx^p} + p\beta \frac{d^p\psi}{dx^{p-1}dy} + \ldots + p\beta' \frac{d^p\psi}{dx\,dy^{p-1}} + \alpha' \frac{d^p\psi}{dy^p},$$

sera encore un covariant de f.*

Pour appliquer ce théorème, nous prendrons $\psi = f$, et nous supposerons ϕ une puissance du covariant quadratique, il viendra alors cette série

$$\sqrt{\Delta}\,\frac{d^2F}{dx\,dy}, \quad \Delta\,\frac{d^4F}{dx^2dy^2}, \quad \Delta\sqrt{\Delta}\,\frac{d^6F}{dx^3dy^3}, \quad \text{etc.},$$

qui aboutit à un invariant, si le degré de F est pair, et à un covariant linéaire si ce degré est impair. Ce covariant linéaire s'évanouit identiquement dans le cas des formes cubiques, car on a alors

$$F = AX^3 + A'Y^3,$$

mais ce cas excepté, il existe bien effectivement. Prenons pour exemple les formes du cinquième degré, le covariant sera alors $\Delta(CX + C'Y)$, et en supposant $C = 0$, $C' = 0$, on ne satisfait plus aux deux relations

$$AB' - 4BC' + 3C^2 = 0, \quad A'B - 4B'C + 3C'^2 = 0,$$

qui seules existent entre les coefficients de F. J'insiste sur ce point en raison de la grande importance des covariants linéaires pour la théorie arithmétique des formes de degrés

canonique, celle du covariant quadratique ψ étant dans le même cas $XY\sqrt{\Delta}$, on voit que les coefficients des termes en U et V, dans la forme

$$\Phi\{X(U-V\sqrt{\Delta}),\quad Y(U+V\sqrt{\Delta})\},$$

seront effectivement d'après notre définition des covariants similaires. Maintenant chacun d'eux par l'application repetée du même principe et de celui qui est fondé sur la différentiation, donnera évidemment naissance à une infinité d'autres, tous compris dans la même forme analytique simple, que nous allons indiquer d'une manière plus précise. Soit pour abréger l'écriture, d'après la notation ingénieuse de M. Cayley,

$$F=(A,B,C,\dots C',B',A')(X,Y)^m,$$

de sorte que la première parenthèse renferme dans leur ordre, les coefficients de la forme ; donc les covariants similaires du même degré que F et qui résultent des méthodes précédentes, seront de la forme

$$\Phi=\sqrt{(\Delta^{2k+1})}(\alpha A,\beta B,\gamma C,\dots -\gamma C',-\beta B',-\alpha A')(X,Y)^m,$$

ou de la suivante

$$\Phi_1=\Delta^{k_1}(\alpha A,\beta B,\gamma C,\dots \gamma C',\beta B',\alpha A')(X,Y)^m,$$

les quantités α, β, γ, etc., étant des constantes numériques. Les autres de degrés $m-2$, $m-4$, etc.... sont de la forme $\dfrac{d^{2i}\Phi}{dX^i dY^i}$, $\dfrac{d^{2i}\Phi_1}{dX^i dY^i}$. Cette remarquable simplicité d'expression que prennent par la substitution canonique une multitude de covariants de la forme f, qu'il eut été impossible d'obtenir jamais en fonction explicite des coefficients de cette forme, justifient ce me semble, l'idée nouvelle des formes canoniques que j'introduis ici.

Section VI.—Recherches ultérieures sur les Invariants des Formes du Cinquième Degré.

Notre point de départ, sera ce théorème auquel conduisent immédiatement les considerations précédentes : soient ϕ et ϕ_1 les covariants de f, qui deviennent respectivement par la substitution canonique les expressions désignées ci-dessus par Φ et Φ_1, ces covariants étant du même degré, on obtiendra un invariant en mettant dans $\phi(y_1,-x)$, $\dfrac{d^m\phi_1}{dx^i dy^{m-i}}$, au lieu de $x^i y^{m-i}$, d'après un théorème enoncé plus haut.

Cela étant, cet invariant exprimé par les coefficients $A, B, C,\ldots$ de la forme canonique, sera

$$I = \Delta^{k+k_1}\sqrt{\Delta}\left(\alpha^2 AA' - m\beta^2 BB' + \frac{m.m-1}{2}\gamma^2 CC' - \text{etc.}\ldots\right).$$

De là se tire une méthode très simple, dont nous allons faire l'application aux formes du cinquième degré pour exprimer les quantités AA', BB', CC', etc., au moyen des coefficients de la forme proposée. Posons

$$f = (a, b, c, c', b', a')\,(x, y)^5,$$

le covariant quadratique, dans le même système de notation, sera

$$\phi = (ab' - 4bc' + 3c^2,\ aa' - 3bb' + 2cc',\ a'b - 4b'c + 3c'^2)\,(x^2, xy, y^2),$$

et en faisant

$$f\left(Ux - V\frac{d\phi}{dy},\ Uy + V\frac{d\phi}{dx}\right) = (f, f_1, f_2, f_3, f_4, f_5)\,(U, V)^5,$$

les diverses formes f, f_1, f_2, etc. seront un groupe de covariants similaires, que nous allons employer à la composition de trois invariants I_0, I_1, I_2, à savoir

I_0 en employant f avec f_1,
I_1 .... id. .... f_3,
I_2 .... id. .... f_5.

déterminant relatif aux équations précédentes, est différent de zéro, et égal à 2^5. Mais les expressions générales données § 4^e^ contiennent en outre le quantité $ACB'^2 - A'C'B^2$, que nous obtiendrons de la manière suivante. Considerons les covariants similaires ayant pour transformées canoniques

$$\sqrt{\Delta}\, \frac{d^2 F}{dX dY} \text{ et } \Delta\, \frac{d^4 F}{dX^2 dY^2}$$

en formant le cube du dernier, on parviendra aux deux formes

$$\sqrt{\Delta}(B,\, C,\, C',\, B')\,(X, Y)^2,$$

et
$$\Delta^3(C^3,\, C^2C',\, CC'^2,\, C'^3)\,(X, Y)^3,$$

d'où l'on tire toujours par le même principe, l'invariant du 18^e^ degré que nous nommerons I_3,

$$I_3 = \Delta^3 \sqrt{\Delta}(BC'^3 - B'C^3).$$

Mais par les relations fundamentales

$$AB' - 4BC' + 3C^2 = 0, \quad A'B - 4B'C + 3C'^2 = 0,$$

on obtient facilement

$$3(BC'^3 - B'C^3) = ACB'^2 - A'C'B^2,$$

d'où enfin

$$ACB'^2 - A'C'B^2 = 3\, \frac{I_3}{\Delta^3 \sqrt{\Delta}}.$$

Voici donc, d'après les formules du § 4^e^, la conclusion de notre théorie pour les formes du cinquième degré.

1°. *L'expression la plus générale des Invariants de ces formes dont le degré* $\mu \equiv 0 \pmod{4}$ *est*

$$I = F\left(\sqrt{\Delta},\ \frac{I_1}{\Delta\sqrt{\Delta}},\ \frac{I_2}{\Delta^2\sqrt{\Delta}}\right),$$

F étant une fonction homogène du degré $\frac{1}{2}\mu$.

2°. *L'expression la plus générale des Invariants dont le degré* $\mu \equiv 2 \pmod{4}$ *est*

$$I' = \frac{I_3}{\Delta^3\sqrt{\Delta}}\, F_1\left(\sqrt{\Delta},\ \frac{I_1}{\Delta\sqrt{\Delta}},\ \frac{I_2}{\Delta^2\sqrt{\Delta}}\right),$$

F_1 étant une fonction homogène de degré $\frac{1}{2}\mu - 2$.

Ainsi un invariant quelconque, ou au moins son produit par une puissance de Δ, est une fonction rationnelle et entière des invariants fondamentaux Δ, I_1, I_2, I_3, des degrés

et 18. Car les fonctions F et F_1 étant homogènes, on peut écrire

$$I = \frac{1}{\Delta^{\frac{5}{4}\mu}} F(\Delta^3, \Delta I_1, I_2) \text{ et } I' = \frac{I_3}{\Delta^{\frac{1}{4}(5\mu-6)}} F_1(\Delta^3, \Delta I_1, I_2).$$

Nous voyons par la se révéler un caractére essentiel des formes de degré superieur au quatrième, et qui consiste en ce que les invariants ne peuvent en général s'exprimer en fonction rationnelle d'un certain nombre d'entre eux supposés algébriquement indépendants. M. Cayley, M. Sylvester, et moi avions long-temps pensé qu'en général les invariants des formes de m^e degré devaient s'exprimer par des fonctions entières de $m-2$ d'entre eux, et c'est même ce qui à empêché M. Sylvester de chercher à démontrer la loi de réciprocité dont il avait aussi présumé l'existence, une contradiction necessaire s'étant manifestée entre cette loi et celle du nombre des invariants fondamentaux. Peut-être cependant, s'il m'est permis d'emettre une conjecture sur un sujet si profond et si difficile, doit-on penser qu'il sera possible d'obtenir pour les formes d'un degré donné, un petit nombre de groupes d'invariants fondamentaux, types d'autant de séries générales dont l'ensemble comprendrait tous les invariants possibles. C'est ainsi par exemple que l'invariant du 18^e degré que nous venons d'obtenir pour les formes du cinquième degré s'offre comme le type de tous les invariants de degré impairement pair de ces formes. Sur ce sujet nous allons encore présenter quelques observations.

Section VII.—Recherche particulière sur l'Invariant I_3 du 18^e *Degré.*

Je me propose de faire voir que le carré de I_3 est non seulement une fonction rationnelle, mais même une *fonction entière* des trois invariants nommés Δ, I_1 et I_2. Soit à cet effet $I_2 - 2I_1\Delta + \Delta^3 = 24J_2$ et $I_1 - \Delta^2 = 8J_1$, j'adopterai pour invariants fondamentaux J_1 et J_2 au lieu de I_1 et I_2, pour la commodité des calculs, et les équations (I) du § 6^e donneront ces expressions très simples

$$CC' = \frac{J_2}{\sqrt{\Delta^5}}, \quad BB' = \frac{J_2 + J_1\Delta}{\sqrt{\Delta^5}}, \quad AA' = \frac{J_2 + 3J_1\Delta + 2\Delta^3}{\sqrt{\Delta^5}}.$$

Cela posé, nous partirons de la rélation suivante,

$$16(ACB'^2 - A'C'B^2)^2$$
$$= (AA'BB' - 16BB'CC - 9C^2C'^2)^2 - 24^2BB'C^3C'^3,$$

qu'on trouvera identique, en vertu des équations fondamentales qu'on a entre les coefficients de la forme canonique savoir

$$AB' - 4BC' + 3C^2 = 0, \quad A'B - 4B'C + 3C'^2 = 0.$$

On peut effectivement d'abord l'écrire ainsi,

$$24^2 BB'C^3C'^3$$
$$= (AA'BB' - 16BB'CC' - 9C^2C'^2)^2 - 16(ACB'^2 - A'C'B^2)^2,$$

ou en de composant en produit la différence des carrés,

$$24^2 BB'C^3C'^3$$
$$= \{AA'BB' - 16BB'CC' - 9C^2C'^2 + 4(ACB'^2 - A'C'B^2)\}$$
$$\{AA'BB' - 16BB'CC' - 9C^2C'^2 - 4(ACB'^2 - A'C'B^2)\}.$$

Maintenant les équations

$$3C^2 = 4BC' - AB', \quad 3C'^2 = 4B'C - A'B,$$

donneront si on les multiplie membre à membre,

$$9C^2C'^2 = 16BB'CC' + AA'BB' - 4(ACB'^2 + A'C'B^2),$$

et en substituant cette valeur de $C^2C'^2$ dans chacun des facteurs, on verra le premier devenir

$$8ACB'^2 - 32BB'CC' = 8B'C(AB' - 4BC') = -24B'C^3,$$

et le second se réduire d'une manière semblable à

$$8A'C'B^2 - 32BB'CC' = 8BC'(A'B - 4BC') = -24BC'^3,$$

d'on suit l'identité annoncée. L'expression du carré de $ACB'^2 - A'C'B^2$, étant alors ramenée à ne plus dépendre que des quantités AA', BB', CC', on trouvera par la substitution des valeurs de ces quantités, un fonction des invariants Δ, J_1, J_2, et en chassant le dénominateur

$$16\Delta^{10}(ACB'^2 - A'C'B^2)^2$$
$$= (-24J_2^2 - 12J_1J_2\Delta + 3J_1^2\Delta^2 + 2J_2\Delta^3 + 2J_1\Delta^4)^2 - 24^2J_2^3(J_2 + J_1\Delta).$$

Or il arrive que le second membre contient en facteur Δ^3, de sorte qu'en supprimant ce facteur il viendra

$$16\Delta^7(ACB'^2 - A'C'B^2)^2 = 16I_3^2 = -24J_2(2J_2^2 + 3J_1^3)$$
$$+ 3\Delta J_1(3J_1^3 - 32J_2^2) - 12\Delta^2J_1^2J_2 + 4\Delta^3(J_2^2 + 3J_1^3) + 4\Delta^4J_1J_2 + 4J_2^2\Delta^5,$$

ce qui est une fonction entière des trois invariants fondamentaux, Δ, J_1, et J_2.

SECONDE PARTIE.

Depuis que la première partie de ces recherches à été terminée, encouragé par la manière si bienveillante dont elles ont été accueillies par mon ami M. Sylvester, j'ai repris avec une nouvelle ardeur l'étude algébrique des formes du cinquième degré, et je vais y consacrer cette seconde partie de mon travail, en réservant en dernier lieu, les considérations arithmétiques que j'ai annoncées dans l'introduction. C'est sur une notion analytique nouvelle, celle des formes-types, qui sera tout-à-l'heure exposée en détail, que se fondent les résultats nouveaux que j'ai obtenus. Cette notion est essentiellement propre aux formes de degrés impairs, avec la seule exception des formes cubiques qui y échappent comme un cas singulier. Pour les formes de degrés pairs il existe quelque chose d'analogue, mais qui jusqu'à présent ne s'est présenté à moi, que d'une manière plus compliquée. Aussi en parlerai-je seulement pour remarquer que les formes biquadratiques font alors exception, de sorte que les formes des premiers quatre degrés, pour des raisons diverses, doivent être considerées comme présentant des cas singuliers dans les théories générales qui ont pour objet les fonctions homogènes à deux indéterminées. C'est donc au seul point de vue algébrique, un champ plus vaste et plus fécund de recherches, qui s'ouvre à partir des formes de cinquième degré, où l'on voit apparâitre le rôle curieux d'éléments analytiques, qui n'existent pas pour les formes de degrés inférieurs. D'ailleurs c'est dans les méthodes simples et faciles qui se présentent dans cette étude, où est l'avenir de la science algébrique, car elle seule peut donner les éléments qui distinguent et caracterisent les divers modes d'existence des racines des équations générales de tous les degrés. J'espére que cette dernière considération recevera sa sanction de ce que nous allons développer en particulier sur les formes du cinquième degré.

Section I.—Des Formes-types.

La notion des formes-types repose sur l'existence des covariants linéaires, dont il à été déja fait mention précédemment, et qu'on obtient de la manière suivante. Soit en employant la notation de Mr. Cayley,

$$f=(a,\ b,\ c,\ \ldots\ c',\ b',\ a')\ (x,\ y)^m,$$

une forme de degré impair,

$$\theta = \{ab' - (m-1)\,bc' + \ldots,\quad aa' - (m-2)\,bb' + \ldots,$$
$$ba' - (m-1)\,b'c + \ldots\}\,(x^2, xy, y^2),$$

le covariant quadratique de f, et Δ l'invariant de θ. Nommons S le substitution au déterminant *un* et aux variables X, Y, qui transforme θ en $\sqrt{\Delta}.XY$, cette même substitution faite dans la proposée f, donnera ce que nous avons nommé la transformée canonique

$$F = (A, B, C, \ldots C', B', A')(X, Y)^m.$$

Ainsi le caractére essentiel de la forme canonique F, est que le covariant Θ analogue à θ, se réduise à $\sqrt{\Delta}\,XY$; les coefficients A, B, etc. ... sont donc liés par les relations

$$AB' - (m-1)\,BC' + \ldots = 0,\quad A'B - (m-1)\,B'C + \ldots = 0.$$

Ceci rappelé, voici comment s'obtient un covariant linéaire λ de la forme f. Elevons θ à la puissance $\frac{1}{2}(m-1)$, ce qui donnera un covariant du degré $m-1$ en x et y, puis mettons y et $-x$ au lieu de x et y; cela fait, en remplaçant un terme quelconque $x^\alpha y^\beta$, par $\dfrac{d^{m-1}f}{dx^\alpha dy^\beta}$, on obtiendra, comme on sait, encore un covariant de f, et ce covariant sera bien du premier degré. Mais il est essentiel d'établir qu'il ne s'évanouit pas identiquement. Soit à cet effet Λ, la transformée de λ, par la substitution S, on aura

$$\Lambda = \sqrt{(\Delta^{\frac{1}{2}(m-1)})}\,\frac{d^{m-1}F}{dx^{\frac{1}{2}(m-1)}dy^{\frac{1}{2}(m-1)}};$$

supposant donc

$$\frac{d^{m-1}F}{dx^{\frac{1}{2}(m-1)}dy^{\frac{1}{2}(m-1)}} = GX + G'Y,$$

Λ ne pourra s'évanouir identiquement qu'autant qu'on aura $G = 0$, $G' = 0$, mais ces relations ne vérifient pas les équations (1), sauf le cas des formes cubiques. Dans ce cas en effet, ayant $F=(A,B,B',A')(X,Y)^3$, Λ sera $3\sqrt{\Delta}(BX+B'Y)$, mais les relations (1)

$$AB' - B^2 = 0,\quad A'B - B'^2 = 0,$$

exigeront que $B = 0$, $B' = 0$. On en déduit effectivement

$$AA'BB' = B^2B'^2,$$

d'ou
$$BB'(AA' - BB') = 0.$$

Si donc le produit BB' n'est pas supposé nul, il faut qu'on

soit $AA' - BB' = 0$, ce qui conduit à la conséquence absurde que l'invariant

$$(AA' - BB')^2 - 4(AB' - B'^2)(A'B - B^2),$$

de la forme cubique est égal à zero. Les covariants linéaires n'ont donc d'existence effective qu'à partir des formes du cinquième degré

$$f = (a, b, c, c', b', a')(x, y)^5,$$

mais pour ces formes il y en aura un nombre infini, dont les degrés par rapport aux coefficients a, b, c, etc. seront la série des nombres impairs 5, 7, 9, etc.

Le covariant du 5^e ordre sera celui que nous venons d'obtenir et dont la transformée par la substitution S est $10\Delta(CX + C'Y)$, le covariant du 7^e ordre résultera du précédent, en y remplaçant x et y, par $\frac{d\theta}{dy}$ et $-\frac{d\theta}{dx}$, d'autres pourront s'obtenir en multipliant les précédents par des invariants de f. En se bornant à prendre pour multiplicateur une puissance de Δ, on obtiendra ainsi des covariants linéaires dont les degrés par rapport aux coefficients de f, seront les nombres $4n + 5$ et $4n + 7$, c. à. d. la série des entiers impairs à commençer par cinq. Nous en concluerons par la loi de réciprocité que toutes les formes dont les degrés sont des nombres impairs à partir du cinq, possédent un covariant linéaire du cinquième degré par rapport à leurs coefficients, et il est très facile d'établir qu'elles n'en possédent pas dont les degrés soient au dessous de cette limite. Mais pour abréger j'omettrai ce détail, et j'arrive immediatement à la définition des formes-types. Soient à cet effet λ et λ_1 deux covariants linéaires distincts pour une même forme f; designons par Σ la substitution

$$\lambda = \xi, \quad \lambda_1 = \eta,$$

et par Φ la transformée de f en ξ et η. Je dis que les coefficients de cette forme Φ, seront tous des invariants de f.

Pour le démontrer voyons ce deviennent les opérations précédentes en prenant pour point de départ une forme f', transformée de f par une substitution quelconque S. Soit δ le déterminant relatif à cette substitution S, λ' et λ_1' les covariants analogues à λ et λ_1. En multipliant par ordres puissances convenables de δ, p. ex. δ^α et δ^β, chacune des équations

$$\lambda' = \xi,$$
$$\lambda_1' = \eta,$$

il résulte de la nature même des covariants, que les premiers membres pourront alors être censés provenir du résultat de la substitution S dans λ et λ'. Ainsi par rapport aux quantités $\delta^\alpha\xi$, $\delta^\beta\eta$, la substitution Σ' analogue à Σ, sera $\Sigma' = \Sigma S$, et son inverse qu'il faudra effectuer dans f, sera $\Sigma'^{-1} = S^{-1}\Sigma$. Or on voit qu'en effectuant en premier la substitution S^{-1}, f' redevient f, et qu'en faisant ensuite la substitution Σ on est ramené, précisément à la forme ϕ, par rapport aux indéterminées $\delta^\alpha\xi$, $\delta^\beta\eta$. De la résulte que les coefficients des formes ϕ, relatives à f, et à une transformée de f, ne diffèrent que par des facteurs qui seront des puissances du déterminant de la substitution, ces coefficients seront donc des invariants de f; et c'est pour cette raison que nous donnons à ϕ le dénomination de forme-type.

Section II.—Calcul de la Forme-type du Cinquième Degré.

La définition que nous venons de donner, ne spécifie pas les covariants linéaires qu'il faut employer dans la substitution qui conduit aux formes-types, il suffit que ces covariants soient bien distincts c. à. d. que le déterminant relatif à la substitution effectuée soit différent de zéro. Mais dans le cas des formes du cinquième degré, que nous allons étudier nous ferons choix des deux covariants linéaires les plus simples, qui sont respectivement du cinquième et du septième degré par rapport aux coefficients de la forme proposée. En effectuant dans ces covariants la substitution S qui transforme f dans la forme canonique F, ils deviendront

$$\lambda = 10\Delta(CX + C'Y), \quad \lambda_1 = 10\Delta\sqrt{\Delta}(CX - C'Y),$$

expressions très simples, qui nous conduisent à faire le calcul de la forme-type, en opérant sur la transformée canonique F, ce qui est permis, puisqu'on parviendra identiquement au même résultat, en prenant pour point de depart toute transformée de f, par une substitution au déterminant *un*. Cela posé, ayant,

$$F = (A, B, C, C', B', A')(X, Y)^5,$$

nous ferons en supprimant un facteur numérique,

$$\Delta(CX + C'Y) = \xi, \quad \Delta\sqrt{\Delta}(CX - C'Y) = \eta,$$

et si nous représentons la transformée en ξ et η par,

$$\phi = (A, B, C, C', B', A')(\xi, \eta)^5,$$

il viendra ces expressions,

$$A = \frac{1}{(2CC'\Delta)^5}\{AC'^5 - A'C^5 + 5BCC'^4 + 5B'C'C^4 + 10C^2C'^2\},$$

$$B = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta}}\{AC'^5 - A'C^5 + 3BCC'^4 - 3B'C'C^4\},$$

$$C = \frac{1}{(2CC'\Delta)^5\Delta}\{AC'^5 + A'C^5 - BCC'^4 - B'C'C^4 - 4C^2C'^2\},$$

$$C' = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta^3}}\{AC'^5 - A'C^5 - BCC'^4 + B'C'C^4\},$$

$$B' = \frac{1}{(2CC'\Delta)^5\Delta^2}\{AC'^5 + A'C^5 - 3BCC'^4 - 3B'C'C^4 + 4C^2C'^2\},$$

$$A' = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta^5}}\{AC'^5 - A'C^5 - 5BCC'^4 - 5B'C'C^4\}.$$

D'après les théorèmes donnés au commencement de ces recherches, on reconnaît tout de suite qu'elles sont bien comme nous l'avons annoncé, des invariants de la forme proposée f, et qu'elles s'exprimeront rationnellement par les fonctions que nous avons nommées Δ_2, J_2, J_3, et par l'invariant du 18e ordre Γ. Mais ici se présente cette circonstance importante, qu'elles contiendront en dénominateur le seul invariant J_3, sans qu'on y voie figurer Δ, comme on pouvait s'y attendre d'après la théorie générale. Pour le faire voir, rappelons d'abord ces rélations qui existent entre les invariants et les coefficients de la forme canonique savoir :

$$AA' = \frac{1}{\sqrt{\Delta^5}}(J_3 + 3\Delta J_2 + \Delta^3),$$

$$BB' = \frac{1}{\sqrt{\Delta^5}}(J_3 + \Delta J_2),$$

$$CC' = \frac{1}{\sqrt{\Delta^5}}J_3,$$

$$ACB'^2 - A'C'B^2 = \frac{1}{\sqrt{\Delta^7}}I.$$

Nous en déduirons les valeurs des quantités $AC'^5 \pm A'C^5$ et $BC'^4 \pm B'C^4$, qui figurent dans les coefficients A, B, etc., par les équations suivantes:

$$36(AC'^5 + A'C^5)$$
$$=(AA'+16CC')(AA'BB'+16BB'CC'-9C^2C'^2)-64AA'BB'CC',$$
$$12(BC'^4 + B'C^4) = -AA'BB' + 16BB'CC' + 9C^2C'^2,$$
$$9(AC'^5 - A'C^5) = (16CC' - AA')(ACB'^2 - A'C'B^2),$$
$$3(BC'^4 - B'C^4) = ACB'^2 - A'$$

Ces équations deviennent effectivement identiques, en vertu des rélations fondamentales qui lient les coefficients de la forme canonique savoir,

$$AB' - 4BC' + 3C^2 = 0, \quad A'B - 4B'C + 3C'^2 = 0.$$

Quant à la méthode très facile par laquelle on les obtient, je pense pouvoir la supprimer pour abréger, car elle se présentera d'elle-même au lecteur qui se sera bien penetré des principes de ces recherches. On en déduit

$$9A' = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta^5}}(CC' - AA')(ACB'^2 - A'C'B^2)$$
$$= -I\frac{\Delta^2 + 3J_2}{2^5J_3^5},$$

$$9C' = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta^3}}(13CC' - AA')(ACB'^2 - A'C'B^2)$$
$$= -I\frac{\Delta^3 + 3\Delta J_2 - 12J_3}{2^5J_3^5},$$

$$9B = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta}}(25CC' - AA')(ACB'^2 - A'C'B^2)$$
$$= -I\frac{\Delta^4 + 3\Delta^2J_2 - 24\Delta J_3}{2^5J_3^5},$$

Le calcul des trois autres coefficients est un peu plus difficile et donne pour resultats,

$$36B' = \frac{1}{(2CC'\sqrt{\Delta})^5\Delta^2}(-81C^3C'^3 + 112BB'C^2C'^2 - 9AA'C^2C'^2$$
$$- 23AA'BB'CC' + A^2A'^2BB')$$
$$= \frac{1}{2^5J_3^5}(\Delta^5J_2 + \Delta^4J_3 + 6\Delta^3J_2^2 - 15\Delta^2J_2J_3 - 3\Delta(10J_3^2 - 3J_2^3) - 54J_3J_2^2),$$

$$36C = \frac{1}{(2CC'\sqrt{\Delta})^5\Delta}(-261C^3C'^3 + 304BB'C^2C'^2 - 9AA'C^2C'^2$$
$$- 35AA'BB'CC' + A^2A'^2BB')$$
$$= \frac{1}{2^5J_3^5}(\Delta^6J_2 + \Delta^5J_3 + 6\Delta^4J_2^2 - 27\Delta^3J_2J_3 - 3\Delta^2(14J_3^2 - 3J_2^3)$$
$$- 90\Delta J_3J_2^2 + 144J_2J_3^2);$$

$$36A = \frac{1}{(2CC'\Delta)^5}(711C^3C'^3 + 496BB'C^2C'^2 - 9AA'C^2C'^2$$
$$- 47AA'BB'CC' + A^2A'^2BB')$$
$$= \frac{1}{2^5J_3^5}(\Delta^7J_2 + \Delta^6J_3 + 6\Delta^5J_2^2 - 39\Delta^4J_2J_3 - 9\Delta^3(6J_3^2 - J_2^3)$$
$$- 126\Delta^2J_3J_2^2 + 288\Delta J_2J_3^2 + 1152J_3^3).$$

Ainsi il est démontré par le calcul, que les coefficients de

[illegible]

[illegible]

[illegible]

$K = \frac{J}{\delta}$, $K = \frac{J}{\delta}$ [illegible]

[illegible]

[illegible] doit être [illegible] dans la théorie de l'équation du cinquième degré qu'a donnée [illegible], mais elle met en évidence les fonctions des coefficients dont dépend essentiellement la nature des racines. [illegible] l'ingénieuse découverte du géomètre Anglais [illegible] jusqu'ici stérile, nous allons pouvoir immédiatement tirer d'importantes conséquences de notre transformée.

Section IV. — Les Invariants de tous les Degrés des Formes du Cinquième Degré, sont des fonctions entières des quatre invariants fondamentaux Δ, J_2, J_4 et I.

Dans la première partie de ces recherches, j'ai obtenu pour les invariants dont le degré est $\equiv 0$ ou $\equiv 2$ (mod. 4) les expressions générales

$$\frac{\Theta(J_2, \Delta J_2, \Delta^2)}{\Delta^{\mu}} \quad \text{et} \quad \frac{I\Theta_1(J_2, \Delta J_2, \Delta^2)}{\Delta^{\mu}},$$

où entre en dénominateur une puissance de Δ, mais on peut aller plus loin et parvenir à des expressions entières, par la considération de la forme-type. En effet, ϕ étant une transformée de f, par une substitution linéaire au déterminant $\frac{\Delta}{2J_2}$, comme il est aisé de le voir, tout invariant de f, s'exprime au moyen d'une fonction semblable des coefficients de ϕ, multiplié par une certaine puissance de J_3. Mais ces coefficients de la forme-type, sont comme nous l'avons établi, des fonctions entières des invariants fondamentaux divisés par une puissance de J_3, donc déjà, tout invariant de la forme proposée est une fonction entière des invariants fondamentaux, au moins une pareille fonction divisée par une puissance de J_3. Distinguant maintenant les deux cas ou le degré des invariants est $\equiv 0$ ou $\equiv 2$ (mod. 4), nous reconnaitrons bien aisément que l'expression générale

$$\frac{F(\Delta, J_2, J_3, I)}{J_3^\nu},$$

ou F est une fonction entière, se réduit dans le premier, à la forme $\frac{H_0(\Delta, J_2, J_3)}{J_3^\nu}$, et dans le second à la forme $\frac{IH_1(\Delta, J_2, J_3)}{J_3^\nu}$, H_0 et H_1 étant pareillement des fonctions entières. Cela suit en effet de ce que le carré et les puissance paires de l'invariant I du 18e degré, s'expriment en fonction entière de Δ, J_2, et J_3. Voici donc, par exemple, pour les invariants dont le degré est multiple de 4, deux expressions differentes, qui doivent être égales

$$\frac{\Theta_0(J_3, \Delta J_2, \Delta^3)}{\Delta^\mu} \text{ et } \frac{H_0(\Delta, J_2, J_3)}{J_3^\nu};$$

or les trois quantités Δ, J_2, J_3, qui y figurent, n'ont entre elles aucune rélation, et doivent être considerés comme absolument indépendantes, l'égalité

$$\frac{H_0(\Delta, J_2, J_3)}{J_3^\nu} = \frac{\Theta_0(J_3, \Delta J_2, \Delta^3)}{\ldots},$$

entraîne donc que H_0 est [illegible] par Δ^μ, *c. à. d.*, que les invariants [illegible] en fonction entière de Δ, J_2, et [illegible] as ou le degré est $\equiv 2$ (mod. 4). [illegible] même, cas il conduit à l'égali[illegible]

$$\frac{I\Theta_1(J_3, \Delta \ldots}{\Delta^\mu}$$

qui après la suppression du facteur I, coincide avec celle qu'on vient d'obtenir, ainsi donc en général, tout invariant d'une forme du cinquième degré, dont le degré par rapport aux coefficients est $\equiv 0 \pmod{4}$, est une fonction entière de Δ, J_2, J_3, et tout invariant dont le degré est $\equiv 2$, est la produit d'une pareille fonction multipliée par l'invariant I du 18[e] degré. Les expressions suivantes

$$\Sigma\alpha.\Delta^i J_2^{i'} J_3^{i''}, \quad I\Sigma\alpha.\Delta^i J_2^{i'} J_3^{i''},$$

ou les quantités α sont numériques, représentent donc tous les invariants des formes du cinquième degré, d'où l'on voit qu'il existe autant d'invariants linéairement indépendants, d'un degré donné m, qu'il y a de solutions entières et positives de l'une or l'autre de ces équations

$$4i + 8i' + 12i'' = m,$$

$$18 + 4i + 8i' + 12i'' = m.$$

On en conclut par la loi de réciprocité, que les formes d'un degré quelconque m, ont autant d'invariants du cinquième degré par rapport à leurs coefficients, qu'il y a de solutions entières et positives des mêmes équations. Ainsi parmi les formes dont le degré est impairement pair, il faut aller jusqu'au 18[e] degré pour rencontrer un invariant du cinquième ordre.

Section V.—Recherche particulière sur le Discriminant des Formes du Cinquième Degré.

MM. Cayley et Sylvester nomment, comme on sait, discriminant d'une forme f, le résultat de l'élimination de $\frac{x}{y}$, entre les deux équations homogènes $\frac{df}{dx} = 0$, $\frac{df}{dy} = 0$. On obtient ainsi pour une forme de degré m un invariant de degré $2(m-1)$, qui égalé à zero exprime que f a un facteur linéaire elevé au carré. Dans le cas des formes du cinquième degré, le discriminant est donc un invariant du 8[e] ordre, et qui d'après la théorie précédente doit être de cette forme $\alpha J_2 + \alpha'\Delta^2$, α et α' étant numériques. Mais nous allons en former l'expression par une méthode particulière et sans supposer les résultats généraux établis dans le précédent §, dont nous voulons offrir ainsi une confirmation dans un cas spécial très important en lui-mêm
A cet effet, nous nous proposerons généralement d'obte

les valeurs des invariants fondamentaux, lorsqu'il existe un facteur linéaire élevé au carré dans la forme proposée, *c. à. d.*, lorsqu'on peut lui donner cette expression

$$f = (0,\ 0,\ a,\ b,\ c,\ d)(x,\ y)^5.$$

En observant qu'on peut mettre $x + ky$, au lieu de x sans que les deux premiers coefficients cessent d'être nuls, disposons de cette quantité k, de manière a faire évanouir le coefficient de xy, dans le covariant quadratique θ. Nous aurons ainsi une transformée

$$f_1 = (0,\ 0,\ a_1,\ b_1,\ c_1,\ d_1)(x,\ y)^5,$$

et il faudra que les nouveaux coefficients vérifient la condition $a_1 b_1 = 0$. Comme nous ne voulons point admettre de facteurs linéaires à la troisième puissance, il faudra faire $b_1 = 0$, et si l'on écrit ainsi f_1 sous la forme

$$f_1 = \left(0,\ 0,\ \frac{4}{a^2},\ 0,\ \frac{3}{b^3},\ e\right)(x,\ y)^5,$$

le covariant θ sera

$$\theta = 12\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

ce qui nous conduit à remplacer encore $\dfrac{x}{a}$ et $\dfrac{y}{b}$ par X et Y. Nous trouverons de le sorte cette transformée des formes à facteur lineaire double

$$pY^5 + q(3XY^4 + 8X^3Y^2),$$

ou p et q sont des constantes quelconques, et qui a pour covariant quadratique

$$\frac{48q^2}{25}(X^2 - Y^2).$$

Cela étant, il suffira de mettre $X + Y$ et $X - Y$, au lieu de X et Y, pour obtenir la transformée canonique, qui sera

$$\begin{aligned} F &= p(X-Y)^5 + q(X-Y)^2(X+Y)\{3(X-Y)^2 + 8(X+Y)^2\}, \\ &= p(X-Y)^5 + q(X-Y)^2(X+Y)(11X^2 + 10XY + 11Y^2), \\ &= (p + 11q,\ -p - \tfrac{1}{5}q,\ p - q,\ -p - q,\ p - \tfrac{1}{5}q,\ -p + 11q)(X, Y)^5. \end{aligned}$$

Voici donc en fonction des deux indéterminées p et q, les valeurs suivantes, propres au cas d'un facteur linéaire élevé au carré dans la forme proposée, savoir,

$$AA' = -p^2 + 121q^2, \quad BB' = -p^2 + \tfrac{1}{25}q^2, \quad CC' = -p^2 + q^2,$$
$$ACB'^2 - A'C'B^2 = \tfrac{48}{5}pq(2p^2 + q^2).$$

On en tire

$$\sqrt{\Delta} = \frac{3.2^{10}}{25} q^2, \quad \frac{J_2}{\sqrt{\Delta^3}} = -\frac{3.2^3}{25} q^2, \quad \frac{J_3}{\sqrt{\Delta^5}} = q^2 - p^2,$$

d'où cette conclusion importante,

$$\Delta^2 + 2^7 J_2 = 0.$$

Le discriminant des formes du cinquième degré est donc obtenu, puisque nous avons un invariant du 8^e ordre $\Delta^2 + 2^7 J_2$ qui s'évanouit lorsqu'on suppose deux racines égales dans ces formes, et il se présente bien sous la forme valide d'après notre théorie générale. Exprimé par les coefficients de la forme canonique, il a cette valeur,

$$\text{discriminant} = \Delta^2 + 2^7 J_2 = \sqrt{\Delta^3}(AA' + 125BB' - 126CC'),$$

de sorte qu'on a un procédé arithmétique facile, pour calculer dans un cas donné cette fonction si importante. Remarquons encore avant d'aller plus loin, la quantité

$$25\Delta^3 - 2^{11} J_3,$$

qui s'évanouit si la forme proposée contient deux facteurs linéaires différents élevés chacune au carré. Si l'on cherche en effet, la discriminant de la forme cubique

$$\frac{F}{(X-Y)^2} = p(X-Y)^3 + q(X+Y)(11X^2 + 10XY + 11Y^2),$$

on la trouvera abstraction faite d'un facteur numérique égal à $q^2(2p^2+q^2)$ et d'après les relations précédentes, cette valeur s'exprime ainsi

$$25\,\frac{25\Delta^3 - 2^{11}J_3}{3^2.2^{20}\Delta^3}.$$

Dans un instant nous allons reconnaître le rôle important que joue cette quantité.*

Section VI.

Expression par les Invariants Fondamentaux, du nombre des racines réelles et imaginaires de toute équation du Cinquième Degré.

La possibilité d'un pareil résultat est une conséquence immédiate de ces deux propriétés de la forme-type, d'être une transformée par une substitution réelle de la forme

* Dans le cas d'une forme contenant au cube un facteur linéaire, tous les invariants s'évanouissent, comme cela résulte d'un théorème générale donné par mon ami M. Cayley dans le Journal de M. Crelle.

proposée, et d'avoir pour coefficients des invariants. Mais on sent combien il y a loin d'une telle possibilité à un résultat effectif, aussi dépuis l'époque ou je communiquais pour la première fois cette vue à mon ami M. Sylvester, avais-je désesperé d'aller plus loin, l'application du théorème de M. Sturm n'étant pas praticable sur l'équation littérale et compliquée qui aurait la forme-type pour son premier membre.

La méthode suivante à laquelle je ne suis parvenu qu'après bien des efforts, me semble peut-être meriter un instant d'attention, car elle offrira si je ne me trompe, une étude algébrique compléte des racines de l'équation générale du cinquième degré, sous la point de vue de la distinction de ces racines comme quantités réelles et imaginaires, lorsqu'on attribue aux coefficients toutes les valeurs réelles possibles. Je ferai précéder cette recherche de quelques lemmes, afin de ne pas interrompre par la suite l'ordre des raisonnements.

Lemmes Préliminaires.

LEMME 1^r. Le produit des carrés des différences des racines d'une équation de degré quelconque $fx = 0$, est positif ou négatif, selon que le nombre des racines imaginaires de cette équation, est $\equiv 0$ ou $\equiv 2$ (mod. 4). Supposons cette proposition vraie pour une équation d'un degré déterminé $fx = 0$, nous allons démontrer qu'elle subsiste pour la nouvelle équation

$$Fx = (x - \alpha)(x - \beta) fx = 0.$$

Soit en effet D et D, les discriminants, ou pour plus de précision, les produits des carrés des différences des racines des équations $F = 0, f = 0$, on trouvera sans difficulté

$$D = (\alpha - \beta)^2 f^2(\alpha) f^2(\beta)\ \mathrm{D}.$$

D'ou l'on voit qu'en supposant réelles les racines α et β, D et D seront de même signe, tandis qu'en les supposant imaginaires conjuguées, D et D seront de signes contraires, car la produit $f(\alpha) f(\beta)$ sera positif, et le facteur $(\alpha - \beta)^2$ négatif. La proposition annoncée se vérifie donc à l'égard de l'équation $F = 0$ si elle a lieu pour l'équation $f = 0$; ainsi elle est générale, puisqu'elle est vraie dans le cas du second degré. Les exemples suivants montreront déjà un usage de cette remarque.

Considérons une forme biquadratique,

$$f = (a, b, c, b', a')(x, y)^4 = a(x - \alpha y)(x - \beta y)(x - \gamma y)(x - \delta y),$$

soit I l'invariant du second ordre

$$aa' - 4bb' + 3c^2,$$

et D le discriminant

$$a^6(\alpha-\beta)^2(\alpha-\gamma)^2 \ldots (\gamma-\delta)^2.$$

Je dis qu'en supposant $I < 0$, la forme proposée aura deux ou quatre racines imaginaires, suivant que D sera négatif ou positif.

On a en effet, ce qui se vérifie très aisément,

$$I = aa' - 4bb' + 3c^2 = \tfrac{a^2}{2}\{(\alpha-\beta)^2(\gamma-\delta)^2 + (\alpha-\gamma)^2(\beta-\delta)^2 + (\alpha-\delta)^2(\beta-\gamma)^2\},$$

donc l'hypothèse $I < 0$, exclut le cas ou toutes les racines sont réelles, et le lemme précédent suffit pour distinguer l'un de l'autre les deux autres cas seuls possibles où le nombre des racines imaginaires est deux ou quatre. Quelque chose d'analogue a lieu aussi pour le cinquième degré, nous allons l'indiquer, bien que nous n'ayons pas à nous en servir par la suite. Soit,

$$f = (a, b, c, c', b', a')(x, y)^5 = a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)(x-\varepsilon y),$$

D le discriminant, $a^8(\alpha-\beta)^2\ldots$ et Δ, l'invariant qui figure dans nos recherches, savoir,

$$(aa' - 3bb' + 2cc')^2 - 4(ab' - 3bc' + 3c^2)(a'b - 3b'c + 3c'^2),$$

on trouvera,

$$\Delta = -\frac{7}{2^5}\,\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\alpha)^2,$$

la signe Σ se rapportant aux termes qu'on déduit de celui que nous avons écrit, par les permutations des racines.

Il s'en suit qu'on supposant Δ positif, la forme aura des racines imaginaires, et comme précédemment, elle en aura deux ou quatre, suivant que D sera négatif ou positif.

En passant remarquons encore cette relation,

$$D = a^8(\alpha-\beta)^2 \ldots (\delta-\varepsilon)^2 = 5^5(\Delta^2 + 2^7 J_3).$$

Lemme 2°. Il a été remarqué § 2, que les coefficients de la forme-type avaient pour commun dénominateur $(2J_3)^5$, d'après cela et pour plus de commodité nous considérons par la suite au lieu de ϕ, la forme $(2J_3)^5\phi$, *c. à. d.* nous ferons

$$\phi = (A, B, C, C', B', A')(\xi, \eta)^5,$$

les coefficients n'offrant plus J_3 en dénominateur, et ayant ainsi pour valeurs,

$$36A = \Delta^7 J_2 + \Delta^6 J_3 + 6\Delta^5 J_2^2 - 39\Delta^4 J_2 J_3 - 9\Delta^3(6J_3^2 - J_2^3)$$
$$- 126\Delta^2 J_3 J_2^2 + 288\Delta J_3^2 J_2 + 1152 J_3^3,$$
$$36C = \Delta^6 J_2 + \Delta^5 J_3 + 6\Delta^4 J_2^2 - 27\Delta^3 J_2 J_3 - 3\Delta^2(14J_3^2 - 3J_2^3)$$
$$- 90\Delta J_3 J_2^2 + 144 J_3^2 J_2,$$
$$36B' = \Delta^5 J_2 + \Delta^4 J_3 + 6\Delta^3 J_2^2 - 15\Delta^2 J_2 J_3 - 3\Delta(10J_3^2 - 3J_2^3)$$
$$- 54 J_3 J_2^2 ;$$
$$9B = - I(\Delta^4 + 3\Delta^2 J_2 - 24\Delta J_3),$$
$$9C' = - I(\Delta^3 + 3\Delta J_2 - 12 J_3),$$
$$9A' = - I(\Delta^2 + 3J_2).$$

Cela posé, on aura ces relations remarquables,

$$(1) \quad \begin{cases} AB' - 4BC' + 3C^2 = 16\Delta J_3^5, \\ A'B - 4B'C + 3C'^2 = - 16 J_3^5, \\ AA' - 3BB' + 2CC' = 0. \end{cases}$$

$$(2) \quad \begin{cases} A - 2\Delta C + \Delta^2 B' = 32 J_3^3, \\ B - 2\Delta C' + \Delta^2 A' = 0. \end{cases}$$

les premières résultent de l'expression du covariant quadratique de ϕ, qu'on obtient bien aisément. Effectivement, cette forme ϕ provient, par le fait de la suppression du dénominateur $(2J_3)^5$, de la transformée canonique F, par la substitution

$$\Delta(CX + C'Y) = 2J_3\xi, \quad \Delta\sqrt{\Delta}(CX - C'Y) = 2J_3\eta,$$

donc son covariant quadratique, proviendra par la même substitution du covariant $\sqrt{\Delta}XY$ rélatif à F, multiplié par la quatrième puissance du déterminant de la substitution, *c. à. d.* par $(2J_3)^4$. Cela donne pour le covariant quadratique de la forme-type, cette expression remarquable,

$$16J_3^5(\Delta\xi^2 - \eta^2),$$

d'ou l'on tire de suite les équations (1).

En recherchant de la même manière le covariant linéaire du cinquième ordre de la forme-type, on obtiendra la valeur $10(2J_3)^{13}\xi$, mais d'après le loi générale de formation (§ 1°) ce covariant sera,

$$2^8 J_3^{10}\left(\Delta^2\frac{d^4\Phi}{d\eta^4} - 2\Delta\frac{d^4\Phi}{d\xi^2 d\eta^2} + \frac{d^4\Phi}{d\xi^4}\right)$$
$$= 2^8 J_3^{10} 10\{(A - 2\Delta C + \Delta^2 B')\xi + (B - 2\Delta C' + \Delta^2 A')\eta\},$$

d'ou l'on conclut les rélations (2).

Il serait très important pour la théorie des formes du cinquième degré, de calculer comme nous venons de la faire, les valeurs d'un plus grand nombre de covariants de la forme-type, on recueillerait ainsi des éléments précieux d'observations qui pourraient éclairer la nature des rapports de ces covariants avec la forme dont ils tirent naissance. Pour le moment nous ne pouvons nous empêcher d'appeler l'attention du lecteur, sur la simplicité des équations que nous venons de trouver entre les coefficients si compliqués de la forme-type, elles vont nous donner une démonstration facile de la proposition suivante, qu'il importe d'établir pour la recherche spéciale que nous avons en vue dans ce §.

Lemme 3^e. L'équation du 4^e degré par rapport à J_2 qu'on forme en égalant à zéro l'invariant du 18^e ordre, a toujours deux racines réelles, et deux racines imaginaires.

D'après la valeur que nous avons obtenue pour I'^2, cette équation est

$$16I'^2 = \Delta^5 J_2^2 + 2\Delta^4 J_2 J_3 + \Delta^3(J_3^2 + 6J_2^3) - 18\Delta^2 J_2^2 J_3 + 9\Delta(J_2^4 - 8J_2 J_3^2) - 24(2J_3^3 + 3J_2^3 J_3) = 0,$$

ou en ordonnant par rapport à J_2,

$$9\Delta J_2^4 + 6(\Delta^2 - 12J_3)\, J_2^3 + (\Delta^5 - 18\Delta^2 J_3)\, J_2^2 + (2\Delta^4 J_3 - 72\Delta J_3^2)\, J_2 + \Delta^3 J_3^2 - 48J_3^3 = 0.$$

Elle est comme on voit assez compliquée, pour qu'on puisse hésiter y appliquer le théorème de M. Sturm, mais heureusement elle admet une transformée très simple. Effectivement pour une valeur de J_2 qui satisfait à cette équation, les coefficients A, C, B', donnent en vertu des équations (1) et (2),

$$AB' + 3C^2 = 16\Delta J_3^5,$$
$$B'C = 4J_3^5,$$
$$A - 2\Delta C + \Delta^2 B' = 32J_3^3,$$

puisque B, C', A', contenant I en facteur, s'annullent. Or en éliminant A et B', on trouve

$$3C^4 - 8\Delta J_3^5 C^2 + 128J_3^8 C - 16\Delta^2 J_3^{10} = 0.$$

Ce résultat paraitra bien remarquable, si l'on a égard à la complication de la valeur de C exprimé en fonction de J_2; quoiqu'il en soit cette fonction étant rationnelle et entière par rapport à J_2, il suffira de raisonner sur l'équation en C, et d'établir qu'elle a bien deux racines réelles et deux

racines imaginaires. Or la premier point résulte de ce que le dernier terme est essentiellement négatif, et le second, de ce qu'en faisant

$$3C^4 - 8\Delta J_3^5 C^2 + 128 J_3^8 C - 16\Delta^2 J_3^{10} = (a, b, c, b', c', a')(C, 1)^4,$$

l'invariant du second ordre,

$$aa' - 4bb' + 3c^2,$$

a la valeur négative,

$$-\tfrac{128}{3}\Delta^2 J_3^{10}. \quad \text{(Lemme 1}^{\text{r}}\text{.)}$$

Des limites entre lesquelles se trouve toujours renfermé l'invariant J_2, de toute forme du cinquième degré à coefficients réels.

La forme-type a laquelle nous avons ramené la forme générale du cinquième degré par une substitution linéaire, ne contenant plus que trois paramètres, Δ, J_2, J_3, on est naturellement conduit à étudier les racines de cette forme considerées comme fonctions de ces paramètres, tandis qu'on n'aurait jamais songé à se proposer la même question sur les racines elles-mêmes de la forme primitive, considerées comme fonctions de cinq quantités arbitraires. Mais de l'abord de cette recherche, se présente une circonstance importante. En considérant pour les coefficients de la forme proposée, des valeurs réelles, les paramètres de la forme-type, qu'on ne devra pas déjà supposer imaginaires, ne peuvent même recevoir toutes les valeurs réelles possibles. Il entre en effet dans la forme-type, l'invariant I du 18e ordre, qui doit être aussi essentiellement réel, de sorte que (étant algébriquement indépendantes) les quantités Δ, J_2, J_3, en tant qu'elles proviennent d'une forme réelle, sont assujeties à cette condition de rendre positive la fonction

$$16I^2 = 9\Delta J_2^4 + 6(\Delta^2 - 12J_3)J_2^3 + (\Delta^5 - 18\Delta^2 J_3)J_2^2 \\ + (2\Delta^4 J_3 - 72\Delta J_3^2)J_2 + \Delta^3 J_3^2 - 48J_3^3.$$

Or quels que soient Δ et J_3, nous avons démontré que l'équation $I^2 = 0$, en prenant J_2 pour inconnue, avait toujours deux racines réelles et deux racines imaginaires. Nommant donc j et j' ces racines réelles, il est aisé de voir qu'en supposant Δ positif, les valeurs de J_2 qui rendront la fonction I réelle, seront nécessairement au dehors de l'intervalle compris entre j et j', tandis qu'en supposant Δ négatif, ces valeurs seront comprises au contraire dans le même intervalle. Une observation très simple confirme c
clusion que J_2 est nécessairement limité quand Δ e
si l'on suppose en effet J_2 très grand, on tr

employant les expressions données précédemment des coefficients A, B, etc., que la forme ϕ devient sensiblement proportionnelle à $(\xi\sqrt{\Delta}+\eta)^5$, de sorte que les cinq racines, se présenteraient toutes comme imaginaires, en devenant égales à la limite, tandis qu'on sait bien que leur commune valeur doit être réelle. Ces limites que nous venons de trouver pour les valeurs de J_2 vont encore se présenter dans une circonstance importante, comme on va voir.

Des limites entre lesquelles les racines de la forme-type sont des fonctions continues de J_2 considéré comme une variable réelle.

Nous nous fonderons pour cette recherche, sur ce théorème si important dans toute l'analyse, que l'illustre géomètre M. Cauchy a démontré sous un point de vue plus général dans les *Nouveaux Exercices de Mathématiques* (tom. II. p. 109), "Les racines d'une équation algébrique dont les coefficients contiennent sous forme rationnelle un paramètre, sont des fonctions continues de ce paramètre, tant qu'en variant suivant une loi donnée, en restant toujours réel par exemple, il n'atteint pas une des valeurs particulières qui font acquérir des racines égales à l'équation proposée. Mais la quantité J_2, que nous considérons comme un paramètre variable entrant dans l'équation que nous voulons etudier, savoir,

$$(A, B, C, C', B', A')(x, 1) = 0,$$

sous un radical carré I, nous ferons $y = Ix$, ce qui donnera l'équation en y,

$$(A, IB, I^2C, I^3C', I^4B', I^5A')(y, 1)^5 = 0,$$

dont tous les coefficients sont rationnels, puisque B, C, A' contiennent déjà I en facteur. Cela posé, nous allons pour appliquer la théorème de M. Cauchy, calculer son discriminant. Or le discriminant D, de la forme primitive

$$f = (a, b, c, c', b', a')(x, y)^5,$$

se reproduisant dans toute transformée, multiplié par la 20^e puissance du déterminant de la substitution, on trouvera d'abord $(2J_2)^{20}.D$ pour le discriminant de ϕ, et $(2J_2)^{20}.I^{20}.D$, pour celui de l'équation en y. Par là nous voyons que les valeurs de J_2 pour lesquelles les racines y deviennent discontinues, sont données par les équations

$$I = 0, \quad D = \Delta^2 + 2^7 J_2 = 0.$$

Et comme le radical carré I, est aussi fonction continue

de J_2, entre les limites déterminées par l'équation $I = 0$, la relation $y = Ix$, montre qu'on peut regarder les racines x elles-mêmes, comme fonctions continues de J_2, tant que cette variable, que nous supposons réelle, n'atteint pas la valeur $-2^{-7}\Delta^2$, ou l'une des quantités nommées précédemment j et j'. Peut-être devons nous faire observer, que nous ne considerons pas un autre genre de discontinuité, le passage à l'infini d'une racine, lorsque le coefficient A s'annulle. Le raison en est, que dans le voisinage d'une valeur réelle de J_2, qui donnerait $A = 0$, les inverses des cinq racines, sont certainement des fonctions continues, et ne pourront passer du réel à l'imaginaire, ou de l'imaginaire au réel, lorsque J_2 aura atteint et dépassé la valeur particulière en question. On voit donc qu'aucun changement dans la mode d'existence des racines de la forme-type, comme quantités réelles et imaginaires, ne correspond à cette discontinuité particulière qui provient du passage par l'infini, et qu'ainsi elle n'est pas à considerer dans notre recherche.*

Sur les valeurs des racines de la forme-type, lorsque J_2 est égal à la limite j ou à la limite j'.

Nous avons précédemment distingué avec soin, dans l'ensemble des valeurs réelles de J_2, les intervalles entre lesquels cette quantité peut être regardée comme provenant d'une forme à coefficients réels. Franchir les limites assignées, sera donc considerer ce que deviennent les racines de la forme-type, pour un état imaginaire des coefficients de la forme primitive. Cependant, si nous supposons toujours J_2 réel, ces valeurs imaginaires qui viendront nécessairement s'offrir, ne seront point entiérement arbitraires, et seront soumises à des conditions spéciales. Or on va voir combien est utile la considération de ces valeurs limitées comme nous le disons, de manière que les invariants du 4^e, du 8^e et du 12^e ordre restent réels, l'invariant du 18^e étant seul affecté du facteur $\sqrt{(-1)}$. Effectivement nous allons pouvoir suivre de la manière la plus facile et la plus claire, comment les racines de la forme-type changent successivement de

* Cette consideration des inverses des racines, sert aussi à établir, quand on recherche la distribution en systèmes circulaires des racines v, d'une équation de la forme $Nv^m + Pv^{m-1} + \ldots = 0$, N, P, ... étant des polynomes entiers en z, que ces systèmes subsistent sans altération, lorsque la contour décrit par la variable z, vient à comprendre un nombre quelconque de points, auxquels correspondent des racines de l'équation $N=0$. Voyez à ce sujet le N°. 37 du mémoire de M. Puiseux, intitulé 'Recherches sur les Fonctions Algébriques,' (*Journal de M. Liouville*, tom. xv.).

nature en passant du réel à l'imaginaire ou de l'imaginaire au réel, lorsque J_2 varie de $-\infty$ à $+\infty$, et par suite établir ce que sont ces racines dans une intervalle donné, résultat important, auquel nous n'aurions pu parvenir en renonçant à ces valeurs de paramètre variable qui supposent nécessairement imaginaires les coefficients de la forme proposée. Voici pour cet objet, les dernières propositions préliminaires que nous avons à démontrer. Je dis d'abord qu'en supposant $D=0$, on aura

$$2^{32}I^2 = (25\Delta^3 - 3.2^{10}J_3)(25\Delta^3 - 2^{11}J_3)^2$$

C'est une conséquence immédiate de la formule

$$\frac{I}{\sqrt{\Delta^7}} = ACB'^2 - A'C'B^2 = \frac{48}{5}pq(2p^2+q^2),$$

donnée § 5. On trouvera en effet le résultat annoncé en élevant au carré et remplaçant p^2 et q^2 par leurs valeurs en J_3 et Δ, telles qu'elles résultent des formules de ce §. Il s'ensuit que pour $D=0$, I sera réel ou imaginaire, suivant la signe de la quantité $25\Delta^3 - 3.2^{10}J_3$, et par conséquent, le discriminant s'évanouira dans l'intervalle des valeurs admises ou des valeurs exclues de J_2, suivant que $25\Delta^3 - 3.2^{10}J_3$, sera positif ou négatif. Cela posé, je vais démontrer, que si la discriminant ne s'évanouit qu'en dehors des limites $J_2 = j$, $J_2 = j'$, les racines de la forme-type présenteront pour ces deux limites, un même nombre de quantités réelles et un même nombre de quantités imaginaires. Deux cas sont à distinguer suivant que Δ est positif ou négatif. Dans l'un et l'autre, les racines de la forme-type seront certainement entre les limites j et j' des fonctions continues de J_2, mais dans le premier il faut exclure les limites car en s'annullant le radical I passe alors brusquement de l'imaginaire au réel, et devient discontinue ; dans le second au contraire, les limites sont comprises, car le radical est réel avant de s'évanouir, et ne devient discontinue en passant à l'imaginaire, que si l'on franchit les limites j ou j'. C'est donc seulement pour ce second cas que notre proposition se trouve immédiatement établie, et sous ce point de vue, le premier exigerait une discussion que la méthode suivante évite, car il n'y figure plus de considérations de continuité.

Lorsque $I=0$, nous avons trouvé précédemment les relations

(1) $AB' + 3C^2 = 16\Delta J_3^5$, $B'C = 4J_3^5$, $A - 2\Delta C + \Delta^2 B' = 32J_3^3$,

et aussi une équation ne contenant que C, et que nous présenterons sous cette forme

$$(2)\quad (3C^2 + 4J_3^5\Delta)(C^2 - 4\Delta J_3^5) = -\,128 J_3^8 C.$$

Cela posé, il s'agit d'en déduire les valeurs des quantités qui déterminent par leurs signes la nature des racines de l'équation

$$(A,\ 0,\ C,\ 0,\ B',\ 0)\,(x,\ 1)^5 = 0.$$

Or ces quantités sont $25C^2 - 5AB'$, en premier lieu, puis les rapports $\dfrac{C}{A}$, $\dfrac{B'}{A}$, mais en leur place il sera préferable de prendre les suivantes $5C^2 - AB'$, AB', AC, ou même celles-ci

$$5C^2 - AB',\quad \frac{5C^2 - AB'}{AB'},$$

et $B'C$, ce qui est permis comme on le verra bien facilement. Mais par l'équation (1), on trouvera

$$5C^2 - AB' = 8(C^2 - 2\Delta J_3^5),$$

et

$$\frac{5C^2 - AB'}{AB'} = 8\,\frac{C^2 - 2\Delta J_3^5}{16\Delta J_3^5 - 3C^2},$$

ce qui nous conduit à déterminer la nature des racines de notre équation, par ces deux fonctions très simples,

$$u = C^2 - 2\Delta J_3^5,\quad v = \frac{C^2 - 2\Delta J_3^5}{16\Delta J_3^5 - 3C^2},$$

car il est inutile de considerer la troisième $B'C$, qui conserve absolument la même valeur pour $J_2 = j$, $J_2 = j'$.

Or en élevant au carré les deux membres de l'équation (2), on introduira partout le carré C^2, et une élimination facile alors, donnera

$$(3)\quad (3u + 10\Delta J_3^5)^2(u - 2\Delta J_3^5)^2 = 128^2 J_3^{16}(u + 2\Delta J_3^5),$$

$$(4)\quad \Delta^3(30v + 5)^2(2v - 1)^2 = 2.32^2 J_3(8v + 1)(3v + 1)^3.$$

Chacune de ces équations aura comme l'équation en C, deux racines imaginaires et deux racines réelles qui correspondent respectivement à $J_2 = j$, $J_2 = j'$, donc pour l'une et pour l'autre, les racines réelles seront de mêmes signes ou de signes contraires, suivant que le dernier terme sera positif ou négatif. Or le dernier terme de (3) est

$$(\tfrac{4}{3})^2\Delta^2 J_3^{20}(25\Delta^3 - 2^{11}J_3),$$

le dernier terme de (4),

$$12^2\,\frac{25\Delta^3 - 3.2^{10}J_3}{25\Delta^3 - 2^{11}J_3},$$

et que Δ soit positif ou négatif, il est aisé de voir que ces quantités sont positives. En effet, pour $\Delta > 0$, elles le sont évidemment si J_3 est négatif, mais si J_3 est positif, la condition $25\Delta^3 - 3.2^{10}J_3 > 0$, qui est l'hypothèse, entraine $25\Delta^3 - 2^{11}J_3 > 0$, et notre proposition est vérifiée. Enfin pour $\Delta > 0$ elle est évidente si J_3 est positif, mais si J_3 est négatif, l'hypothèse qui est alors $25\Delta^3 - 3.2^{10}J_3 < 0$, entraine $25\Delta^3 - 2^{11}J_3 < 0$. Donc aux deux limites j et j', les trois quantités qui déterminent la nature des racines de la forme-type, ont individuellement les mêmes signes, et ces racines présentent dans ces deux cas un même nombre de quantités réelles et imaginaires.

Ce que deviennent successivement les racines de la forme-type lorsque J_2 varie de $-\infty$ à $+\infty$.

Nous distinguerons quatre cas principaux dans cette recherche, que nous traiterons dans l'ordre suivant:

Premier Cas :	$\Delta > 0$,	$25\Delta^3 - 3.2^{10}J_3 > 0.$
Second :	$\Delta < 0$,	$25\Delta^3 - 3.2^{10}J_3 > 0.$
Troisième :	$\Delta > 0$,	$25\Delta^3 - 3.2^{10}J_3 < 0.$
Quatrième :	$\Delta < 0$,	$25\Delta^3 - 3.2^{10}J_3 < 0.$

Premier Cas.

Les valeurs admises de J_2 forment alors deux séries, l'une de $-\infty$ à j, la seconde de j' à $+\infty$, (en nommant j la plus petite des quantités j et j') et la condition $25\Delta^3 - 3.2^{10}\Delta^3 > 0$, signifie comme il a été dit plus haut, que le discriminant s'évanouira necessairement, pour une valeur de J_2 comprise dans l'une des séries indiquées, nous admettrons pour fixer les idées que ce soit dans la première.

Cela posé, faisons croître J_2 par degrés insensibles à partir de $-\infty$; tant que le discriminant $D = \Delta^2 + 2^7J_2$, ne viendra pas à s'annuller, les cinq racines resteront des fonctions continues, et aucun changement ne surviendra dans leur nature. Mais pour $J_2 = -2^{-7}\Delta^2$, deux d'entre elles et deux seulement deviendront égales, de sorte que dans le voisinage de cette valeur, elle pourront passer du réel à l'imaginaire

or de l'imaginaire au réel, en devenant discontinues, tandis que les trois autres resteront au contraire des fonctions continues de J_2.

En raison de cette circonstance essayons d'en déterminer la nature. Pour cela nous nous placerons précisément dans ce cas particulier ou $D=0$. Divisant la forme-type par le facteur linéaire qu'elle contient alors au carré, nous obtiendrons une forme cubique, dont il faudra calculer le discriminant. Mais dans ce but, nous pouvons remplacer la forme-type Φ, par la transformée canonique F, puisqu'elle s'en déduit en faisant une substitution au déterminant réel $2J_3$. Alors un calcul très facile qui a été exécuté §5 (in finem) conduit abstraction faite d'un facteur positif à la fonction deja consideré plus haut

$$25\Delta^3 - 2^{11}J_3.^*$$

Il a été remarqué qu'elle était positive dans ce premier cas ou nous nous trouvons maintenant, ou l'on a les conditions

$$\Delta > 0, \quad 25\Delta^3 - 3.2^{10}J_3 > 0.$$

Ainsi de ces trois racines fonctions continues de J_2, entre les limites $J_2 = -\infty$, $J_2 = j$, une seule est réelle et les deux autres sont imaginaires. Cela posé, il s'agirait de reconnaitre pour des valeurs de J_2, infiniment voisines de $-2^{-7}\Delta^2$, la nature des deux autres racines qui sont égales pour $J_2 = -2^{-7}\Delta^2$. Cette question centre dans les principes connus, mais nous pouvons l'éviter en rapellant le premier lemme ou il a été établi que la seule condition $D > 0$ assurait l'existence de deux racines imaginaires et de trois racines réelles. Puisqu'il y a dans l'équation deux racines imaginaires quelque soit J_2, il faudra que les deux racines qui deviennent égales quand le discriminant s'évanouit, soient réelles, tant qu'il est négatif, et passent en devenant discontinues à l'imaginaire, lorsqu' après s'être annullé le discriminant devient positif. Maintenant J_2 continuant à croître, la forme-type offrira toujours quatre racines imaginaires et une racine réelle jusqu' à ce qu'on parvienne à la limite $J_2 = j$, à partir de laquelle on entre dans l'intervalle des valeurs exclues du paramètre. Alors les coefficients qui contiennent en facteur le radical carré deviennent dans tout cet intervalle, imaginaires, cependant nous allons encore suivre les racines en les faisant dépendre d'une équation

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* J'ai pris suivant l'usage, le discriminant d'une forme cubique, de signe contraire au produit des carrès des différences des racines de cette forme.

à coefficients réels. Pour cela, faisans dans la proposée

$$(A, B, C, C', B', A')(x, 1)^5 = 0, \quad y = x\sqrt{(-1)},$$

nous aurons dans l'intervalle compris entre j et j', la transformée à coefficients réels

$$\{A, \sqrt{(-1)}, B, -C, -\sqrt{(-1)}\,C', B', \sqrt{(-1)}\,A'\}(y, 1)^5 = 0.$$

Dans cet intervalle, et les limites comprises les cinq racines y, seront fonctions continues de J_2, ainsi leur nature dépend de leurs valeurs initiales, par ex. pour $J_2 = j$. Mais il est bien à remarquer qu' alors, les quatre racines qui sont imaginaires, peuvent avoir leurs parties réelles nulles ; deux cas differents peuvent donc se présenter, les valeurs initiales des racines y, seront toutes réelles, ou bien quatre d'entre elles seront imaginaires et une seule réelle.

C'est une question curieuse et délicate, de reconnaître si les deux cas sont possibles, ou lequel peut seulement avoir lieu. Pour le résoudre, je remarquerai que l'équation en y, pour $J_2 = j$ par ex. est de cette forme,

$$(A, 0, -C, 0, B', 0)(y, 1) = 0,$$

et que la quantité $\frac{B'}{A}$, est nécessairement positive. En effet si elle était négative, on voit bien aisément que cette équation aurait nécessairement deux racines imaginaires et très racines réelles. Et la même chose a lieu pour $J_2 = j'$, d'ou il suit que la signe commun aux deux racines réelles de l'équation en

$$\nu = \frac{C^2 - 2\Delta J_3^5}{16\Delta J_3^5 - 3C^2} = \frac{1}{8}\,\frac{5C^2 - AB'}{AB'},$$

sera celui de la quantité $5C^2 - AB'$, aux deux limites. Or l'équation en ν, a ses deux racines positives, car son premier membre, comme nous l'avons vu, est positif pour $\nu = 0$, et par la substitution ou la trouvera négatif au contraire pour $\nu = \frac{1}{2}$, donc nous avons une racine comprise entre zero et $\frac{1}{2}$, et l'autre racine qui est nécessairement de même signe, sera donc aussi positive. Etant ainsi assurés de signe des deux quantités $5C^2 - AB'$, considérons l'équation en u, qui a pour racines

$$u = \tfrac{1}{8}(5C^2 - AB') = C^2 - 2\Delta J_3^5.$$

Cette équation est

$$(3u + 10\Delta J_3^5)^2(u - 2\Delta J_3^5)^2 = (128)^2 J_3^{10}(u + 2\Delta J_3^5),$$

or ou faisant $u = \Delta J_3^5 U$, elle deviendra

$$\Delta^3 (3U + 10)^2 (U - 2)^2 = (128)^2 J_3 (U + 2),$$

ou $$\Delta^3 (3U + 10)^2 (U - 2)^2 - (128)^2 J_3 (U + 2) = 0.$$

Mais dans cette transformée, le premier membre est positif pour $U = 0$, puisque par hypothèse on a

$$25\Delta^3 - 2^{11} J_3 > 0,$$

et pour $U = 2$, il sera négatif. Donc les deux racines réelles et de même signe de cette transformée sont positives. Or pour qu'il en soit de même, comme nous l'avons trouvé d'ailleurs des deux racines $u = \Delta J_3^5 U$, la quantité J_3 doit être positive. C'est la donc une conséquence nécessaire de la supposition faite, que le discriminant s'évanouit entre les limites $J_2 = -\infty$, $J_2 = j$, et on en déduit par l'équation $B'C = 4J_3^5$ que la rapport $\dfrac{C}{B'}$ est positif. Donc enfin l'équation en y, pour $J_2 = j$ et $J_2 = j'$, a ses racines toutes réelles, et le premier des deux cas dont nous avions admis le possibilité a seul lieu.

Au de la de la limite $J_2 = j'$, les coefficients de l'équation en x, redeviennent réels, et dans cette seconde série des valeurs admises de paramètre, jusqu'à $J_2 = +\infty$, les cinq racines restent indéfiniment des fonctions continues, et offrent toujours une quantité réelle et quatre quantités imaginaires, dont les valuers initiales sont les produits du facteur $\sqrt{(-1)}$, par des quantités réelles.

Enfin considérons le cas ou le discriminant s'évanouit entre les limites $J_2 = +\infty$, $J_2 = j''$, et faisons alors de croître la paramètre variable, de $+\infty$ à $+\infty$. Tout-à-fait comme précédemment, nous trouverons dans l'intervalle compris entre les limites, $+\infty$ et j'', trois racines qui seront fonctions continues de J_2. Deux d'entre elles seront imaginaires et la troisième réelle, à cause de la condition $25\Delta^3 - 2^{11} J_3 > 0$. Quand aux deux autres qui deviennent égales quand le discriminant s'évanouit, elles seront imaginaires tant que le discriminant D restera positif, et passeront à l'état réel en devenant discontinues, lorsque D après s'être annulé deviendra négatif. Nous parvenons ainsi à la limite $J_2 = j''$, avec deux racines imaginaires et trois racines réelles. Pour suivre ultérieurement les racines, dans l'intervalle des valeurs exclues, de $J_2 = j''$ à $J_2 = j$, nous ferons encore $y = x\sqrt{(-1)}$ ... à coefficients réels, aura dans toute ce ... fonctions continues de J_2.

Quand a leur nature, elle résulte cette fois sans ambiguité des valeurs initiales, qui offrent trois quantités réelles, et deux quantités imaginaires, produits du facteur $\sqrt{(-1)}$, multiplié par des quantités réelles. Seulement nous observerons que J_3 doit être nécessairement dans ce cas, negatif; cela résulte très facilement de la discussion faite précédemment et nous pensons inutile de nous y arrêter. Enfin lorsque la paramètre décroit de la limite j à $-\infty$, nous rétrouvons pour les cinq racines des fonctions continues, parmi les quelles deux sont imaginaires et les trois autres réelles.

Second Cas.

Les valeurs admises de J_2 forment une seule série de j à j', et la condition

$$25\Delta^3 - 3.2^{10}J_3 > 0,$$

signifie que le discriminant s'évanouit dans cet intervalle. Faisant donc croître J_2 par degrés insensibles à partir de la limite j, tant qu'on n'atteindra pas la valeur $-2^{-7}\Delta^2$, pour la quelle D s'annule, les cinq racines demeureront des fonctions continues, et aucun changement ne surviendra dans leur nature. Mais pour $D=0$, deux d'entre elles présenteront alors une discontinuité en devenant égales, tandis que les trois autres resteront des fonctions continues jusqu'à la limite j'. En raisonnant comme dans le cas précédent, on verra que la nature de ces trois racines dépend encore de l'expression $25\Delta^3 - 2^{11}J_3$, qui maintenant peut-être positive ou négative. Supposons le d'abord positive; c'est admettre dans l'intervalle compris entre j et j', l'existence de deux racines imaginaires et d'une racine réelle. Donc tant que le discriminant avant de s'évanouir restera négatif, les deux autres racines de l'équation seront réelles, est lorsque D deviendra positif après s'être annulé, elles passeront en devenant discontinues à l'état imaginaire. Ainsi donc dans ce cas, deux racines imaginaires et trois racines réelles à l'origine $J_2=j$, et quatre racines imaginaires avec une racine réelle à la limite supérieure $J_2=j'$. Maintenant si nous faisons encore $y=x\sqrt{(-1)}$, pour arriver à une transformée à coefficients réels entre les limites $J_2=j$, $J_2=-\infty$, d'une part, $J_2=j'$, $J_2=+\infty$, de l'autre, il est clair que dans ces deux intervalles les racines y, ne présenteront plus aucune discontinuité, et demeureront respectivement ce qu'elles sont aux deux origines. Or pour $J_2=j$ nous savons avoir sur les cinq racines x trois quantités

réelles et deux imaginaires, donc il en sera de même pour les racines y. Et puisqu'il en est ainsi, l'expression $5C'^2 - AB'$ est positive, alors nous en concluerons qu'elle sera négative pour $J_2 = j'$, car la dernier terme de l'équation en u, étant

$$(\tfrac{4}{5})^2 \Delta J_3^{20}(25\Delta^3 - 2^{11}J_3),$$

à cause de $\Delta < 0$, les deux racines u, sont de signes contraires. Donc les racines x présentant quatre quantités imaginaires pour $J_2 = j'$, il en sera de même des racines y.

Supposons en second lieu,

$$25\Delta^3 - 2^{11}J_3 < 0\,;$$

c'est admettre trois racines réelles comme fonctions continues de j à j'. Alors les deux autres racines qui sont égales quand D s'annulle, seront imaginaires pour $D < 0$, et deviendront réelles quand D passera à l'état positif. Ainsi comme tout-à-l'heure, deux racines imaginaires et trois racines réelles à l'origine, $J_2 = j$, mais cinq racines réelles à la limite $J_2 = j'$. Pour ce qui comme les quantités $y = x\sqrt{(-1)}$, de $J_2 = j$ à $J_2 = -\infty$, elles seront fonctions continues, et dans tout cet intervalle présenteront comme à l'origine, deux quantités imaginaires, et trois réelles. De $J_2 = j'$ à $J_2 = +\infty$, elles seront encore continues, mais une seule sera réelle, les quatre autres imaginaires, et ayant pour valeurs initiales les produits du facteur $\sqrt{(-1)}$, multiplié par des quantités réelles.

Troisième Cas.

Les deux derniers cas peuvent se ramener par la considération suivante au deux premiers.

Concevons que dans la forme-type, en change Δ et J_3 en $-\Delta$ et $-J_3$, en conservant J_2 avec son signe, on vérifier que les coefficients A, B, C, C', B', A', deviendront respectivement $-A$, $B\sqrt{(-1)}$, C, $-C'\sqrt{(-1)}$, $-B'$, $A'\sqrt{(-1)}$; donc en mettant à la place de x, $x\sqrt{(-1)}$, et multipliant encore la transformée par $\sqrt{(-1)}$, on trouvera exactement le même résultat qu'en changeant les signes des invariants Δ et J_3. Or les conditions caractéristiques des deux derniers cas savoir,

$$\Delta > 0,\ \ 25\Delta^3 - 3.2^{10}J_3 < 0,\quad \text{et}\quad \Delta < 0,\ \ 25\Delta^3 - 3.2^{10}J_3 < 0,$$

reproduisant par le changement de signe de Δ et J_3, celles des deux premiers. Ainsi du second, nous allons déduire la troisième, et du premier la quatrième, avec ce seul

changement, que tout ce qui a été dit des quantités x et y, devra être transporté aux quantités y et x, x étant toujours l'inconnue de l'équation proposée, et y désignant $x\sqrt{(-1)}$. Cela donne les conclusions suivantes, en comment par le 3ᵉ cas. Alors les valeurs admises du paramètre, formant les deux séries de $-\infty$ à j et de j' à $+\infty$.

Dans la première, des cinq racines x, deux sont imaginaires et trois réelles, dans la seconde, quatre sont imaginaires, une seule est réelle, et d'ailleurs dans les deux séries elles restent toutes fonctions continues du paramètre. Pour les racines y, c'est dans l'intervalle compris de j à j' qu'elle dépendent d'une équation à coefficients réels, et deux cas sont à distinguer suivant que $25\Delta^3 - 2^{11}J_3$ est négatif ou positif. Dans le premier, sur les trois racines qui sont fonctions continues de j à j', une est réelle, et deux sont imaginaires. Quand aux deux autres racines qui deviennent discontinues pour $D = 0$, elles sont réelles si D est négatif et imaginaires lorsque D est positif. Enfin si $25\Delta^3 - 2^{11}J_3$ est positif, les trois racines qui sont fonctions continues sont réelles, et les deux autres sont imaginaires pour $D < 0$ et réelles pour $D > 0$.

Quatrième Cas. Résumé.

En nous bornant pour abréger aux racines x, on voit

$J_2 = j'$ à $J_2 = +\infty$. Ces remarques faites, nous pouvons maintenant rapprocher les divers résultats que nous venons d'obtenir ; nous formerons ainsi le tableau suivant, qui offre l'expression par les invariants fondamentaux, du nombre des racines réelles et imaginaires de l'équation génerale du cinquième degré.

$$\Delta^2 + 2^7 J_2 < 0 \ldots\ldots$$

trois racines réelles, deux racines imaginaires.

$$\Delta^2 + 2J_2^7 > 0 \begin{cases} \Delta < 0,\ 25\Delta^3 - 3.2^{10} J_3 < 0,\ J_3 > 0 \\ \hfill \text{Cinq racines réelles,} \\ \Delta < 0,\ 25\Delta^3 - 3.2^{10} J_3 > 0,\ 25\Delta^3 - 2^{10} J_3 < 0 \\ \hfill \text{Cinq racines réelles,} \\ \Delta > 0, \ldots \text{Une racine réelle, quatre imaginaires,} \\ \Delta < 0,\ 25\Delta^3 - 3.2^{10} J_3 > 0,\ 25\Delta^3 - 2^{11} J_3 > 0 \\ \hfill \text{Une racine réelle, quatre imaginaires.} \end{cases}$$

On comprend facilement, comment dans certaine cas la nombre des conditions a pu se réduire. Par exemple, avec $\Delta^2 + 2^7 J_2 > 0$ et $\Delta > 0$, on trouve une racine réelle et quatre racines imaginaires, lorsque $25\Delta^3 - 3.2^{10} J_3$ est positif, et aussi lorsqu'il est négatif, on peut donc ne conserver que les deux premières conditions. Enfin nous remarquerons, dans l'un des cas ou il y a cinq racines réelles, que les conditions $\Delta < 0$, $J_3 > 0$, entrainent la suivante, $25\Delta^3 - 3.2^{10} J_3 < 0$, qu'on pourra supprimer si l'on veut. La simplicité de ces résultats ne semble-t-elle pas indiquer que la théorème de M. Sturm, si beau dans sa généralité, est loin de fournir l'expression définitive, des conditions de réalité des racines des équations algébriques ?

Section VII.

Sur la réduite du 6^e^ degré de l'équation générale du 5^e^ degré.

Lagrange a fait voir que la résolution par radicaux de l'équation du 5^e^ degré, dépend lorsqu'elle est possible, de la détermination d'une racine commensurable, d'une équation du 6^e^ degré dont les coefficients dépendent rationnellement de ceux de la proposée. Mais jamais le calcul de cette réduite du 6^e^ degré n'a été effective en général. La raison en est que les fonctions du cinq lettres les plus simples qui n'ont que six valeurs, étant au moins du second degré par rapport à l'une de ces lettres, les coefficients de

la réduite se présenteriont comme des fonctions des cinq coefficients de l'équation proposée, montant jusqu'au 12º degré, et contiendraient par suite plusieurs certaines de termes. Or on va voir qu'on peut vaincre cette difficulté à l'aide des résultats que nous avons obtenus sur les invariants des formes du 5ᵉ degré. Faisons en effet

$$f=(a, b, c, c', b', a')(x, y)^5=(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)(x-\varepsilon y),$$

et considérons la fonction suivant des racines

$$J = a^4(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\alpha)^2 \\ + a^4(\alpha-\gamma)^2(\beta-\delta)^2(\gamma-\varepsilon)^2(\delta-\alpha)^2(\varepsilon-\beta)^2,$$

on reconnaitre bien facilement qu'elle est susceptible seulement de six valeurs, et en second lieu qu'elle est un invariant de la forme f. Il en résulte que les coefficients de l'équation du 6ᵉ degré en t, seront des fonctions rationnelles et entières de cas invariants fondamentaux, Δ, J_2, J_3, car l'invariant du 18ᵉ ordre n'y entrera pas, les degrés par rapport aux coefficients de f étant multiples de 4.

Ainsi qu'on représente cette équation en t par

$$t^6+(1)\,t^5+(2)\,t^4+(3)\,t^3+(4)\,t^2+(5)\,t+(6)=0,$$

(1), (2), etc. seront respectivement des fonctions linéaires des quantités plancées en regard, dans le tableau suivant:

(1) Δ

que l'une des valeurs de ces fonctions des racines qui sans être symétrique par rapport à cinq d'entre elles n'ont cependant que six déterminations possibles, est alors nécessairement rationnelle.

Je ne terminerai pas ces recherches sur les formes du cinquième degré, sans rappeler que mon ami M. Sylvester, avait obtenu avant moi, dans son beau mémoire sur le calcul des formes, la notion des invariants du 4^e^, du 8^e^, et du 12^e^ ordre. En donnant aux formes du cinquième degré, cette expression élégante,

$$ax^5 + by^5 + cz^5,$$

sous la condition $x + y + z = 0$, M. Sylvester a trouve pour ces invariants les valeurs

$$a^2b^2 + a^2c^2 + b^2c^2 - 2abc(a+b+c), \quad a^2b^2c^2(ab+ac+bc), \quad a^4b^4c^4,$$

qui sont des fonctions symétriques très simples des trois éléments a, b, c. Enfin l'invariant du 18^e^ ordre qui joue un role si important dans ma théorie, s'est aussi présenté dans ses recherches, élevé au carré et indiquant lorsqu'il s'évanouit, l'impossibilité de la réduction à la forme citée,

$$ax^5 + by^5 + cz^5.$$

Exprimé en a, b, c, il a pour valeur

$$a^5b^5c^5(a-b)(a-c)(b-c),$$

expression encore bien simple, et qui montre sous des points de vues très différents comment on est conduit aux mêmes notions analytiques, dans cette vaste et féconde théorie des formes.

ON THE SEPARATION OF THE VARIABLES IN DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

By W. H. L. Russell.

The method of solving differential equations of the first order by separating the variables, was introduced soon after the invention of the infinitesimal calculus. John Bernoulli applied it to the integration of homogeneous equations. The process underwent great improvement in the hands of Euler. One of the equations he treated was the following:

$$ydx(c + nx) - dy(y + a + bx + nx^2) = 0,$$

in which the variables are separated by putting

$$y = \frac{z(a + bx + cx^2)}{c + nx - z}.$$

Since his time Jacobi has given the complete integral of the differential equation

$$(A + A'x + A''y)(x\,dy - y\,dx) - (B + B'x + B''y)\,dy + (C + C'x + C''y)\,dx = 0,$$

by a very elegant process which will be found in *Crelle's Journal.* I propose in the following paper to investigate more completely than has hitherto been done, a method by which we may ascertain how far such equations are integrable by the assumption

$$y = \frac{p_1 + p_2 z + p_3 z^2 + \dots}{q_1 + q_2 z + q_3 z^2 + \dots}$$

Let the differential equation be

$$U\frac{dy}{dx} = V, \text{ where } U \text{ and } V \text{ are functions of } (x) \text{ and } (y).$$

Let $y = \dfrac{p + qz}{r + sz}$, where p, q, r, s are functions of (x).

We may expand (v) in terms of (z), ascending in powers of (z), and put

$$v = v_0 + v_1 z + v_2 z^2 + v_3 z^3 + \&c.$$

Now in the transformed equation the variables are to be separated. Hence it must be of the form $QXZ = QX'Z'$, where Q is a function of (x) and (z); X, X' are functions of (x) only, Z, Z' are functions of (z) only. Consequently we shall have

$$u = w(r + sz)^2 (c_0 + c_1 z + c_2 z^2 + \&c.) \ldots\ldots\ldots\ldots (\text{I}).$$

or $$u = w(r + sz)(c_0 + c_1 z + c_2 z^2 + \ldots) \ldots\ldots\ldots\ldots (\text{II}),$$

or else $$u = w(c_0 + c_1 z + c_2 z^2 + \ldots) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{III}),$$

where (w) is a function of (x) only. Let us take the form (I). Then the equation becomes

$$w(rq - sp)(c_0 + c_1 z + \ldots)\frac{dz}{dx} = v_0 + c_0\left(p\frac{dr}{dx} - r\frac{dp}{dx}\right)w$$

$$+\left\{v_1 + wc_1\left(p\frac{dr}{dx} - r\frac{dp}{dx}\right) + w\left(q\frac{dr}{dx} + p\frac{ds}{dx} - s\frac{dp}{dx} - r\frac{dq}{dx}\right)\right\}z + \&c.$$

$= X_0 + X_1 z + X_2 z^2 + \ldots$ when X_0, X_1 ... are functions of (x). Hence the condition that the variables be separated in the second member of the equation in easily seen to be

$A_0 X_0 = A_1 X_1 = A_2 X_2 = \ldots$ when A_0, $A_1 \ldots$ are any constants.

Since X_0, X_1, X_2 are rational and entire functions of (x), we may put

$$X_0 = a_0 + b_0 x + c_0 x^2 + \ldots, \quad X_1 = a_1 + b_1 x + c_1 x^2 + \&c.,$$
$$X_2 = a_2 + b_2 x + c_2 x^2 + \ldots$$

Hence we see that the conditions of separation are

$$\frac{b_0}{a_0} = \frac{b_1}{a_1} = \frac{b_2}{a_2} = \ldots \qquad \frac{c_0}{a_0} = \frac{c_1}{a_1} = \frac{c_2}{a_2} = \ldots$$

I need scarcely remark that the variables are already separated in the first member of the equation. Hence these equations contain the solution of the Problem, under the condition implied by equation (I). In cases (II) and (III) the investigation will be exactly analogous.

I shall now proceed to some examples. For the present I shall assume p, q, r, s to be of the first degree and to be given by the following equations:

$$p = a + bx, \quad q = a' + b'x, \quad r = m + nx, \quad s = m' +$$

Let the differential equation

$$(rq - sp)^2 \frac{dy}{dx} = (\mu + \phi x)\, q^2 - (\mu' + \phi' x)\, pq + (\mu'' + \phi'' x)\, p^2$$
$$- \left\{2(\mu + \phi x)\, qs - (\mu' + \phi' x)(rq + sp) + 2(\mu'' + \phi'' x)\, rp\right\} y$$
$$+ \left\{(\mu + \phi x)\, s^2 - (\mu' + \phi' x)\, rs + (\mu'' + \phi'' x)\, r^2\right\} y^2.$$

Then the variables are separated by assuming $y = \dfrac{p + qz}{r + sz}$, provided that

$$\frac{\mu + na - mb}{\phi} = \frac{\mu' + na' + n'a - m'b - mb'}{\phi'} = \frac{\mu'' + a'n' - m'b'}{\phi''},$$

and the transformed equation is

$$\phi(rq - sp)\frac{dz}{dx} = (\mu + na - mb + \phi x)(\phi + \phi' z + \phi'' z^2).$$

Let the differential equation be

$$(q - sy)(rq - sp)\frac{dy}{dx} = \mu q^2 - \mu' pq + \mu'' p^2$$
$$- \left\{2\mu qs - \mu'(rq + sp) + 2\mu'' rp\right\} y + \left\{\mu s^2 - \mu' rs + \mu'' r^2\right\} y^2.$$

Then the variables are separated by the same assumption as before, provided that

$$\frac{\mu m + na - mb}{\mu n} = \frac{\mu m' + \mu' m + a'n + n'a - bm' - mb'}{\mu n' + \mu' n}$$
$$= \frac{\mu' m' + \mu'' m + a'n' - b'm'}{\mu n' + \mu'' n} = \frac{m'}{n'},$$

and the transformed equation is

$$\mu n(rq - sp)\frac{dz}{dx} = (\mu m + na - mb + \mu n x)$$
$$\left\{\mu n + (\mu n' + \mu' n)\, z + (\mu' n' + \mu'' n)\, z^2 + \mu'' n z^3\right\}.$$

As instances of this transformation, I take the following particular cases of the equation which was mentioned at the commencement of this paper as treated by Jacobi.

$$(1 + xy)\frac{dy}{dx} = 1 - y + y^2.$$

t, $y = \dfrac{1 + x + z}{1 - xz}$, and we have

$$(1 + x + x^2)\frac{dz}{dx} = - xz(1 + z + z^2).$$

Again, let $(1+xy)\dfrac{dy}{dx} = 2 - 2y + y^2.$

Let $y = \dfrac{1+2x-z}{1+x+xz}$, and we have

$$(1+2x+2x^2)\frac{dz}{dx} = -x(1+z)(1+z^2).$$

Let the equation be

$$x(q-sy)(rq-sp)\frac{dy}{dx} = (\mu+\phi x)q^2 - (\mu'+\phi' x)pq + (\mu''+\phi'' x)p^2$$
$$-\{2(\mu+\phi x)qs - (\mu'+\phi' x)(rq+sp) + 2(\mu''+\phi'' x)rp\}y$$
$$+\{(\mu+\phi x)s^2 - (\mu'+\phi' x)rs + (\mu''+\phi'' x)r^2\}y^2.$$

Then if

$$\frac{\mu n + \phi m + an - bm}{m\mu} = \frac{m'\phi + n'\mu + m\phi' + n\mu' + a'n + an' - m'b - mb'}{m'\mu + m\mu'}$$
$$= \frac{m'\phi' + n'\mu' + m\phi'' + n\mu'' + n'a' - m'b'}{m'\mu' + m\mu''} = \frac{m'\phi'' + \mu''n'}{m'\mu''},$$

and $$\frac{n\phi}{m\mu} = \frac{n'\phi + n\phi'}{m'\mu + m\mu'} = \frac{n'\phi' + n\phi''}{m'\mu' + m\mu''} = \frac{n'\phi''}{m'\mu''},$$

the transformed equation will be

$$m\mu x(rq-sp)\frac{dz}{dx} = \{m\mu + (n\mu + \phi m + an - bm)x + n\phi x^2\}$$
$$\{m\mu + (m'\mu + m\mu')z + (m'\mu' + m\mu'')z^2 + m'\mu'' z^3\}.$$

Let the equation be

$$(rq-sp)(ry-p)\frac{dy}{dx} = \mu q^2 - \mu' pq + \mu'' p^2$$
$$-\{2\mu qs - \mu'(rq+sp) + 2\mu'' rp\}y + (\mu s^2 - \mu' rs + \mu'' r^2)y^2.$$

Then if $y = \dfrac{p+qz}{r+sz}$, and

$$\frac{m}{n} = \frac{m\mu' + m'\mu + na - mb}{\mu' n + \mu n'} = \frac{m\mu' + m\mu'' + na' + n'a - bm' - b'm}{n'\mu' + n\mu''}$$
$$= \frac{m'\mu'' + n'a' - m'b'}{\mu'' n'},$$

the transformed equation will be

$$\mu nz(rq - sp)\frac{dz}{dx} = \{\mu(m + nx)\}$$
$$\{\mu n + (\mu' n + \mu n')z + (n'\mu' + n\mu'')z^2 + \mu'' n' z^3\}.$$

Let

$$x(rq - sp)(ry - p)\frac{dy}{dx} = (\mu + \phi x)q^2 - (\mu' + \phi' x)qp + (\mu'' + \phi'' x)p^2$$
$$- \{2(\mu + \phi x)qs - (\mu' + \phi' x)(rq + sp) + 2(\mu'' + \phi'' x)rp\}y$$
$$+ \{(\mu + \phi x)s^2 - (\mu' + \phi' x)rs + (\mu'' + \phi'' x)r^2\}y^2.$$

Then, if $\dfrac{m\mu}{n\phi} = \dfrac{m\mu' + m'\mu}{n\phi' + n'\phi} = \dfrac{m\mu'' + \mu' m'}{n\phi'' + \phi' n} = \dfrac{\mu'' m'}{\phi'' n'}$,

$$\frac{\mu n + \phi m}{n\phi} = \frac{n\mu' + m\phi' + m'\phi + n'\mu + na - mb}{n\phi' + n'\phi}$$
$$= \frac{\mu'' n + \phi'' m + \phi' m' + n'\mu' + a'n + an' - m'b - mb'}{n\phi'' + n'\phi'}$$
$$= \frac{\mu'' n' + \phi'' m' + n'\phi' - b'm'}{\phi'' n'},$$

the transformed equation will be

$$n\phi\, xz(rq - sp)\frac{dz}{dx} = (m + nx)(\mu + \phi x)$$
$$\{n\phi + (n\phi' + \phi n')z + (n\phi'' + \phi' n')z^2 + \phi'' n' z^3\}.$$

Let us now assume the quantities $p, q, r, s,$ to rise above the first degree, and let $p = a + bx + cx^2$, $q = a' + b'x + c'x^2$, $r = m + nx + hx^2$, $s = m' + n'x + h'x^2$), and let the equation be

$$(q - sy)(rq - sp)\frac{dy}{dx} = (\mu + \phi x)q^2 - (\mu' + \phi' x)pq + (\mu'' + \phi'' x)p^2$$
$$- \{2(\mu + \phi x)qs - (\mu' + \phi' x)(rq + sp) + (\mu'' + \phi'' x)rp\}y$$
$$+ \{(\mu + \phi x)s^2 - (\mu' + \phi' x)rs + (\mu'' + \phi'' x)r^2\}y^2.$$

Then, if

$$\frac{\mu m + an - bm}{\phi h} = \frac{\mu' m + \mu m' + a'n + an' - m'b - mb'}{\phi' h + \phi h'}$$
$$= \frac{a'n' - b'm' + \mu' m' + \mu'' m}{\phi' h' + \phi'' h} = \frac{\mu'' m'}{\phi'' h'},$$

Again, let $(1 + xy)\dfrac{dy}{dx} = 2 - 2y + y^2$.

Let $y = \dfrac{1 + 2x - z}{1 + x + xz}$, and we have

$$(1 + 2x + 2x^2)\frac{dz}{dx} = -x(1 + z)(1 + z^2).$$

Let the equation be

$$x(q - sy)(rq - sp)\frac{dy}{dx} = (\mu + \phi x)q^2 - (\mu' + \phi' x)pq + (\mu'' + \phi'' x)p^2$$

$$- \{2(\mu + \phi x)qs - (\mu' + \phi' x)(rq + sp) + 2(\mu'' + \phi'' x)rp\}y$$

$$+ \{(\mu + \phi x)s^2 - (\mu' + \phi' x)rs + (\mu'' + \phi'' x)r^2\}y^2.$$

Then if

$$\frac{\mu n + \phi m + an - bm}{m\mu} = \frac{m'\phi + n'\mu + m\phi' + n\mu' + a'n + an' - m'b - mb'}{m'\mu + m\mu'}$$

$$= \frac{m'\phi' + n'\mu' + m\phi'' + n\mu'' + n'a' - m'b'}{m'\mu' + m\mu''} = \frac{m'\phi'' + \mu''n'}{m'\mu''},$$

and $$\frac{n\phi}{m\mu} = \frac{n'\phi + n\phi'}{m'\mu + m\mu'} = \frac{n'\phi' + n\phi''}{m'\mu' + m\mu''} = \frac{n'\phi''}{m'\mu''},$$

the transformed equation will be

$$m\mu x(rq - sp)\frac{dz}{dx} = \{m\mu + (n\mu + \phi m + an - bm)x + n\phi x^2\}$$

$$\{m\mu + (m'\mu + m\mu')z + (m'\mu' + m\mu'')z^2 + m'\mu'' z^3\}.$$

Let the equation be

$$(rq - sp)(ry - p)\frac{dy}{dx} = \mu q^2 - \mu' pq + \mu'' p^2$$

$$- \{2\mu qs - \mu'(rq + sp) + 2\mu'' rp\}y + (\mu s^2 - \mu' rs + \mu'' r^2)y^2.$$

Then if $y = \dfrac{p + qz}{r + sz}$, and

$$\frac{m}{n} = \frac{m\mu' + m'\mu + na - mb}{\mu'n + \mu n'} = \frac{m\mu' + m\mu'' + na' + \ldots}{n'\mu' + \ldots}$$

$$= \frac{m'\mu'' + n'a' - m'b'}{\mu''n'},$$

Let

$$x^3(rq-sp)^2\frac{dy}{dx}=(\mu+\phi x^4)q^2-(\mu'+\phi' x^4)qp+(\mu''+\phi'' x^4)p^2$$
$$-\{2(\mu+\phi x^4)qs-(\mu'+\phi' x^4)(rq+sp)+2(\mu''+\phi'' x^4)rp\}y$$
$$+\{(\mu+\phi x^4)s^2-(\mu'+\phi' x^4)rs+(\mu''+\phi'' x^4)r^2\}y^2.$$

Then, if

$$\frac{na-mb}{\mu}=\frac{na'+n'a-bm'-b'm}{\mu}=\frac{a'n'-b'm'}{\mu''},$$

$$\frac{ah-cm}{\mu}=\frac{a'h+h'a-cm'-c'm}{\mu'}=\frac{a'h'-c'm'}{\mu''},$$

$$\frac{\phi+bh-nc}{\mu}=\frac{\phi'+b'h-c'n+bh'-cn'}{\mu'}=\frac{\phi''+b'h'-c'n'}{\mu''},$$

the transformed equation will be

$$\mu x^2(rq-sp)\frac{dz}{dx}=\{\mu+(an-bm)x^2+2(ah-cm)x^3$$
$$+(\phi+bh-nc)x^4\}\{\mu+\mu' z+\mu'' z^2\}.$$

Let $(rq-sp)(ry-p)^2\dfrac{dy}{dx}=\mu(q-sy)^3+\mu'(ry-p)^3.$

Then, if

$$\frac{m}{n}=\frac{m'}{n'}=\frac{an-bm}{2(ah-cm)}=\frac{m\mu'+a'n+an'-m'b-bm'}{\mu'n+2(a'h+ah'-m'c-c'm)}$$
$$=\frac{m'\mu'+a'n'-b'm'}{n'\mu'+2(a'h'-c'm')},$$

$$\frac{m}{h}=\frac{m'}{h'}=\frac{an-bm}{bh-cn}=\frac{m\mu'+a'n+an'-m'b-bm'}{\mu'h+b'h+bh'-cn'-c'n}$$
$$=\frac{m'\mu'+a'n'-b'm'}{\mu'h'+b'h'-c'n'},$$

the transformed equation will be

$$\mu m z^2(rq-sp)\frac{dz}{dx}=\mu(m+nx+hx^2)$$
$$\{m\mu+m'\mu z+(an-bm)z^2+(m\mu'+a'n+an'-m'b-mb')z^3$$
$$+(m'\mu'+a'n'-b'm')z^4\}.$$

Let now the differential equation be

$$x^2(q-sy)(rq-sp)\frac{dz}{dx}=(\mu+\phi x^2)q^2-(\mu+\phi x^2)qp+(\mu+\phi x^2)p^2$$
$$-\left\{2(\mu+\phi x^2)qs-(\mu'+\phi'x^2)(rq+sp)+2(\mu''+\phi''x^2)rp\right\}y$$
$$+\left\{(\mu+\phi x^2)s^2-(\mu'+\phi'x^2)rs+(\mu''+\phi''x^2)r^2\right\}y^2.$$

Then, if $\quad \dfrac{n}{m}=\dfrac{n\mu'+n'\mu}{m\mu'+m'\mu}=\dfrac{n\mu''+n'\mu'}{m\mu''+m'\mu'}=\dfrac{n'}{\mu''},$

$$na-mb=a'n+an'-m'b-mb'=a'n'-m'b'=0,$$

$$\frac{m\phi-2cm}{m\mu}=\frac{m\phi'+m'\phi-2cm'-2c'm}{m\mu'+m'\mu}=\frac{m\phi''+m'\phi'-2c'm'}{m\mu''+m'\mu'}=\frac{m'\phi''}{\mu''m'}$$

$$-\frac{cn}{m\mu}=\frac{n\phi'+n'\phi-cn'-c'n}{m\mu'+m'\mu}=\frac{n\phi''+n'\phi'-c'n'}{m\mu''+m'\mu'}=\frac{\phi''n'}{\mu''m'}.$$

Then, if $y=\dfrac{p+qz}{r+sz}=\dfrac{a+bx+cx^2+(a'+b'x+c'x^2)z}{m+nx+(m'+n'x)z}$,

the transformed equation will be

$$m\mu x^2(rq-sp)\frac{dz}{dx}=\left\{m\mu+n\mu x+(n\phi-2cm)x^2-cnx^4\right\}$$
$$\left\{m\mu+(m\mu'+\mu m')z+(m\mu''+m'\mu')z^2+\mu''m'z^3\right\}.$$

I now proceed to investigate the criterion of integrability when we assume

$$y=\frac{p_1+p_2z+p_3z^2}{r_1+r_2z+r_3z^2}.$$

Let $U\dfrac{dy}{dx}=V$, as before, be the given equation; and let $\dfrac{V}{U}=\dfrac{v}{u}$, where (v) and (u) are rational and entire functions of (x) and (z). The transformed equation will be

$$u\left\{(r_1p_2-p_1r_2)+2(r_1p_3-p_1r_3)z+(p_2r_3-r_2p_3)z^2\right\}\frac{dz}{dx}$$
$$=(r_1+r_2z+r_3z^2)^2v$$
$$+\left\{p_1\frac{dr_1}{dx}-r_1\frac{dp_1}{dx}\right\}u+\left\{p_1\frac{dr_2}{dx}+p_2\frac{dr_1}{dx}-r_1\frac{dp_2}{dx}-r_2\frac{dp_1}{dx}\right\}uz$$
$$+\left\{p_2\frac{dr_2}{dx}+p_1\frac{dr_3}{dx}+p_3\frac{dr_1}{dx}-r_2\frac{dp_2}{dx}-r_3\frac{dp_1}{dx}-r_1\frac{dp_3}{dx}\right\}uz^2$$
$$+\left\{p_2\frac{dr_3}{dx}+p_3\frac{dr_2}{dx}-r_2\frac{dp_3}{dx}-r_3\frac{dp_2}{dx}\right\}uz^3+\left\{p_3\frac{dr_3}{dx}-r_3\frac{dp_3}{dx}\right\}uz^4.$$

Let

$$x^3(rq-sp)^2\frac{dy}{dx}=(\mu+\phi x^4)q^2-(\mu'+\phi'x^4)qp+(\mu''+\phi''x^4)p^2$$
$$-\{2(\mu+\phi x^4)qs-(\mu'+\phi'x^4)(rq+sp)+2(\mu''+\phi''x^4)rp\}y$$
$$+\{(\mu+\phi x^4)s^2-(\mu'+\phi'x^4)rs+(\mu''+\phi''x^4)r^2\}y^2.$$

Then, if

$$\frac{na-mb}{\mu}=\frac{na'+n'a-bm'-b'm}{\mu}=\frac{a'n'-b'm'}{\mu''},$$
$$\frac{ah-cm}{\mu}=\frac{a'h+h'a-cm'-c'm}{\mu'}=\frac{a'h'-c'm'}{\mu''},$$
$$\frac{\phi+bh-nc}{\mu}=\frac{\phi'+b'h-c'n+bh'-cn'}{\mu'}=\frac{\phi''+b'h'-c'n'}{\mu''},$$

the transformed equation will be

$$\mu x^2(rq-sp)\frac{dz}{dx}=\{\mu+(an-bm)x^2+2(ah-cm)x^3$$
$$+(\phi+bh-nc)x^4\}\{\mu+\mu'z+\mu''z^2\}.$$

Let $(rq-sp)(ry-p)^2\dfrac{dy}{dx}=\mu(q-sy)^3+\mu'(ry-p)^3.$

Then, if

$$\frac{m}{n}=\frac{m'}{n'}=\frac{an-bm}{2(ah-cm)}=\frac{m\mu'+a'n+an'-m'b-bm'}{\mu'n+2(a'h+ah'-m'c-c'm)}$$
$$=\frac{m'\mu'+a'n'-b'm'}{n'\mu'+2(a'h'-c'm')},$$
$$\frac{m}{h}=\frac{m'}{h'}=\frac{an-bm}{bh-cn}=\frac{m\mu'+a'n+an'-m'b-bm'}{\mu'h+b'h+bh'-cn'-c'n}$$
$$=\frac{m'\mu'+a'n'-b'm'}{\mu'h'+b'h'-c'n'},$$

the transformed equation will be

$$\mu mz^3(rq-sp)\frac{dz}{dx}=\mu(m+nx+hx^2)$$
$$\{m\mu+m'\mu z+(an-bm)z^2+(m\mu'+a'n+an'-m'b-mb')z^3$$
$$+(m'\mu'+a'n'-b'm')z^4\}.$$

Let now the differential equation be

$$x^2(q-sy)(rq-sp)\frac{dz}{dx}=(\mu+\phi x^2)q^2-(\mu+\phi x^2)qp+(\mu+\phi x^2)p^2$$
$$-\left\{2(\mu+\phi x^2)qs-(\mu'+\phi' x^2)(rq+sp)+2(\mu''+\phi'' x^2)rp\right\}y$$
$$+\left\{(\mu+\phi x^2)s^2-(\mu'+\phi' x^2)rs+(\mu''+\phi'' x^2)r^2\right\}y^2.$$

Then, if $\dfrac{n}{m}=\dfrac{n\mu'+n'\mu}{m\mu'+m'\mu}=\dfrac{n\mu''+n'\mu'}{m\mu''+m'\mu'}=\dfrac{n'}{\mu''}$,

$$na-mb=a'n+an'-m'b-mb'=a'n'-m'b'=0,$$

$$\frac{m\phi-2cm}{m\mu}=\frac{m\phi'+m'\phi-2cm'-2c'm}{m\mu'+m'\mu}=\frac{m\phi''+m'\phi'-2c'm'}{m\mu''+m'\mu'}=\frac{m'\phi''}{\mu''m'}$$
$$-\frac{cn}{m\mu}=\frac{n\phi'+n'\phi-cn'-c'n}{m\mu'+m'\mu}=\frac{n\phi''+n'\phi'-c'n'}{m\mu''+m'\mu'}=\frac{\phi''n'}{\mu''m'}.$$

Then, if $y=\dfrac{p+qz}{r+sz}=\dfrac{a+bx+cx^2+(a'+b'x+c'x^2)z}{m+nx+(m'+n'x)z}$,

the transformed equation will be

$$m\mu x^2(rq-sp)\frac{dz}{dx}=\left\{m\mu+n\mu x+(n\phi-2cm)x^3-cnx^4\right\}$$
$$\left\{m\mu+(m\mu'+\mu m')z+(m\mu''+m'\mu')z^2+\mu''m'z^3\right\}.$$

I now proceed to investigate the criterion of integrability when we assume

$$y=\frac{p_1+p_2z+p_3z^2}{r_1+r_2z+r_3z^2}.$$

Let $U\dfrac{dy}{dx}=V$, as before, be the given equation; and let $\dfrac{V}{U}=\dfrac{v}{u}$, where (v) and (u) are rational and entire functions of (x) and (z). The transformed equation will be

$$u\left\{(r_1p_2-p_1r_2)+2(r_1p_3-p_1r_3)z+(p_3r_2-r_3p_2)z^2\right\}\frac{dz}{dx}$$
$$=(r_1+r_2z+r_3z^2)^2v$$
$$+\left\{p_1\frac{dr_1}{dx}-r_1\frac{dp_1}{dx}\right\}u+\left\{p_1\frac{dr_2}{dx}+p_2\frac{dr_1}{dx}-r_1\frac{dp_2}{dx}-r_2\frac{dp_1}{dx}\right\}uz$$
$$+\left\{p_2\frac{dr_2}{dx}+p_1\frac{dr_3}{dx}+p_3\frac{dr_1}{dx}-r_2\frac{dp_2}{dx}-r_3\frac{dp_1}{dx}-r_1\frac{dp_3}{dx}\right.$$
$$+\left\{p_2\frac{dr_3}{dx}+p_3\frac{dr_2}{dx}-r_2\frac{dp_3}{dx}-r_3\frac{dp_2}{dx}\right\}uz^3+\left\{p_3\frac{dr_3}{dx}\right.$$

Let $v = v_0 + v_1 z + v_2 z^2 + \dots$, when v_0, v_1, v_2 are functions of (x), $u = u_0 + u_1 z + u_2 z^2 + \dots$

Then, if we substitute these in the given equation, we may write it thus:

$$(x_0 + x_1 z + {}'x_2 z^2 + \dots)\frac{dz}{dx} = (x_0' + x_1' z + x_2' z^2 + \dots).$$

Then, in order that the variables in this equation may be separated, we put

$$Ax_0 = Bx_1 = Cx_2 = \&\text{c.},$$
$$A'x_0' = B'x_1' = C'x_2' = \&\text{c.},$$

where A, B, &c., A', B', &c. are any constants. Then, by equating like powers of (x) in these equations, we obtain the required conditions of integrability.

Let the differential equation be

$$(r_1 + r_2\rho + r_3\rho^2)^2 \frac{dy}{dx} = (\mu + \phi x) + (\mu' + \phi' x)\rho + (\mu'' + \phi'' x)\rho^2$$
$$+ (\mu^{(3)} + \phi^{(3)} x)\rho^3 + (\mu^{(4)} + \phi^{(4)} x)\rho^4,$$

where (ρ) is a root of the equation

$$\rho^2(r_3 y - p_3) + \rho(r_2 y - p_2) + (r_1 y - p_1) = 0,$$

and $\quad p_1 = a + bx, \quad p_2 = a' + b'x, \quad p_3 = a'' + b''x,$

$\quad r_1 = m + nx, \quad r_2 = m' + n'x, \quad r_3 = m'' + n''x.$

Then the conditions that the variables be separated in the transformed equation will be

$$\frac{a'n + b'm - bm' - an'}{a'm - am'} = \frac{a''n + b''m - an'' - m''b}{ma'' - m''a} = \frac{b''m' + a''n' - a'n'' - b'm''}{a''m' - a'm''},$$

$$\frac{b'n - n'b}{a'm - am'} = \frac{b''n - n''b}{ma'' - m''a} = \frac{b''n' - n''b'}{a''m' - a'm''}.$$

And
$$\frac{\mu + na - mb}{\phi} = \frac{\mu' + n'a - m'b + na' - mb'}{\phi'}$$
$$= \frac{\mu'' + n'a' - m'b' + n''a - m''b + na'' - mb''}{\phi''}$$
$$= \frac{\mu^{(3)} + a'n'' - m'b'' + a''n' - m''b'}{\phi^{(3)}} = \frac{\mu^{(4)} + n''a'' - m''b''}{\phi^{(4)}}.$$

Let the differential equation be

$$(r_1 + r_2\rho + r_3\rho^2)\frac{dy}{dx} = \mu + \phi x + (\mu' + \phi' x)\rho + (\mu'' + \phi'' x)\rho^2,$$

where ρ, r_1, p_1, &c., are the same as in the last example.

Then the conditions that the variables be separated in the transformed equation are as follow:

$$\frac{\mu m + na - mb}{\phi n} = \frac{\mu m' + \mu' m + an' + a'n - mb' - m'b}{\phi n' + \phi' n}$$

$$= \frac{\mu m'' + \mu' m' + m\mu'' + an'' - m''b + a''n - mb'' + n'a' - m'b'}{\phi n'' + \phi' n' + n\phi''}$$

$$= \frac{\mu' m'' + m'\mu'' + a'n'' + a''n' - m'b'' - m''b'}{\phi' n'' + \phi'' n'} = \frac{\mu'' m'' + a''n'' - m''b''}{\phi'' n''},$$

$$\frac{\phi m + \mu n}{\phi n} = \frac{\phi m' + n'\mu + m\phi' + \mu' n}{\phi n' + \phi' n}$$

$$= \frac{m''\phi + n''\mu + \phi' m' + \mu' n + n\mu'' + \phi'' m}{\phi n'' + \phi' n' + n\phi''}$$

$$= \frac{\phi' m'' + \mu'\phi'' + n'\mu'' + m'\phi''}{\phi' n'' + n'\phi''} = \frac{\phi'' m'' + n''\mu''}{\phi'' n''}.$$

REMARKS ON INTEGRATION.

By the Rev. Robert Carmichael, A.M.,

Trinity College, Dublin.

It is proposed, in the following paper, to offer some practical remarks upon the subject of integration, more especially in reference to three capital defects under which many of the methods in common use appear to me to labour. The first defect in the methods, to which allusion is made, is their extremely artificial character, which occasions much embarrassment to the student at first, and considerable difficulty in his effort to retain them. The second great defect in these methods is, that they seem wholly unsusceptible of useful generalization. The third defect is less common, and consists in this, that some of the processes employed are circuitous, terms being introduced which subsequent operations cause to disappear. The method here put forward appears to be free from these defects, and, in so far as it is calculated to reduce and simplify the labours of the student, a practical good.

In illustration, I shall apply the method to the well-known equations

$$xD^2y + Dy + y = 0,$$

(Gregory's *Examples*, chap. v., Ex

and $$D^2y + \frac{2}{x}Dy + \left(n^2 - \frac{2}{x^2}\right)y = 0,$$

(Gregory's *Examples*, chap. IV., sect. 2, Ex. 10.)

In quoting from the manual just named, I would desire to express a sense of its general merit as well as personal obligation to its study.

If we multiply the first equation by x and the second by x^2, they become respectively

$$(xD)^2.y + xy = 0,$$

$$(xD-1)(xD+2).y + n^2x^2y = 0.$$

Now the common type of these equations is

$$F(xD)y + Mx^m y = 0;$$

or, more generally,

$$F(xD)y + Mx^m y = X.$$

Let us suppose that

$$X = \Sigma A_a x^a,$$

and proceed to solve the more general type.

Operating on both sides with the inverse of $F(xD)$, we get

$$y + \frac{1}{F(xD)}Mx^m.y = \frac{1}{F(xD)}X + \frac{1}{F(xD)}0,$$

or the required solution is at once

$$y = \begin{cases} \Sigma \dfrac{A_a x^a}{F(a)} \left\{1 - \dfrac{Mx^m}{F(a+m)} + \dfrac{(Mx^m)^2}{F(a+2m)\,F(a+m)} - \&c.\right\} \\ + \\ \Sigma\, C_\alpha x^\alpha \left\{1 - \dfrac{Mx^m}{F(\alpha+m)} + \dfrac{(Mx^m)^2}{F(\alpha+2m)\,F(\alpha+m)} - \&c.\right\}, \end{cases}$$

the coefficients within brackets in the first and second great terms differing merely in the substitution of α for a.

In this method no mathematical artifice is employed, and the result seems to be obtained in the most direct manner. That the method admits of easy generalization can be readily now shown.

Let the partial differential equation to be solved be represented by the type

$$F(\nabla)z + \Theta_m z = \Omega,$$

where Θ_m is an homogeneous function in x and y of the m^{th} degree, Ω is a mixed function of x, y, and ∇ is the index-symbol

$$xD_x + yD_y.$$

Operate with

$$\frac{1}{F(\nabla)},$$

having broken up Ω into sets of homogeneous functions; there results

$$\left(1 + \frac{1}{F(\nabla)}\,\Theta_m\right)z = \Sigma\,\frac{\Theta_a}{F(a)} + \Sigma u_\alpha,$$

where u_α is a homogeneous function of the given degree α, but arbitrary in form, and operating on both sides with the inverse of

$$\left(1 + \frac{1}{F(\nabla)}\,\Theta_m\right),$$

we get at once, as before,

$$z = \begin{cases} \Sigma \dfrac{\Theta_a}{F(a)} \left\{1 - \dfrac{\Theta_m}{F(a+m)} + \dfrac{\Theta_m^{\ 2}}{F(a+2m)\,F(a+m)} - \&c.\right\} \\ + \\ \Sigma u_\alpha \left\{1 - \dfrac{\Theta_m}{F(\alpha+m)} + \dfrac{\Theta_m^{\ 2}}{F(\alpha+2m)\,F(\alpha+m)} - \&c.\right\}. \end{cases}$$

Let us now apply the method to the first example proposed, in its modified form, namely,

$$(xD)^2.y + xy = 0,$$

premising that its susceptibility of some such method of integration was suggested in the year 1847 by the Rev. Charles Graves, Professor of Mathematics in this University.

Operating on both sides of this equation with $\frac{1}{(xD)^2}$, we get

$$y + \frac{1}{(xD)^2}\,xy = C_1 \log x + C_2;$$

whence

$$y = \left\{1 - \frac{1}{(xD)^2}\,x + \frac{1}{(xD)^2}\,x\,.\,\frac{1}{(xD)^2}\,x - \&c.\right\}.(C_1 \log x + C_2),$$

or

$$y = \left\{\begin{array}{l} C_1\left\{1 - x\,\dfrac{1}{(1+xD)^2} + x^2\,\dfrac{1}{(2+xD)^2\,(1+xD)^2} - \&c.\right\}\log x, \\ \qquad\qquad\qquad + \\ C_2\left\{1 - \dfrac{x}{1^2} + \dfrac{x^2}{1^2.2^2} - \dfrac{x^3}{1^2.2^2.3^2} + \dfrac{x^4}{1^2.2^2.3^2.4^2} - \&c.\right\}. \end{array}\right.$$

But

$$\frac{1}{(1+xD)^2}\log x = \frac{1}{1^2}\left(1 - \frac{2}{1}\,xD\right)\log x = \frac{1}{1^2}\,(\log x - 2),$$

$$\frac{1}{(2+xD)^2\,(1+xD)^2}\log x = \frac{1}{1^2.2^2}\left\{1 - 2\left(\frac{1}{1} + \frac{1}{2}\right)xD\right\}\log x$$

$$= \frac{1}{1^2.2^2}\left\{\log x - 2\left(\frac{1}{1} + \frac{1}{2}\right)\right\}, \&c.$$

or

$$y = \begin{cases} (C_1 \log x + C_2)\left\{1 - \dfrac{Mx^m}{m^2} + \dfrac{(Mx^m)^2}{m^2.(2m)^2} - \dfrac{(Mx^m)^3}{m^2.(2m)^2.(3m)^2} + \text{\&c.}\right\} \\ + \\ 2C_1\left\{\dfrac{1}{m^3} Mx^m - \dfrac{m(2+1)}{m^3.(2m)^3}(Mx^m)^2 + \dfrac{m^2(6+3+2)}{m^3.(2m)^3.(3m)^3}(Mx^m)^3 - \text{\&c.}\right\} \end{cases}$$

Similarly, if it be proposed to integrate the partial differential equation

$$\nabla_2 . z + \nabla . z + \Theta_m . z = 0,$$

where $\qquad \nabla_2 = x^2 D_x^{\,2} + 2xy D_x D_y + y^2 D_y^{\,2},$

a corresponding reduction gives

$$z + \frac{1}{\nabla^2}\,\Theta_m . z = u_0\,\frac{\log x + \log y}{2} + v_0,$$

and the symbolic solution is

$$z = \left(1 - \frac{1}{\nabla^2}\,\Theta_m + \frac{1}{\nabla^2}\,\Theta_m . \frac{1}{\nabla^2}\,\Theta_m - \text{\&c.}\right)\left(u_0\,\frac{\log x + \log y}{2} + v_0\right).$$

Hence

$$z = \begin{cases} u_0\left\{1 - \Theta_m\,\dfrac{1}{(m+\nabla)^2} + \Theta_m^{\,2}\,\dfrac{1}{(m+\nabla)(2m+\nabla)^2} - \text{\&c.}\right\}\dfrac{\log x + \log y}{2}, \\ + \\ v_0\left\{1 - \dfrac{\Theta_m}{m^2} + \dfrac{\Theta_m^{\,2}}{m^2.(2m)^2} - \dfrac{\Theta_m^{\,3}}{m^2.(2m)^2\,(3m)^2} + \text{\&c.}\right\}; \end{cases}$$

and finally,

$$z = \begin{cases} \left(u_0\,\dfrac{\log x + \log y}{2} + v_0\right)\left\{1 - \dfrac{\Theta_m}{m^2} + \dfrac{\Theta_m^{\,2}}{1^2.2^2} - \dfrac{\Theta_m^{\,3}}{1^2.2^2.3^2} + \text{\&c.}\right\}, \\ + \\ 2u_0\left\{\dfrac{1}{m^3}\,\Theta_m - \dfrac{m(2+1)}{m^3.(2m)^3}\,\Theta_m^{\,2} + \dfrac{m^2(6+3+2)}{m^3.(2m)^3.(3m)^3}\,\Theta_m^{\,3} - \text{\&c.}\right\}. \end{cases}$$

Proceeding now to apply the same method to the modified form of the second example,

$$(xD - 1)(xD + 2)\,y + n^2x^2 . y = 0,$$

we get

$$y + \frac{1}{(xD-1)(xD+2)}\,n^2x^2 . y = C_1 x + \frac{C_2}{x^2}.$$

Consequently the solution is given by

$$y = \left\{1 - \frac{1}{(xD-1)(xD+2)}\, n^2x^2 \right.$$

$$\left. + \frac{1}{(xD-1)(xD+2)}\, n^2x^2 \cdot \frac{1}{(xD-1)(xD+2)}\, n^2x^2 - \&c. \right\} \left(C_1 x + \frac{C_2}{x^2} \right),$$

or

$$y = \left\{ \begin{array}{c} C_1 x \left\{ 1 - \dfrac{n^2x^2}{2.5} + \dfrac{(n^2x^2)^2}{2.5.4.7} - \dfrac{(n^2x^2)^3}{2.5.4.7.6.9} + \&c. \right\}, \\ + \\ \dfrac{a^2C^2}{1.2} \left\{ 1.2.\dfrac{1}{n^2x^2} + 1 - \dfrac{n^2x^2}{1.4} + \dfrac{(n^2x^2)^2}{1.4.3.6} - \&c. \right\}, \end{array} \right.$$

which may be condensed into the known form, but in its present shape can be generalized.

In connexion with the integration of equations with variable coefficients, and more particularly with the first example, the following important remark has been made by Gregory (*Examples*, p. 314). "After all, however, when these equations are of the second or higher orders, the number of cases in which they are integrable is very limited, and there seems to be no great prospect of the number being much increased. A little consideration will point out the reason of this. When we speak of an equation being integrable, we mean that the dependent variable can be expressed in terms of the independent variable by means of a finite series of functions of that quantity, the forms of such functions being limited to those known as algebraical and transcendental. Now it has been seen that the simplest forms of differential equations involve the highest transcendents which we recognize as known functions, such as e^{ax} or $\cos nx$; and it is to be expected, that when the equations become more complicated their integrals must involve higher transcendents to which we have not affixed particular names, and which we do not look on as known forms. This indeed is found to be the case, as for example in the equation

$$xD^2y + Dy + y = 0,$$

which in its integral involves the transcendent

$$\psi(x) = 1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \&c.$$

It would appear then, that before we are able t
any further progress in the solution of differentia

we must create new transcendents in the same way as the ordinary transcendents e^x, $\cos x$, $\log x$, &c. have been created; we must study their properties, and endeavour to express the integrals of differential equations by means of them. The first part of this task has for some time past occupied the attention of mathematicians, and great progress has been made in it, though much still remains to be done. The second part has also been the object of study, though not to the same extent as the other; and several mathematicians have applied themselves with success to the expression of the integrals of differential equations by means of definite integrals which are the representatives of new transcendents. Thus, for instance, in the case cited above, the transcendent

$$1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \&c. = \frac{1}{\pi}\int_0^\pi d\theta \cos(2 \sin\theta x^{\frac{1}{2}}).$$

Examples of such integrals will be found in Crelle's *Journal*, vol. x. p. 92; vol. XII. p. 144; vol. XVII. p. 363."

Now it appears to me that, until evaluated, the integral

$$\frac{C_2}{\pi}\int_0 d\theta \cos(2 \sin\theta x^{\frac{1}{2}})$$

must be considered to be quite as symbolic as the equivalent

$$\left\{1 + \frac{1}{(xD)^2}x\right\}^{-1}.C_2.$$

Indeed we may regard all symbolic condensations, as well as definite integrals, in the light of representatives of new transcendents.

For instance, if

$$U = A_0 + A_1 x + A_2 x^2 + \&c.,$$

where A_0, A_1, A_2, &c., are constants, it may easily be proved that

$$F(xD).U = F(0)A_0 + F(1)A_1 x + F(2)A_2 x^2 + \&c.$$

$$\frac{1}{F(xD)}.U = \frac{1}{F(0)}A_0 + \frac{1}{F(1)}A_1 x + \frac{1}{F(2)}A_2 x^2 + \&c.,$$

and it seems to me that the left-hand members may fairly be regarded as such representatives.

5, Trinity College, Dublin, March 1854.

[redacted]

ON THE CURVATURE OF CURVES IN SPACE.

By William Spottiswoode, M.A., F.R.S.

Let L, L_1, be the characteristics of the two surfaces, by the intersection of which the curve is formed; and let U, V, W, U_1, V_1, W_1, be the partial differential coefficients of L, L_1, with respect to x, y, z, respectively. Then writing

$$\left.\begin{aligned} &U : V : W \\ &= VW_1 - V_1W = WU_1 - W_1U : UV_1 - U_1V \end{aligned}\right\} \ldots (1),$$

the equations to a tangent line take the form

$$\left.\begin{aligned} &dx : dy : dz \\ &= U : V : W \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

if θ be the common ratio of this system, there result

$$\left.\begin{aligned} \frac{d^2x}{ds^2} &= \frac{dU}{ds}\theta + U\frac{d\theta}{ds} \\ \frac{d^2y}{ds^2} &= \frac{dV}{ds}\theta + V\frac{d\theta}{ds} \\ \frac{d^2z}{ds^2} &= \frac{dW}{ds}\theta + W\frac{d\theta}{ds} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

whence

$$\begin{vmatrix} \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \\ \frac{dU}{ds} & \frac{dV}{ds} & \frac{dW}{ds} \\ U & V & W \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (4).$$

Now $$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0 \ldots\ldots\ldots\ldots (5),$$

or $$U\frac{d^2x}{ds^2} + V\frac{d^2y}{ds^2} + W\frac{d^2z}{ds^2} = 0 \ldots\ldots\ldots\ldots (6)$$

whence, writing

$$U^2 + V^2 + W^2 = P^2 \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2}$$

$$= P\frac{dU}{ds} - U\frac{dP}{ds} : P\frac{dV}{ds} - V\frac{dP}{ds} : P\frac{dW}{ds} - W\frac{dP}{ds} \ldots (8);$$

or, writing

$$\left.\begin{aligned} \Omega &= \begin{vmatrix} \mathrm{UVW} \\ \mathrm{U'V'W'} \\ \mathrm{U_1V_1W_1} \end{vmatrix} = -(U\mathrm{U'} + V\mathrm{V'} + W\mathrm{W'}) \\ \Omega_1 &= \begin{vmatrix} \mathrm{U_1V_1W_1} \\ \mathrm{U_1'V_1'W_1'} \\ \mathrm{UVW} \end{vmatrix} = U\mathrm{U_1'} + V\mathrm{V_1'} + W\mathrm{W_1'} \end{aligned}\right\} \ldots (9),$$

these ratios

$$\left.\begin{aligned} &= (\mathrm{W}\Omega_1 + \mathrm{W_1}\Omega)V - (\mathrm{V}\Omega_1 + \mathrm{V_1}\Omega)W \\ &: (\mathrm{U}\Omega_1 + \mathrm{U_1}\Omega)W - (\mathrm{W}\Omega_1 + \mathrm{W_1}\Omega)U \\ &: (\mathrm{V}\Omega_1 + \mathrm{V_1}\Omega)U - (\mathrm{U}\Omega_1 + \mathrm{U_1}\Omega)V \end{aligned}\right\} \ldots\ldots (10);$$

or, if ϕ be the angle between the two surfaces,

$$\left.\begin{aligned} &= \Omega_1\mathrm{P}(\mathrm{PU_1} - \mathrm{P_1U}\cos\phi) + \Omega\mathrm{P_1}(\mathrm{P_1U} - \mathrm{PU_1})\cos\phi \\ &: \Omega_1\mathrm{P}(\mathrm{PV_1} - \mathrm{P_1V}\cos\phi) + \Omega\mathrm{P_1}(\mathrm{P_1V} - \mathrm{PV_1})\cos\phi \\ &: \Omega_1\mathrm{P}(\mathrm{PW_1} - \mathrm{P_1W}\cos\phi) + \Omega\mathrm{P_1}(\mathrm{P_1W} - \mathrm{PW_1}\cos\phi) \end{aligned}\right\} \ldots (11).$$

But if ρ, ρ_1, be the radii of curvature of normal sections of the two surfaces passing through the tangent line to the curve, it will be found that

$$\Omega : \Omega_1 = \frac{\mathrm{P}}{\rho} : \frac{\mathrm{P_1}}{\rho_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

so that (11) may be again transformed into the following:

$$\left.\begin{aligned} &\left(\frac{1}{\rho_1\cos\phi} - \frac{1}{\rho}\right)\mathrm{PU_1} + \left(\frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}\right)\mathrm{P_1V} \\ &: \left(\frac{1}{\rho_1\cos\phi} - \frac{1}{\rho}\right)\mathrm{PV_1} + \left(\frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}\right)\mathrm{P_1V} \\ &: \left(\frac{1}{\rho_1\cos\phi} - \frac{1}{\rho}\right)\mathrm{PW_1} + \left(\frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}\right)\mathrm{P_1W} \end{aligned}\right\} \ldots (13).$$

Now $\rho\cos\phi$ is the radius of curvature of the oblique section of the first surface, made by the normal pla

the second which passes through the tangent line; so that $\dfrac{1}{\rho\cos\phi} - \dfrac{1}{\rho_1}$ is the difference of the curvatures of the two curves formed by the intersection of that plane with the two surfaces; similarly $\dfrac{1}{\rho_1\cos\phi} - \dfrac{1}{\rho}$ is the difference of the curvatures of the two curves formed by the intersection of the normal plane to the first surface, which passes through the tangent line to the given curve, with the two surfaces themselves:

If ρ be the radius of absolute curvature of the given curve,

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \text{............} (14),$$

and consequently

$$C = \frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}, \quad C_1 = \frac{1}{\rho_1\cos\phi} - \frac{1}{\rho} \text{......} (15),$$

the common ratio of the systems (8) and (13) becomes

$$= \rho PP_1 (C^2 + 2CC_1\cos\phi + C_1^2)^{\frac{1}{2}} \text{............} (16);$$

or again,

$$= \rho^2 PP_1 \left(\frac{C}{\rho} + \frac{C_1}{\rho_1}\right):$$

hence

$$\frac{1}{\rho} = \frac{1}{\sin\phi}\left\{\frac{1}{\rho^2} + \frac{1}{\rho_1^2} - \frac{2\cos\phi}{\rho\rho_1}\right\}^{\frac{1}{2}} \text{............} (17),$$

which is an expression for the radius of absolute curvature of the curve in terms of the radii of curvature of the two surfaces. When the surfaces touch one another

$$\sin\phi = 0, \quad \cos\phi = 1,$$

and consequently $\rho = 0$, unless $\rho = \rho_1$; that is to say (as will be found on substituting the ordinary values of ρ and ρ_1), unless for some normal section $\dfrac{d^2x}{ds^2}, \dfrac{d^2y}{ds^2}, \dfrac{d^2z}{ds^2}$, are the same for both surfaces; in other words, unless the surfaces touch in more points than one. This might have been cipated *à priori*, since the curve would otherwise reduce to a single point. If $\rho = \rho_1$ the value of ρ becomes erminate, because the condition $\rho = \rho_1$ indicates that radii of curvature of *any* common normal section of two surfaces passing through the point in question are qual, and as this curvature may vary with the direction

of the normal plane, there are in fact an infinite number of branches of the curve passing through the point, each of which may have its own curvature. And on referring to (17) it will be found that

$$\frac{2}{\rho} = \frac{1}{\rho} + \frac{1}{\rho_1},$$

and consequently if for any one or more sections

$$\rho = \rho_1,$$

(*i.e.* if along any one or more sections the surfaces have their convexities turned in the same direction,)

$$\rho = \rho_1 = \rho = 1,$$

i.e. the radius of absolute curvature of that branch of the curve is the same as the common radius of curvature of the normal section of the two surfaces on which that branch lies; but if for any one or more sections,

$$\rho + \rho_1 = 0,$$

i.e. if along any one or more sections the surfaces have their convexities turned in different directions, *i.e.*

$$\rho = 0,$$

either there is no branch in that direction, or the branch has a point of inflexion or of suspended curvature.

If the two surfaces cut one another at right angles

$$\sin\phi = 1, \quad \cos\phi = 0,$$

and consequently

$$\frac{1}{\rho^2} = \frac{1}{\rho^2} + \frac{1}{\rho_1^2}.$$

It is well known that if three surfaces cut one another at right angles they cut one another in their principal sections and in their lines of curvature; hence, if ρ, ρ_1, ρ_2, be the radii of absolute curvature of the three curves so formed, and $R, R^1, R_1, R_1^1, R_2, R_2^1$, the principal radii of curvature of the three surfaces respectively, there will result

$$\frac{1}{\rho^2} = \frac{1}{R_1^2} + \frac{1}{R_2'^2}, \quad \frac{1}{\rho_1^2} = \frac{1}{R_2^2} + \frac{1}{R'^2}, \quad \frac{1}{\rho_2^2} = \frac{1}{R^2} + \frac{1}{R_1'^2},$$

and consequently

$$\frac{1}{\rho^2} + \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} = \frac{1}{R^2} + \frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R'^2} + \frac{1}{R_1'^2} + \frac{1}{R}$$

[redacted]

In the case of an umbilicus

$$\mathrm{R} = \mathrm{R}', \quad \mathrm{R}_1 = \mathrm{R}_1', \quad \mathrm{R}_2 = \mathrm{R}_2';$$

and consequently, writing

$$\frac{1}{\rho} + \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{2}{\Sigma},$$

$$\frac{1}{\mathrm{R}^2} = 2\left(\frac{1}{\rho_1} - \frac{1}{\Sigma}\right)\left(\frac{1}{\rho_2} = \frac{1}{\Sigma}\right),$$

$$\frac{1}{\mathrm{R}_1^2} = 2\left(\frac{1}{\rho_2} - \frac{1}{\Sigma}\right)\left(\frac{1}{\rho} - \frac{1}{\Sigma}\right),$$

$$\frac{1}{\mathrm{R}_2^2} = 2\left(\frac{1}{\rho} - \frac{1}{\Sigma}\right)\left(\frac{1}{\rho_1} - \frac{1}{\Sigma}\right);$$

which may be written also as follows,

$$\frac{1}{\mathrm{R}^2}\left(\frac{1}{\rho} - \frac{1}{\Sigma}\right) = \frac{1}{\mathrm{R}_1^2}\left(\frac{1}{\rho_1} - \frac{1}{\Sigma}\right) = \frac{1}{\mathrm{R}_2^2}\left(\frac{1}{\rho_2} - \frac{1}{\Sigma}\right)$$

$$= -\frac{1}{(\mathrm{R}^2 + \mathrm{R}_1^2 + R_2^2)\Sigma}.$$

ON THE CONJUGATE LINES OF SURFACES.

By W. Walton.

Let $F = 0$ be the algebraical equation to a surface, free from radical and negative indices. In elementary works on Solid Geometry (see Gregory's *Solid Geometry*, chap. XIII.) it is shewn that the existence of a tangent cone at any point of a surface, or of an edge of regression, or in fact, generally, of plural tangency, corresponds to the coexistence of the four equations

$$F = 0, \quad \frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 0.$$

It appears however to have escaped the notice of writers on Geometry, that these analytical conditions sometimes correspond to the existence of a *conjugate curved line*, that is, to a curved line entirely detached from the surface, but forming part of the geometrical locus of its equation, bearing in fact the same relation to the surface which a conjugate point bears to a curve.

Imagine the existence of a conjugate line: let x, y, z, be the coordinates of any one of its points. Suppose x_1, y_1, z_1, to be written in the equation $F=0$ in lieu of x, y, z, y_1 and z_1 being coordinates of a point in the plane of y, z, nearly but not quite in contact with the projection of the conjugate curve upon that plane. Then x_1 in the equation $F(x_1, y_1, z_1)=0$ will be impossible, and therefore have two values of the forms

$$\alpha+\beta\sqrt{(-1)}, \quad \alpha-\beta\sqrt{(-1)}.$$

Hence, when one of these roots of the equation in x_1 becomes α, the other will also become α. Thus we see that in the equation

$$F(x, y, z)=0$$

x will have two equal values for every pair of values of y, z. Hence

$$\frac{dF}{dx}=0.$$

Similarly we see that

$$\frac{dF}{dy}=0, \quad \frac{dF}{dz}=0.$$

Thus the existence of a conjugate line implies the coexistence of the four equations

$$F=0, \quad U=0, \quad V=0, \quad W=0,$$

where U, V, W, denote the partial differential coefficients of F with regard to x, y, z, respectively.

Putting

$$\frac{d^2F}{dx^2}=u, \quad \frac{d^2F}{dy^2}=v, \quad \frac{d^2F}{dz^2}=w,$$

$$\frac{d^2F}{dy\,dz}=u', \quad \frac{d^2F}{dz\,dx}=v', \quad \frac{d^2F}{dx\,dy}=w',$$

we have, twice differentiating the equation to the surface, and writing λ, μ, ν, for the direction-cosines of the tangent line at any point of the conjugate line,

$$\lambda^2u+\mu^2v+\nu^2w+2\mu\nu u'+2\nu\lambda v'+2\lambda\mu w'=0 \ldots (\text{A}).$$

Since each of the ratios between the quantities λ, μ, ν, can have only one value for a given point x, y, z, in the conjugate line, the algebraical state of this equation must be such as to establish this restriction. Thus we see that

the sufficient and necessary conditions for the existence of a conjugate line are the coexistence of the four equations

$$F = 0,\quad U = 0,\quad V = 0,\quad W = 0,$$

and the singleness of values of the ratios between λ, μ, ν, as implied by the equation (A).

Ex. Take the equation

$$F = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)^2 - \left(\frac{xyz}{m^3} - 1\right)^2 \cdot \left(\frac{xyz}{n^3} - 1\right),$$

where n is supposed to be greater than m.

It is easily ascertained that the conditions

$$F = 0,\quad U = 0,\quad V = 0,\quad W = 0,$$

are satisfied by the two simultaneous equations

$$xyz = m^3,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and that, under these circumstances, $\frac{1}{k^6}$ being substituted for $\frac{n^3 - m^3}{m^6 n^3}$,

$$u = \frac{8x^2}{a^4} + \frac{2y^2z^2}{k^6},$$

$$v = \frac{8y^2}{b^4} + \frac{2z^2x^2}{k^6},$$

$$w = \frac{8z^2}{c^4} + \frac{2x^2y^2}{k^6},$$

$$u' = \frac{8yz}{b^2c^2} + \frac{2x^2yz}{k^6}$$

$$v' = \frac{8zx}{c^2a^2} + \frac{2y^2zx}{k^6},$$

$$w' = \frac{8xy}{a^2b^2} + \frac{2z^2xy}{k^6}.$$

Hence the equation (A) becomes

$$4\left(\frac{\lambda x}{a^2} + \frac{\mu y}{b^2} + \frac{\nu z}{c^2}\right)^2 + \frac{1}{k^6}(\lambda yz + \mu zx + \nu xy)^2 = 0,$$

which can be satisfied only by putting

$$\frac{\lambda x}{a^2} + \frac{\mu y}{b^2} + \frac{\nu z}{c^2} = 0,$$

and $$\frac{\lambda}{x} + \frac{\mu}{y} + \frac{\nu}{z} = 0.$$

Thus the ratios between λ, μ, ν, have only single values, and therefore the curve of intersection of the two surfaces, represented by the equations

$$xyz = m^3, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

is a conjugate or isolated line.

12th July, 1854.

THE GEOMETRY OF QUATERNIONS.

By John Paterson, of Albany (N. Y.)

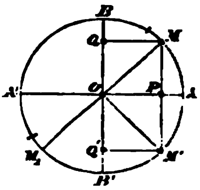
On the Extension of the principle of Perpendicularity to the Three dimensions of Space.

1. The remarkable discovery of a new analytical element, denominated *quaternion*, by Sir W. R. Hamilton of Dublin, appears destined to mark an epoch in the history of geometry. As the discovery consists essentially in an extension of the principle of perpendicularity from a single plane, to the system of three mutually rectangular planes that serve for the admeasurement of the three dimensions of space, it may be of use to investigate the genesis of the diagonal of a rectangular parallelopipedon, by the same kind of operations that generate the diagonal of a rectangular parallelogram.

2. The principle of perpendicularity consists in the dynamico-geometrical fact, that a straight line, considered as the measure of the result of an operation, has at the same time two distinct values, namely, the value of its own magnitude or length in the direction in which it lies, and the value zero in the direction perpendicular thereto. As an immediate consequence of this fact, if the line be inclined to the system of two rectangular axes to which it is refe[rred] for measurement, it will have a finite value on each [axis] equal to its projection thereon; that is, its product b[y]

cosine of the angle it makes with that axis, which again is equal to its product by the sine it makes with the other axis.

If we regard such line, in the different positions which may be given to it about an origin in a plane, when referred to a fixed system of rectangular axes, as the measure of an operation of the first order, such as that of simple addition or of simple multiplication, by a linear factor, it can be readily understood either as the measure of a rectilinear force of the first order acting in its direction, and beginning at the origin O; or as the measure of the result of a circular force of the same order, producing uniform rotation about the centre O, and beginning at the axis OA. In both cases, the line in the position OM is measured at once by its projections OP and OQ: in the former case, OM is the measure both of the operation (which consists in performing a multiplication by this line OM; 1_μ being multiplicand and $1_\lambda = OM$ multiplier, $1_\lambda.1_\mu = OM \times 1_\mu$), and of the result of the operation; but in the latter case, the operation consists in performing a multiplication by the angle AOM or its measure the arc $AM = \theta.1_r$, the radius $OA = 1_r$ (or even $1_r.1_\mu$) being the multiplicand, $\theta.1_r.1_e.1_\mu = AM \times OA \times 1_\mu = OM \times 1_\mu$; so that here the measure of the result OM differs from the measure of the operation itself, which is the arc AM; but the measure of the result, $OM = OP + OQ$, is all the notation records.



3. If we begin with a linear multiplication at the origin O, the indeterminate radius OA may be adopted as linear unit $1_e = 1_\lambda$ when written as a multiplier; and 1_μ being multiplicand, $1_\lambda . 1_\mu$ expresses the transfer of 1_μ from OA, in which position it is usually written $+1.1_e.1_\mu = OA \times 1_\mu$, where $+1$ is the coefficient of a positive linear operation, and $+1.1_e$ is the measure of such operation; which notation stands directly contrasted with $-1.1_\lambda.1_\mu = -1.1_e.1_\mu = OA' \times 1_\mu$, as representing the transfer of 1_μ from O to A', a negative linear operation, having -1.1_e as its measure, and -1 as coefficient of the same.

Next taking $OA \times 1_\mu = +1.1_e.1_\mu$ as multiplicand, and the arc $AM = \theta.1_r$ as multiplier, the product $\theta.1_r.1_e.1_\mu = OM$ in which position $OM = AM \times 1_e = \theta.1_r.1_e$ has the two

sures $OP = +1.\cos\theta.1_\varepsilon$ and $OQ = +\sqrt{(-1)}\sin\theta.1_\varepsilon$, the last notation for a reason that will presently appear. A second multiplication by any new arc whatever, $\theta'.1_\rho = MN$, will transfer the radius into the position ON, where its measures on the horizontal and perpendicular axes are given by the expressions $\cos(\theta+\theta')\,1_\varepsilon$ and $\sin(\theta+\theta')\,1_\varepsilon$ (the last affected by the symbol $\sqrt{(-1)}$, as may be traced upon the diagram.

4. If $\theta' = \theta = \frac{1}{2}\pi = 90°$, a right angle, the first multiplication will transfer the loaded radius or multiplicand $1_\varepsilon.1_\mu = 1_\rho.1_\mu$ from the position OA to the position OB; whence the second multiplication carries it to the position OA', where it is expressed by the notation $-1.1_\varepsilon.1_\mu$, the measure of the *result* of a negative operation; and since $\frac{1}{2}\pi.1_\rho = 90°.1_\rho$ is the factor that gives the position OB by a simple multiplication, and the result $OA' = -1.1_\varepsilon$ when the multiplication is repeated (the square of the operation), it is plain that the former result may be denoted as the square root of the latter, that is, $OB = \sqrt{(-1)}\,1_\varepsilon$. On this convention, however, it is to be borne in mind that when the symbol $\sqrt{(-1)}$ is the coefficient of a multiplier, it governs a right angle, or the arc of 90°; but when it expresses a product or result, it is the unit coefficient of a straight line perpendicular to the rectilinear multiplicand to which it has been applied to produce that result. Thus, when applied to the multiplicand 1_ε in the position $OA = +1.1_\varepsilon$, the multiplier $\sqrt{(-1)}\,1_\rho = \text{arc}\,AB = 90°$ (= the measure of the operation) gives the product $OB = +\sqrt{(-1)}\,1_\varepsilon$ (= the measure of the result of the operation): a second application of the multiplier $\sqrt{(-1)}\,1_\rho$, gives the result $OA' = -1.1_\varepsilon$; a third application, the result $OB' = -\sqrt{(-1)}\,1_\varepsilon$; and a fourth application, the result $OA = +1.1_\varepsilon$, identical with the primitive position.

5. θ being any arc whatever AM, if θ' be its conjugate or complementary arc to the circumference, the product $\theta'.\theta.1_\rho.1_\varepsilon.1_\mu$ transfers $1_\varepsilon.1_\mu$ from OA to OM by the first multiplier $\theta.1_\rho$, and from OM through $MA'A$ to OA by second multiplier $\theta'.1_\rho$. Since this gives $\theta'.\theta.1_\rho.1_\varepsilon.1_\mu = +1.1_\varepsilon.1_\mu$, we have $\theta'.\theta.1_\rho = +1$, and therefore $\theta' = \dfrac{1}{\theta}$. The conjugate factor $\dfrac{1}{\theta}.1_\rho$ then represents the multiplying arc $MBA'B'A$, complementary of AM to the circumference, and passes t
radius through the former to its primitive position OA;

as the same result would be given by retrograde rotation from M to A, it is seen that the negative factor $-\theta.1_r$ produces the same result as the positive factor $\frac{1}{\theta}.1_r$, but through a different path of operation.

6. The negative arc $-\theta.1_r$ will be represented by AM'; and as $OQ = +\sqrt{(-1)}\sin\theta.1_r$ is the sine of the positive arc AM, we have $OQ' = -\sqrt{(-1)}\sin\theta.1_r$ for the sine of the negative arc AM'. Taking the general unit radius 1_r (irrespective of position), we can write

$$\{1+\sqrt{(-1)}\sin\theta\}\,1_r = B'O + OQ = B'Q,$$

and
$$\{1-\sqrt{(-1)}\sin\theta\}\,1_r = B'O - Q'O = B'Q'.$$

The product of these factors is thus effected, term by term: $\{1+\sqrt{(-1)}\sin\theta\}\,1_r$ being the multiplicand or passive factor, the term 1_r is carried by the first term $+1.1_r$ of the multiplier or active factor, through the circumference, which is a perfect natural unit of space, back to its primitive position $B'O$, thereby fulfilling the condition $1^2.1_r = 1_r$; and similarly the same unit multiplier carries the term $+\sqrt{(-1)}\sin\theta.1_r$ through the circumference, back to its primitive position OQ. The second term $-\sqrt{(-1)}\sin\theta.1_r$ of the multiplier carries the radius 1_r from the position OA (for 1.1_r is completely indeterminate in its first position), through three right angles, into the position OB', and reduces its magnitude to $\sin\theta.1_r$; and finally, the same multiplying term, applied to the passive term $+\sqrt{(-1)}\sin\theta.1_r = OQ$, carries it, through three right angles, into the position OA, and reduces its magnitude to $\sin^2\theta.1_r$. The complete product then stands

$$\{1^2+\sqrt{(-1)}\sin\theta-\sqrt{(-1)}\sin\theta+\sin^2\theta\}\,1_r = (1+\sin^2\theta)\,1_r,$$

which may be immediately constructed for any unit radius and any arc θ; or if θ be determined by the condition that $1+\sin^2\theta = 0$, we find that $\sin\theta = \pm\sqrt{(-1)}$, which requires that $\theta = \pm\frac{1}{2}\pi$. Instead of the unit radius $1_r = 1_r$, $\cos\theta$ might be taken for the first term of both factors, and then the multiplication gives

$$\{\cos\theta+\sqrt{(-1)}\sin\theta\}\,\{\cos\theta-\sqrt{(-1)}\sin\theta\} = \cos^2\theta+\sin^2\theta,$$

a result otherwise known to be equal to the square of unity; and in this way we obtain a demonstration of the forty-seventh proposition of the first book of Euclid, by the method of the calculus of operations. Our demonstration, however, is more general than that of Euclid; for in the

complete equation $(a^2+b^2)\,1_e^{\,2} = R^2 1_e^{\,2}$ (where 1_e is the linear unit, and $R1_e$, $a1_e$, and $b1_e$ the hypothenuse and sides of a right-angled triangle), the *spacial* factor $1_e \times 1_e$ may have many different forms, while in Euclid it represents an area only.

7. As $1^2 = 1$, if the multiplicand be the linear unit 1_e, and the multiplier the abstract number 1, we have

$$(\cos^2\theta + \sin^2\theta)\,1_e = 1^2.1_e = 1_e = OA = OM$$

the diagonal of the rectangular parallelogram, of which the sides are $OP = \cos\theta.1_e$ and $OQ = \sin\theta.1_e$, where the coefficients $\cos\theta$ and $\sin\theta$ are numerical, and the products $\cos\theta.1_e$ and $\sin\theta.1_e$ are *magnitudinal* only; but as also $(+1)^2 = +1$, we have

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}\,\{\cos\theta - \sqrt{(-1)}\sin\theta\}\,1_e = (+1)^2\,1_e = +1.1_e$$
$$= OA = OM \times OM',$$

the product of the two diagonals whose sides are respectively

$$OP = +1\,\cos\theta.1_e \text{ and } OQ = +\sqrt{(-1)}\,\sin\theta.1_e,$$

and $\quad OP = +1\,\cos\theta.1_e$ and $OQ' = -\sqrt{(-1)}\,\sin\theta.1_e$,

where the *directional* coefficients $+1$ and $\sqrt{(-1)}$ also appear, and the complete terms involve both magnitude and direction.

8. The diagonal of a rectangular parallelopipedon furnishes the spacial analogue of the preceding formula, and is expressed by

$$(\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')\,1_e = (+1)^2\,1_e = +1.1_e;$$

$\theta', \theta'', \theta'''$, being arcs of mutually perpendicular great circles, of the sphere of which the unit radius $1_r = 1_e$ forms the diagonal of the parallelopipedon.

If R be any arithmetical coefficient, and $R1_e$ the magnitude of the diagonal of a rectangular parallelopipedon, the equation

$$R^2(\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')\,1_e = +1.R^2 1_e$$

denotes a concrete biquaternion in its most general form, and is the product of the two simple concrete quaternions denoted by the equations

$$R\{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}\,1_e = +1\,.R1_e, \text{ and}$$

$$R\{-\sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta'''\}\,1_e = \frac{1}{+1}.R1_e = +1.R1_e,$$

the latter being the conjugate or reciprocal of the former, and each the diagonal of a rectangular parallelopipedon. But as the directional coefficient is wholly determined within the sphere to radius unity, we may make $R = 1$, and generate the biquaternion unity by the actual multiplication of the left-hand members of these two equations. That this operation really consists in extending the application of the principle of perpendicularity from a single plane to the three mutually perpendicular coordinate planes of general space, must appear through the following investigations.

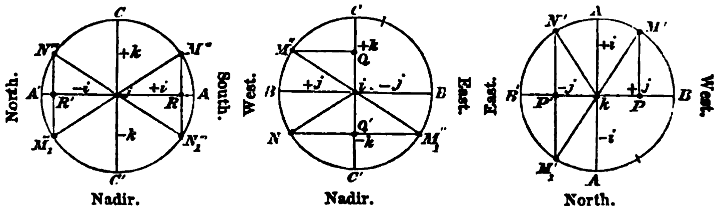
9. Let $OM = 1_r = 1_e$ be the unit radius of a sphere, situated in the superior south-west quadrant formed by the mutual intersections of the horizontal, meridional, and equatorial planes; and let its projections on the three positive axes (western, southern, and ascending) be OP, OQ, and OR. Let three planes be drawn through OM, perpendicular respectively to the three coordinate axes; and draw radii to the respective intersections M', M'', and M''', of these planes with the sides of one triquadrantal triangle ABC formed by the circumferences of the horizontal, meridional, and equatorial circles: each angle of each of these three radii with the axes will be equal to the angle of OM with the same axis, and hence arise the following fundamental relations of admeasurement:

$$\begin{aligned} OP &= OM' \times \sin AOM'.1_e = +\sqrt{(-1)}\sin\theta'.1_e, \\ OQ &= OM'' \times \sin BOM''.1_e = +\sqrt{(-1)}\sin\theta''.1_e, \\ OR &= OM''' \times \sin COM'''.1_e = +\sqrt{(-1)}\sin\theta'''.1_e; \end{aligned}$$

which immediately point out the following as their conjugates:

$$\begin{aligned} OP' &= ON' \times \sin AON'.1_e = -\sqrt{(-1)}\sin\theta'.1_e, \\ OQ' &= ON'' \times \sin BON''.1_e = -\sqrt{(-1)}\sin\theta''.1_e, \\ OR' &= ON''' \times \sin CON'''.1_e = -\sqrt{(-1)}\sin\theta'''.1_e. \end{aligned}$$

III.	II.	I.
Meridional Plane.	*Equatorial Plane.*	*Horizontal Plane.*
Face West.	Face South.	Face Zenith.
Zenith.	Zenith.	South.



Nadir.	Nadir.	North.

First diagram. A, the southernmost point of the horizontal plane, is the origin of arcs θ', and the rotation is right-handed through B, &c.

Second diagram. B, the westernmost point of the equatorial plane, is the origin of arcs θ'', and the rotation is right-handed through C, &c.

Third diagram. C, the highest point of the meridional plane, is the origin of arcs θ''', and the rotation is right-handed through A, &c.

These several origins and directions of operation establish, by aid of the principle of perpendicularity, the following measures of the results of quadrantal steps of rotation:

I.	II.	III.
$OA = +1'.1_{,} \quad = +i;$	$OB = +1''.1_{,} \quad = +j;$	$OC = +1'''.1_{,} \quad = +k;$
$OB = +\sqrt{(-1)}'.1_{,} = +j;$	$OC = +\sqrt{(-1)}''.1_{,} = +k;$	$OA = +\sqrt{(-1)}'''.1_{,} = +i;$
$OA' = -1'.1_{,} \quad = -i;$	$OB' = -1''.1_{,} \quad = -j;$	$OC' = -1'''.1_{,} \quad = -k;$
$OB' = -\sqrt{(-1)}'.1_{,} = -j.$	$OC' = -\sqrt{(-1)}''.1_{,} = -k.$	$OA' = \sqrt{(-1)}'''.1_{,} = -i.$

10. The multiplication of the two simple quaternions

$$\{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}\,1_{,} = +1.1_{,} = OM, \text{ and}$$

$$\{-\sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta'''\}\,1_{,} = \frac{1}{+1}.1_{,} = ON \text{ or } OM,$$

(for the right-hand member is really indeterminate in its direction, and may have any required position to satisfy the conditions of the operation), is now to be effected term by term, to shew the genesis of the biquaternion

$$(\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')\,1_{,} = (+1)^2.1_{,} = +1.1_{,} = OM.$$

The notation of the process with the left-hand members stands thus:

$$\begin{array}{lll} +\sqrt{(-1)}\sin\theta' & +\sqrt{(-1)}\sin\theta'' & +\sqrt{(-1)}\sin\theta''' \\ \underline{-\sqrt{(-1)}\sin\theta'} & \underline{-\sqrt{(-1)}\sin\theta''} & \underline{-\sqrt{(-1)}\sin\theta'''} \\ +1\sin^2\theta' & +1\sin^2\theta'' & +1\sin^2\theta''' \end{array}$$

$$\left\{\begin{array}{l}\{+\sqrt{(-1)}\sin\theta'\}\{-\sqrt{(-1)}\sin\theta''\} \\ \{+\sqrt{(-1)}\sin\theta''\}\{-\sqrt{(-1)}\sin\theta'\}\end{array}\right\} + \left\{\begin{array}{l}\{+\sqrt{(-1)}\sin\theta'''\}\{-\sqrt{(-1)}\sin\theta'\} \\ \{+\sqrt{(-1)}\sin\theta'\}\{-\sqrt{(-1)}\sin\theta'''\}\end{array}\right\}$$

$$+\left\{\begin{array}{l}\{+\sqrt{(-1)}\sin\theta''\}\{-\sqrt{(-1)}\sin\theta'''\} \\ \{+\sqrt{(-1)}\sin\theta'''\}\{-\sqrt{(-1)}\sin\theta''\}\end{array}\right\}.$$

The operations may be as follow :

(1) The multiplicand $1_{,}$ has the value $+1'.1_{,}$ in the position OA; and the application of the multiplier $+\sqrt{(-\ )}\sin\theta'.1_r$ carries it right-handedly, through a right angle, into the position OB, at the same time reducing its value to

$$+\sqrt{(-1)}\sin\theta'.1_{,} = OP.$$

The second multiplier $-\sqrt{(-1)}\sin\theta'.1_r$ being now applied, carries the new multiplicand $+\sqrt{(-1)}\sin\theta'.1_{,}$ through three right angles, and reduces its value to $+1.\sin^2\theta'.1_{,}$ upon the axis OA. As the spacial multiplier in each instance is really the angle or arc, the *directional* effects alone of these two multiplications are equivalent to the rotation of the radius $1_{,} = OA$, first through the angle $\theta' = AOM'$ (the measure of which is the arc $\theta'.1_r = AM'$), and then through the conjugate angle $M'OBA'B'A$, the measure of which is the reciprocal arc

$$\frac{1}{\theta'}.1_r = (2\pi - \theta')\,1_r = M'BA'B'A = ABA'B'N',$$

of which the sine is $-\sqrt{(-1)}\sin\theta'.1_r = OP'$.

(2) The multiplicand $1_{,}$ has the value $+1''.1_{,}$ in the position OB; and the multiplier $+\sqrt{(-1)}\sin\theta''.1_r$ (measure of the arc $\theta''.1_r = BM''$) will carry it right-handedly through a right angle, reducing its value to $+\sqrt{(-1)}\sin\theta''.1_{,} = OQ$ upon the radius OC. From this position, the second multiplier $-\sqrt{(-1)}\sin\theta''.1_r$ carries the new multiplicand $+\sqrt{(-1)}\sin\theta''.1_{,}$ through three right angles, and reduces its value to $+1.\sin^2\theta''.1_{,}$ upon the axis OB. In directional effect alone, the two multiplications are equivalent to the multiplication of the radius $1_{,} = OB$, first by the arc $\theta''.1_r = BM''$, and then by its reciprocal $\frac{1}{\theta''}.1_r = (2\pi - \theta'')\,1_r = M''CB'C'B = BCB'C'N''$, the sine of which is $OQ' = -\sqrt{(-1)}\sin\theta''.1_r$.

(3) Lastly, the multiplicand $1_{,}$ has the value $+1'''.1_{,}$ in the position OC; whence the multiplier $+\sqrt{(-1)}\sin\theta'''.1_r$ (measure of the arc $\theta'''.1_r = CM'''$) transfers it through a right angle, and reduces its value to $+\sqrt{(-1)}\sin\theta'''.1_{,} = OR$ upon the radius OA. From this position, the second multiplier $-\sqrt{(-1)}\sin\theta'''.1_r$ carries it through three right angles, and reduces its value to $+1.\sin^2\theta'''.1_{,}$ upon the axis OC. On directional operation, the two are equivalent to the multiplication of the radius $1_{,} = OC$, first by the arc $\theta'''.1_r = CM'''$,

[redacted]

and then by its reciprocal $\frac{1}{\theta'''}.1_r = M'''AC'A'C = CAC'A'N'''$, the sine of which is $OR' = -\sqrt{(-1)}\sin\theta'''.1_r$.

The sum of the results of these three operations of multiplication is therefore $(+\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')1_e$, which is otherwise known to be equal to 1.1_e; and this disposes of the first line of one product, leaving the second line (a double one, enclosed in braces) to be examined.

(1) The multiplicand 1_e is carried by the factor $+\sqrt{(-1)}\sin\theta'.1_r$, from OA (fig. I.), through a right angle, into the position OB; whence the factor $-\sqrt{(-1)}\sin\theta''.1_r$ carries it through three right angles, upon OC' (fig. II.). The multiplying angles being here mutually perpendicular, their product does not give the result $+1 = \{+\sqrt{(-1)}\}\{-\sqrt{(-1)}\}$; but the result $\{+\sqrt{(-1)}\}\{-\sqrt{(-1)}\}\sin\theta'.\sin\theta''.1_e$ is a measure of the radius $OM_1 = -1.1_e$ on the negative axis OC'; and similarly it is found that the double factor in the first line comprised in each of the two other pairs of braces, gives a measure of OM_1 on a negative axis, namely, one on OB' and one on OA'; so that these three partial measures form the complete measure of $OM_1 = -1.1_e$, as the result of the factors written in the first line between the braces.

(2) Then in the second line, the multiplicand 1_e is carried by the factor $+\sqrt{(-1)}\sin\theta''.1_r$, from OC' (fig. II.), through one right angle, into the position OB (fig. I.); whence the factor $-\sqrt{(-1)}\sin\theta'.1_r$ carries it through three right angles, upon OA, where it constitutes a first partial measure of $OM = +1.1_e$. The double factor of the second line in each of the two other braces similarly gives a partial measure of OM on a positive axis, namely, one on OC and one on OB; so that all three together form the complete measure of $OM = +1.1_e$, and the results of the two lines between the braces balance in the equation $-1.1_e + 1.1_e = 0$.

11. The right-hand member of the equation of quaternions denotes the radius independent of direction, while the several terms of the left-hand member denote the respective measures of the radius upon that axis which is perpendicular to the axis from which its angle is counted. The diagonal OM of the parallelopipedon has the three measures OP, OQ, and OR, which are severally the sines of the AM', BM'', CM''', that serve to express the pri quaternion

$$\{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}1_e = 1_e = O.$$

[redacted]

The radii OM', OM'', OM''', are obtained by revolving OM perpendicularly to each coordinate plane respectively, until its intersection with the same.

If a great circle be drawn through the points A and M, and $\alpha.1_r$ be written for the arc AM that measures the angle AOM, the reciprocal of AM will be the arc $\frac{1}{\alpha}.1_r = AMA'N$ that measures the angle $(2\pi - \alpha) = AOMA'N = MOA'NA$. The radius ON would represent the quaternion conjugate to OM; but this radius ON, being revolved perpendicularly towards each coordinate plane till it meets the same, will determine three radii ON', ON'', and ON''' (the last being substituted for ON_1, because θ''' *decreases* as α *increases*), whose measures are seen to be respectively OP', OQ', and OR', which are precisely the measures of the radii OM_1', OM_1'', and OM_1''', respectively the opposites of the radii OM', OM'', and OM''' that constitute the primitive quaternion OM; and finally these radii OM_1', OM_1'', and OM_1''', are severally determined from OM_1, the opposing radius to OM, just as OM', OM'', and OM''' were determined from OM, and ON', ON'', and ON''' from ON, being the projections of OM_1 and ON upon the negative axes. Then since it has been seen that $OP' = -\sqrt{(-1)}\sin\theta'.1_s$, $OQ' = -\sqrt{(-1)}\sin\theta''.1_s$, and $OR' = -\sqrt{(-1)}\sin\theta'''.1_s$, it is manifest that our conjugate quaternion may be represented by

$$OM_1 = \{-\sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta'''\}\,1_s = 1_s,$$

independent of direction.

12. The rationale of the operation is thus given: Multiplication by a geometrical (that is, a spatial) factor is explained by the transfer of the multiplicand through a distance equal to the extent of the multiplier, such extent being counted from the origin of admeasurement. The result of the multiplication (or value of the product), or of several successive multiplications, is determined by the measure of the distance the multiplicand is finally carried from the origin. If the circumference be taken for spatial multiplier, and the unit radius 1_s loaded with the unit mass 1_μ at its further extremity for multiplicand, the distance of 1_μ from the centre of revolution will be unity, and one complete revolution or multiplication restores it to the same origin and distance from the centre of revolution, and consequently satisfies the equation $1^n.1_s.1_\mu = 1.1_s.1_\mu$. It is obvious, also, that the circumference may be divided into any number

of parts for multipliers, and successive multiplication by all these parts would restore the primitive direction and distance, while each single multiplication would place the unit of mass 1_μ in a position in which its measure on the primitive axis would have a different value.

13. If the unit diagonal OM of a rectangular parallelopipedon, regarded as the measure of a force of the first order, be referred to three rectangular coordinate axes, its projections on these axes will be the measures of its components in their directions, and may be expressed by $+\sqrt{(-1)}\sin\theta'.1_e$, $+\sqrt{(-1)}\sin\theta''.1_e$, and $+\sqrt{(-1)}\sin\theta'''.1_e$; each being equal, in another view, to the revolution of the radius $OM = 1_e$ through the corresponding angle θ', θ'', θ''', accompanied by a reduction of the value of the coefficient 1 to $\sin\theta'$, $\sin\theta''$, $\sin\theta'''$, so as to satisfy the equation

$$1.1_e = \{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}1_e.$$

The given quantities are the magnitude of the diagonal, and the angles it makes with the axes of coordinates; and in terms of these data are the magnitudes of the components determined. If now the resultant of the components in the direction of the diagonal be determined, it will give an identical equation; the process consisting in projecting back each component through the angle of its first projection from the diagonal, which is obviously equivalent to effecting the products

$$\{+\sqrt{(-1)}\sin\theta'\}\{-\sqrt{(-1)}\sin\theta'\} = +1\sin^2\theta',$$
$$\{+\sqrt{(-1)}\sin\theta''\}\{-\sqrt{(-1)}\sin\theta''\} = +1\sin^2\theta'',$$
$$\text{and}\quad \{+\sqrt{(-1)}\sin\theta'''\}\{-\sqrt{(-1)}\sin\theta'''\} = +1\sin^2\theta'''.$$

Here the components and their angles with the direction of the resultant are the data, and the magnitude of that resultant is both known and redetermined by the projections through the negative angles $(\frac{1}{2}\pi - \theta')$, $(\frac{1}{2}\pi - \theta'')$, $(\frac{1}{2}\pi - \theta''')$, the cosines of which are $-\sqrt{(-1)}\sin\theta'$, $-\sqrt{(-1)}\sin\theta''$, and $-\sqrt{(-1)}\sin\theta'''$. But revolution through each of these negative angles is evidently equivalent in result to a positive revolution through its complement to the circumference $(2\pi - \theta')$, $(2\pi - \theta'')$, $(2\pi - \theta''')$; and this leads to the effectuation of the product

$$\{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}$$
$$\{-\sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta'''\}$$
$$= (+1\sin^2\theta' + 1\sin^2\theta'' + 1\sin^2\theta''') = 1.$$

14. If, instead of using the angle θ''' counted from the zenithal axis (fig. III.), we write θ for its complement counted from the meridional axis, we fall upon the expression

$$\{\cos\theta + \sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta''\}\,\{\cos\theta - \sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta''\}$$
$$= \cos^2\theta + \sin^2\theta' + \sin^2\theta'' = 1,$$

which is the spherical analogue of the more familiar expression

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}\,\{\cos\theta - \sqrt{(-1)}\sin\theta\} = \cos^2\theta + \sin^2\theta = 1$$

for the circle. In operating the multiplications in this case, the same regard to the property of perpendicularity, both between arcs and between axes, must be had as in the case we have preferred to treat on account of its perfect symmetry.

15. The following deduction exhibits the quaternion in a form under which it has been obtained analytically.

Take the diameter M_1M of the sphere, and regard each radius M_1O and OM as positive unity: we can then write

$$\{1 + \sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}\,1_e$$
$$= M_1O + OP + OQ + OR = M_1O + OM = 2.1_e,$$
$$\{1 - \sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta''')\}\,1_e$$
$$= M_1O - OP' - OQ' - OR' = M_1O - OM_1 = 0.1_e,$$

and the effectuation of the product gives

$$1 + \sin^2\theta' + \sin^2\theta'' + \sin^2\theta''' = 0.$$

Then, if we make the radius $= m_1$, $\sin\theta' = m_2$, $\sin\theta'' = m_3$, $\sin\theta''' = m_4$, we get the analytical expression

$$m_1^2 + m_2^2 + m_3^2 + m_4^2 = 0$$

alluded to, which has also been shown to be the product

$$(m_1 + im_2 + jm_3 + km_4)\,(m_1 - im_2 - jm_3 - km_4) = 0,$$

where $i^2 = j^2 = k^2 = -1$; and to these coefficients i, j, k, belongs yet the very remarkable property, that

$$ij = -ji, \quad ik = -ki, \quad \text{and} \quad jk = -kj,$$

which may be accounted for in the following manner.

(1) Take for the primitive quaternion OM any one of the three unit axes, as $OA = +1'.1_e = +i$: then observing

that the operation, which in Sir W. R. Hamilton's lectures is defined to be a linear multiplication by the axis of rotation, is here exhibited as multiplication by a right angle perpendicular to that axis, we may trace the steps as they follow:

$$\begin{aligned}
(+1')\,1_e &= \quad\; +i = OA \;\ldots\ldots\; \text{I.}\\
(+1')\,\{+\sqrt{(-1)'}\}\,1_e &= ik = +j = OB \;\ldots\ldots\; \text{I.}\\
\{+\sqrt{(-1)'}\}\,\{+\sqrt{(-1)''}\}\,1_e &= ji = +k = OC \;\ldots\ldots\; \text{II.}\\
\{+\sqrt{(-1)'}\}\,(-1'')\,1_e &= ki = -j = OB' \;\ldots\ldots\; \text{II.}\\
\{+\sqrt{(-1)'}\}\,\{-\sqrt{(-1)''}\}\,1_e &= -ji = -k = OC' \;\ldots\ldots\; \text{II.}\\
\{+\sqrt{(-1)'}\}\,(+1'')\,1_e &= -ki = +j = OB \;\ldots\ldots\; \text{II.}\\
(+1'')\,(-1')\,1_e &= jk = -i = OA' \;\ldots\ldots\; \text{I.}\\
(+1'')\,\{-\sqrt{(-1)'}\}\,1_e &= -ik = -j = OB' \;\ldots\ldots\; \text{I.}\\
(+1'')\,(+1')\,1_e &= -jk = +i = OA \;\ldots\ldots\; \text{I.}
\end{aligned}$$

(2) Take $OB = +1''.1_e$ for primitive quaternion, and

$$\begin{aligned}
(+1'')\,1_e &= \quad\; +j = OB \;\ldots\; \text{II.}\\
(+1'')\,\{+\sqrt{(-1)''}\}\,1_e &= ji = +k = OC \;\ldots.\; \text{II.}\\
\{+\sqrt{(-1)''}\}\,\{+\sqrt{(-1)'''}\}\,1_e &= kj = +i = OA \;\ldots\; \text{III.}\\
\{+\sqrt{(-1)''}\}\,(-1''')\,1_e &= ij = -k = OC' \;\ldots\; \text{III.}\\
\{+\sqrt{(-1)''}\}\,\{-\sqrt{(-1)'''}\}\,1_e &= -kj = -i = OA' \;\ldots\; \text{III.}\\
\{+\sqrt{(-1)''}\}\,(+1''')\,1_e &= -ij = +k = OC \;\ldots.\; \text{III.}\\
(+1''')\,(-1'')\,1_e &= ki = -j = OB' \;\ldots\; \text{II.}\\
(+1''')\,\{-\sqrt{(-1)''}\}\,1_e &= -ji = -k = OC' \;\ldots\; \text{II.}\\
(+1''')\,(+1'')\,1_e &= -ki = +j = OB \;\ldots.\; \text{II.}
\end{aligned}$$

(3) Lastly, take OC for primitive quaternion, and

$$\begin{aligned}
(+1''')\,1_e &= \quad\; +k = OC \;\ldots..\; \text{III.}\\
(+1''')\,\{+\sqrt{(-1)'''}\}\,1_e &= kj = +i = OA \;\ldots.\; \text{III}\\
\{+\sqrt{(-1)'''}\}\,\{+\sqrt{(-1)'}\}\,1_e &= ik = +j = OB \;\ldots.\; \text{I.}\\
\{+\sqrt{(-1)'''}\}\,(-1')\,1_e &= jk = -i = OA' \;\ldots.\; \text{I.}\\
\{+\sqrt{(-1)'''}\}\,\{-\sqrt{(-1)'}\}\,1_e &= -ik = -j = OB' \;\ldots\; \text{I.}\\
\{+\sqrt{(-1)'''}\}\,(+1')\,1_e &= -jk = +i = OA \;\ldots.\; \text{I.}\\
(+1')\,(-1''')\,1_e &= ij = -k = OC' \;\ldots\; \text{III.}\\
(+1')\,\{-\sqrt{(-1)'''}\}\,1_e &= -kj = -i = OA' \;\ldots\; \text{III.}\\
(+1')\,(+1''')\,1_e &= -ij = +k = OC \;\ldots.\; \text{III.}
\end{aligned}$$

The diagonal of the cube whose side is unity, is the single quaternion competent to these three processes.

16. Triple products will appear as follow:

(1) $OB = +1.1_{,}$ ...... II., I.

$OA' = k1_{,} = \sqrt{(-1)}'.1_{,}$ ...... I., III.

$OC = kj1_{,} = \sqrt{(-1)}'.\sqrt{(-1)}'''.1_{,}$ ...... III., II.

$OB' = kji1_{,} = \sqrt{(-1)}'.\sqrt{(-1)}'''.\sqrt{(-1)}''.1_{,} = -1.1_{,}$...II., I.

(2) $OB = +1.1_{,}$ ...... II.

$OC = i1_{,} = \sqrt{(-1)}''.1_{,}$ ...... II., III.

$OA = ij1_{,} = \sqrt{(-1)}''.\sqrt{(-1)}'''.1_{,}$ ...... III., I.

$OB = ijk1_{,} = \sqrt{(-1)}''.\sqrt{(-1)}'''.\sqrt{(-1)}'.1_{,} = +1.1_{,}$...I., II.

(3) $OB' = -1.1_{,}$ ...... II.

$OC' = i1_{,} = \sqrt{(-1)}''.1_{,}$ ...... II., III.

$OA' = ij1_{,} - \sqrt{(-1)}''.\sqrt{(-1)}'''.1_{,}$ ...... III., I.

$OB' = ijk1_{,} = \sqrt{(-1)}''.\sqrt{(-1)}'''.\sqrt{(-1)}'.1_{,} = -1.1_{,}$...I., II.

(4) $OB' = -1.1_{,}$ ...... II., I.

$OA = k1_{,} = \sqrt{(-1)}'.1_{,}$ ...... I., III.

$OC' = kj1_{,} = \sqrt{(-1)}'.\sqrt{(-1)}'''.1_{,}$ ...... III., II.

$OB = kji1_{,} = \sqrt{(-1)}'.\sqrt{(-1)}'''.\sqrt{(-1)}''.1_{,} = +1.1_{,}$......II.

Which results show that $ijk = -kji$. (See Hamilton's *Lectures*, p. 208, &c.)

17. In the *London, Edinburgh, and Dublin Philosophical Magazine* for October 1853, p. 283, it is related that Sir W. R. Hamilton has recently employed in some formulæ a fourth imaginary unit h, in addition to i, j, k, appropriated to the three coordinate planes of space; and as this additional imaginary is distinct from the other three, and commutative with them, the author of the article infers that it must be extraspatial. This supposition is not at all necessary: the three symbols of perpendicularity, i, j, k, are restricted each to its *special* plane; while h may hold a *general* value, applicable to each and all of the planes.

N.B. The quaternion has yet other forms than those above given, but here is enough for an example of the

method I have followed in obtaining them. This method arises from carrying out the principles of *the calculus of operations*, as I understand the meaning of the term. Any compound function is the result of certain simple operations, the traces of which are retained in its expression. For instance, the function ψ^n expresses the *abstract* result of n multiplications; and this result will be rendered *concrete* by introducing the unit of space, when it becomes $\psi^n . 1_s^n$, or $\psi^n . 1_e^n$ if 1_e the linear unit be used. We must begin, then, by inquiring into the result of n operations with *linear* or *angular* unity, or both. When the relation between the diagonal and sides of a rectangular parallelogram, or between the diagonal and sides of a rectangular parallelopipedon, is brought to mind, in which we have

$$1^2_e = (\cos^2\theta + \sin^2\theta)\, 1_e = 1_e$$

in the first case, and

$$1^2_e = (\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')\, 1_e = (\cos^2\theta' + \cos^2\theta'' + \cos^2\theta''')\, 1_e = 1_e$$

in the second, together with the decomposition of each of these compound coefficients of the linear unit into a pair of binomial factors, it is readily perceived that the concrete function of the second degree $\psi^2 . 1_e^2$ contains values that cannot be evolved from the abstract or merely numerical ψ^2. Attention to the element of space, necessarily involved in the measurement of the result of any operation whatever, has furthermore led to the discovery of the actual genesis of the differential coefficient, and thereby thrown open the entire theory of mathematical development.

THE ATTRACTION OF ELLIPSOIDS CONSIDERED GEOMETRICALLY.

By Matthew Collins, B.A.

The attraction of an ellipsoid A on a point P on its surface or within it, in a direction perpendicular to one of its principal planes B, is proportional to the distance of the attracted point P from that principal plane.

1. When P is on the surface, draw PP', a chord of A, perpendicular to B, and through P and P' draw planes parallel to B, cutting the principal axis CC' perpendicu[lar to B in]
p and p'; then describe through p, p' an ellipsoi[d con-]
centric, similar, and similarly placed to A, and its

on p will be equal to the attraction of A on P in a direction perpendicular to B: for through PP' draw two planes E, F, containing a very small angle, and through pp' draw two planes e, f, parallel to the former; then let a cone of revolution, whose axis is PP' and vertex P, cut E, F along the lines PE, PE', PF, PF'; and let another such cone, very close to the former, also cut E, F along PE_1, PE_1', PF_1, PF_1'; and through p draw in the planes e, f the straight lines pe, pe', pf, pf', and pe_1, pe_1', pf_1, pf_1', respectively parallel to the foregoing: then, as the sections of the two similar ellipsoids A and a, by the parallel planes E, e, are necessarily similar ellipses, and as the chord PP', parallel to an axis (CC') of the greater ellipse, is equal to the homologous axis pp' of the less; therefore, by Airy's *Tract* on the Figure of the Earth, Props. 2, 3,

$$PE + PE' = pe + pe':$$

and therefore, by Airy's 4th Prop., the sum of the attractions exerted on P along PP' by the two small pyramids $PEFE_1F_1$, $PE'F'E_1'F_1'$, is equal to the sum of the attractions exerted on p along pp' by the two corresponding small pyramids $pefe_1f_1$, $pe'f'e_1'f_1'$: and since there are obviously as many pairs of pyramids in the double wedge $PP'EFE'F'$ as there are corresponding pairs of pyramids (whose solid angles are also equal to those of the former) in the double wedge $pp'efe'f'$. And as, moreover, each double wedge of A has a corresponding double wedge of a, therefore the whole attraction of a on p is equal to the attraction of A on P along PP'; but, since a is similar to A, the attraction of a on p is to attraction of A on C as pp' $(= PP')$ to CC' (*Principia*, Prop. 87, Cor. 1); and so the attraction of A on P perpendicular to B, which was proved equal to the attraction of a on p, is therefore $= \dfrac{\frac{1}{2}PP'}{\frac{1}{2}CC'} \times$ attraction of A on C; which, since A and CC' are constant, $\propto \frac{1}{2}PP'$, viz. the distance of P from B.

The general equation of a surface of the second order being

$$Q(xyz) = A + Bx + Cy + Dz + Ex^2 + \&c. = 0,$$

the equation of the diametral plane bisecting all chords ¬llel to the straight line $x = mz$ and $y = nz$, is known to be

$$md_xQ + nd_yQ + d_zQ = 0,$$

., not containing the absolute constant term A, shews

that if *any* straight line $ABB'A'$ cuts two surfaces of the second order, whose equations differ only in the constant term, the intercepts AB, $A'B'$ will be equal, since the chords AA', BB' are bisected in the same point by the diametral plane conjugate to it, which plane is the same for both surfaces. Now, the equations of two ellipsoids, which are concentric, similar and similarly placed, differ only in the absolute constant term, and therefore the intercepts of *any* straight line cutting two such ellipsoids are equal; and hence it follows that a shell, bounded by concentric, similar and similarly placed ellipsoidal surfaces, exerts no attraction on a point P situated anywhere within it, or upon its interior surface. See Airy's *Tract* on the Figure of the Earth, Prop. 12, or the *Principia*, Props. 70 and 91, Cor. 3, Book I.

(2) When P is within the ellipsoid A, we have then only to describe through P another ellipsoid A' concentric, similar and similarly placed to A; and since the shell between A and A' exerts no attraction on P, as is already proved, therefore the attraction of A on P is the same as that of A' on P; then, as in (1), draw PP' a chord of A' perpendicular to B, and through P and P' draw planes parallel to B, cutting the principal axis CC' perpendicularly in p and p', and then describe through p and p' another ellipsoid a, concentric, similar and similarly situated to A or A': and by (1), the attraction of A (or A') on P perpendicularly to B is equal to the whole attraction of a on p, and therefore $= \dfrac{\frac{1}{2}PP'}{\frac{1}{2}CC'} \times$ attraction of A on C, which, as already observed, $\propto \frac{1}{2}PP'$, viz. the distance of P (now supposed within A) from B.

[The foregoing elementary geometrical demonstration is that alluded to and promised in the Note at the bottom of page 51 of the No. of this Journal for February 1854.]

The preceding proposition shews that the attraction of an ellipsoid on any point on its surface or within it, can be got at once from the attraction of the ellipsoid on a point placed at the extremity of an axis; and this latter attraction is found, and reduced to elliptic functions, as follows.

Let O be the centre and $OA = a$, $OB = b$, $OC = c$, the semiaxes; and let the attracted point C be the vertex, and CO the axis of a cone of revolution D whose semiangle is θ, and let $\theta + d\theta$ be the semiangle of another such cone E.

[redacted]

very close to D, and having the same vertex and axis; and let O be the common vertex of two other cones D' and E' parallel to the former. Conceive the portion of the ellipsoid between D and E to be divided into elementary pyramids by planes passing through CO: let f be the length of a side of one of these little pyramids, which is a chord of the ellipsoid and a side of D, and let g be the parallel side of D', which is a radius of the ellipsoid; and let f' and g' be the projections of f and g upon c, and let ω be the small angle between two consecutive f's (or g's); then the attraction of the little pyramid, whose side is f, on its vertex C is equal to $f\omega d\theta$, (vide Airy's *Tract* on the Figure of the Earth, Prop. 4), and therefore its component along CO is equal to $f'\omega d\theta$, therefore

$$= 2c \times \frac{f'}{2c}\,\omega d\theta;$$

but $\dfrac{f'}{2c} = \dfrac{g'^2}{c^2}$, therefore the said component $= \dfrac{2}{c}\,g'^2\omega d\theta$, therefore

$$= \frac{4}{c} \times \frac{1}{2}\,g^2 \cos^2\theta\,\omega d\theta.$$

Now $\frac{1}{2}g^2\omega$ is the area on D' included between the two consecutive g's, and the sum of all such elements is the entire surface of D', which we shall still name D'; therefore the attraction of the slice of the ellipsoid between D and E upon C along CO is $= \dfrac{4D'\cos^2\theta\,d\theta}{c}$. Now the projection of D' on the plane of ab is obviously an ellipse, whose area D'' is $= D'\sin\theta$. Let now r and r' be the sides of D' (or radii of the ellipsoid) lying in the planes of ca and cb, then the semiaxes of D'' are plainly the projections of r and r', or $r\sin\theta$ and $r'\sin\theta$, and therefore $D'' = \pi rr'\sin^2\theta$; and so $D' = \dfrac{D''}{\sin\theta}$, therefore $= \pi rr'\sin\theta$: and so the attraction of the slice on C along CO is $= \dfrac{4\pi}{c}\,rr'\cos^2\theta\sin\theta d\theta$. Now $\dfrac{1}{r^2} = \dfrac{\cos^2\theta}{c^2} + \dfrac{\sin^2\theta}{a^2}$ and $\dfrac{1}{r'^2} = \dfrac{\cos^2\theta}{c^2} + \dfrac{\sin^2\theta}{b^2}$, and therefore the differential of the required attraction on C is

$$= \frac{4\pi abc\cos^2\theta\sin\theta d\theta}{(c^2\sin^2\theta + a^2\cos^2\theta)^{\frac{1}{2}}(c^2\sin^2\theta + b^2\cos^2\theta)^{\frac{1}{2}}}.$$

This expression, given in Poisson's *Mécanique*, vol. I. p. [illegible]

[redacted]

by putting $\cos\theta = u = (1+v)^{-\frac{1}{2}}$, becomes transformed into

$$\frac{4\pi abc u^2 du}{\{c^2+(a^2-c^2)u^2\}^{\frac{1}{2}}\{c^2+(b^2-c^2)u^2\}^{\frac{1}{2}}}$$

$$= \frac{2\pi c\,dv}{(1+v)\sqrt{\left\{(1+v)\left(1+\frac{c^2}{a^2}v\right)\left(1+\frac{c^2}{b^2}v\right)\right\}}},$$

where $v = \tan^2\theta$.

Now, supposing $a > b > c$, let OA' and OB', portions of OA and OB, be the semiaxes of the focal ellipse whose plane is perpendicular to the least semiaxis OC, then

$$OA'^2 = a^2 - c^2 \quad\text{and}\quad OB'^2 = b^2 - c^2;$$

let OQ be perpendicular to the tangent PTQ which touches this ellipse in T and meets OA' at the point P, and let the angle $OPT = \phi$, and $\rho^2 = c^2 + (b^2-c^2)u^2 = c^2 + OB'^2\cos^2\theta$, and $\rho'^2 = c^2 + (a^2-c^2)u^2 = c^2 + OA'^2\cos^2\theta$; and if the point P be taken such that $\frac{OP}{OA'} = \frac{\rho}{c}$ or $OP = \frac{OA'}{c}\rho$, then the equation $OA'^2\cos^2 A'OQ + OB'^2\cos^2 B'OQ = OQ^2$, $\therefore = OP^2\sin^2\phi$, gives

$$(a^2-c^2)\sin^2\phi + (b^2-c^2)\cos^2\phi = \frac{a^2-c^2}{c^2}\{c^2+(b^2-c^2).\cos^2\theta\}\sin^2\phi,$$

which gives

$$(a^2-c^2)\cos^2\theta = \cos^2\phi\,\{c^2 + (a^2-c^2).\cos^2\theta\},$$

i.e. $OA'^2\cos^2\theta = \rho'^2\cos^2\phi$, and therefore $\cos\phi = \frac{OA'}{\rho'}\cos\theta$;

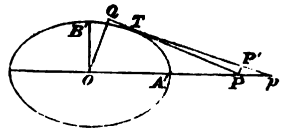
and since $\quad OP = \frac{OA'}{c}\rho = \frac{OA'}{c}(c^2 + OB'^2\cos^2\theta)^{\frac{1}{2}}$,

therefore

$$Pp = d.OP = \frac{OA'}{c}(c^2 + OB'^2\cos^2\theta)^{-\frac{1}{2}}.OB'^2\cos\theta\sin\theta.d\theta$$

$$= \frac{OA'.OB'^2}{c\rho}\cos\theta\sin\theta.d\theta.$$

Let PP' be perpendicular to the consecutive tangent from p, then $pP' = Pp\cos\phi$, therefore



$$= \frac{OA'^2.OB'^2}{c\rho'}\cos^2\theta\sin\theta.d\theta,$$

but the attraction of the slice of the ellipsoid betwe
and E on C along CO was found above

$$= \frac{4\pi abc \cos^2\theta \sin\theta . d\theta}{\rho\rho'}.$$

And moreover, since $P'p$ is obviously $= d.B'TP$, therefore

$$= d(B'TP - \text{const. } B'TA'), \quad i.e. = d(TP - \text{arc } TA');$$

therefore the attraction of the slice

$$= \frac{4\pi abc^2}{OA'^2.OB'^2} \times P'p, \quad i.e. = \frac{4\pi abc^2}{(a^2-c^2)(b^2-c^2)} d(TP - \text{arc } TA');$$

and therefore the attraction of the whole ellipsoid on C is

$$= \frac{4\pi abc^2}{(a^2-c^2)(b^2-c^2)} \times (T_1P_1 - \text{arc } T_1A'),$$

where P_1 and T_1 denote the ultimate positions of P and T corresponding to $\theta = 0$; and since, by construction,

$$OP = \frac{OA'}{c}\rho = \frac{OA'}{c}\{c^2 + (b^2 - c^2).\cos^2\theta\};$$

therefore, when $\theta = 0$, $OP_1 = \frac{b}{c} OA'$, and therefore

$$P_1T_1 = \frac{a(b^2-c^2)}{bc} = \frac{a^2(b^2-c^2)}{abc}.$$

Hence also the differential of the ellipsoid's attraction on B, *i.e.* the attraction on B along BO of a portion of the ellipsoid comprised between two cones of revolution whose vertex is B and axis BO, and semiangles θ and $\theta + d\theta$, is

$$= \frac{4\pi abc \cos^2\theta \sin\theta . d\theta}{(b^2\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}} (b^2\sin^2\theta + a^2\cos^2\theta)^{\frac{1}{2}}}$$

$$= \frac{4\pi abc u^2 du}{\{b^2 - (b^2-c^2).u^2\}^{\frac{1}{2}} \{b^2 + (a^2-b^2).u^2\}^{\frac{1}{2}}},$$

where, as before, $u = \cos\theta$. Now, to represent this geometrically, let OA'' and OC', portions of OA, OC, be the semiaxes of the focal hyperbola whose plane is perpendicular to OB, then

$$OA''^2 = a^2 - b^2 \text{ and } OC'^2 = c^2 - b^2 = -(b^2 - c^2) = -OB'^2;$$

and putting now

$$\rho^2 = b^2 - (b^2 - c^2)\cos^2\theta \text{ and } \rho'^2 = b^2 + (a^2 - b^2)\cos^2\theta,$$

and taking the point P on the primary axis OA'' so that $\frac{OP}{OA''} = \frac{\rho}{b}$, so that, as $\rho < b$, P will lie between O and A''; then, drawing PT touching the hyperbola, we find, as before, *mutatis mutandis*, the whole of the attraction on B

$$= \frac{4\pi ab^2 c}{(a^2 - b^2)(b^2 - c^2)} (T_1 P_1 - \text{arc } T_1 A''),$$

P_1 and T_1 being now, as before, the ultimate positions of P and T corresponding to $\theta = 0$, so that, as before,

$$OP_1 = \frac{c}{b} OA''.$$

The whole attraction on A cannot be represented similarly, because there is no focal conic perpendicular to OA; but the equation $\frac{A}{a} + \frac{B}{b} + \frac{C}{c} = 4\pi$, discovered by the late Professor M'Cullagh, will then serve to A, where A, B, C denote the whole attractions of the ellipsoid on the points A, B, C.

Let a, b, c, be the semiaxes of an homogeneous fluid ellipsoid, and A, B, C the attractions on points at the ends of a, b, c, caused partly by the ellipsoid's own attractions on its parts, partly by centrifugal force of revolution about an axis ($2c$), or by the action of an extraneous force directed towards the centre, and varying as the distance from the centre; then the ellipsoid will preserve its shape if $Aa = Bb = Cc$.

For then the whole forces, acting on any particle xyz of the mass in directions parallel to a, b, c, will obviously be

$$\frac{Ax}{a}, \frac{By}{b}, \text{ and } \frac{Cz}{c}.$$

Now, dividing these by $Aa = Bb = Cc$, they are as

$$\frac{x}{a^2}, \frac{y}{b^2}, \text{ and } \frac{z}{c^2};$$

but when the point xyz is on the surface, these last are as the cosines of the angles that the normal at the point xyz makes with the axes, since the equation of the tangent plane is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1.$$

Thus the components of the force on the point xyz at the surface are as the cosines of the angles that the normal at this point makes with the axes; therefore the direction of the resultant force is coincident with the normal, or perpendicular to the surface itself, which is the necessary condition of equilibrium, the general formula

$$dp = \rho(Xdx + Ydy + Zdz)$$

obviously becomes in this case

$$dp = -\rho\left(\frac{Ax}{a}\,dx + \frac{By}{b}\,dy + \frac{Cz}{c}\,dz\right)$$

$$= -\frac{\rho Aa}{2}\left(\frac{2xdx}{a^2} + \frac{2ydy}{b^2} + \frac{2zdz}{c^2}\right),$$

and therefore $p = C' - \frac{\rho Aa}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$ at the surface $p = 0$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, therefore $C' = \frac{\rho Aa}{2}$, and therefore the pressure $= \frac{\rho Aa}{2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)$, which, at the centre, where $x = y = z = 0$, becomes $\frac{\rho Aa}{2}$. (See the *Principia*, Prop. 20, Book III.; M'Laurin's *De Causa Phys. Flux et Reflux*, Maris, Prop. 1; Airy's *Tract* on the Figure of the Earth, Props. 14, 15, 16.)

Let R and r be the radii of two homogeneous concentric globes, A and a the attractions of each on a point on the surface of the other, then $\frac{A}{R^3} = \frac{a}{r^3}$, whatever be the law of attraction as a function of the distance.

Let O be their common centre, OrR a radius meeting them at r and R, bc a chord of less parallel to OR: produce Ob, Oc to meet the large globe at B, C, then BC will be parallel to bc or to OR; and if b describe any little figure b' on the surface of r, it is evident B will describe a similar figure B' on the surface of R, and the areas S, s of the normal sections of the cylinders C and c, simultaneously described by bc and BC, will obviously be to each other as $B' : b'$, therefore as $R^2 : r^2$. Now, by Euclid, Prop. 4, Book I., $Br = bR$ and $Cr = cR$; therefore, by Airy's *Tract* on the Figure of the Earth, Prop. 18 (generalized), attraction of cylinder C on the point r along rO : attraction of cylinder c on R along $RO :: S : s :: R^2 : r^2$; and as this

fixed proportion holds true for each corresponding pair of cylinders, therefore by taking their sums we shall also have $A : a :: R^2 : r^2$. (See Poisson's *Mécanique*, vol. I. p. 201.)

P.S.—Several of the propositions given in the present paper, and in that I have inserted in the February No. of this Journal, were given in 1846 by the late Professor M'Cullagh at his lectures. I have, however, departed in several places from his method of proof, and supplied some connecting links which seemed most necessary.

Mechanics' Institute, Liverpool, April 15, 1854.

ON THE AREA OF THE CYCLOID.

To the Editor of the Cambridge and Dublin Mathematical Journal.

Sir,—The determination of the area of the Cycloid, so easily effected by modern analysis, was regarded by the geometricians of the seventeenth century as a problem of no small difficulty. Mersenne was the first who attempted a solution: he was however unsuccessful. It was proposed by him in despair to Roberval in 1628, who also failed in his attempt at that time. About seven years afterwards, however, Roberval overcame the difficulty and communicated his good fortune to Mersenne.

In a letter to Descartes, Mersenne made mention of Roberval's discovery of the area of the Cycloid as a great feat in geometry, simply stating the result obtained by Roberval, without giving any clue to the method. Descartes, solving the problem himself with little difficulty, communicated his method in reply to Mersenne, with some supercilious remarks about the supposed difficulty of the problem. Fermat and other mathematicians of that day exercised their ingenuity in the same question. A solution of the problem by pure geometry, which was some time ago communicated to me by Mr. R. L. Ellis of Trinity College, possesses so great a superiority over any of the geometrical methods of these early mathematicians which I have seen, that I think it may be acceptable to those readers of your Journal who take an interest in the history of mathematics.

"The motion of the generating circle may be resolved into two uniform motions, a motion of translation parallel to the directrix and of rotation round its own cer
area generated by the describing point may be
as generated by these two motions: that of transl

affects the motion of rotation, and the area due to the latter is the same as if the former did not exist, that is, it is equal to the area of the generating circle. Contrariwise the motion of rotation does affect the area due to that of translation, inasmuch as in virtue of it the distance of the describing point from the directrix is varied: the mean distance, viewed as depending on the motion of rotation, is equal to the radius of the generating circle, and the corresponding area is therefore a rectangle, the base of which is the space slided over and altitude that radius; and, as this space is the circumference of the generating circle, the area in question is equal to twice the area of that circle: on the whole, therefore, the area of the cycloid is equal to three times that of the generating circle.

"The reason is just the same as that by which what are called Guldinus's properties are established. We here resolve the motion of a describing point into motions parallel and perpendicular to the abscissa; the latter generates no area, the former generates a rectangular area having for its base the abscissa and for its altitude the mean value of the ordinates; that is, the ordinate of the centre of gravity of the arc, which is a known result. The only difference to be attended to in the two cases relates to the mode in which the average is to be taken."

Mr. Ellis has remarked, that the same method may be extended to the determination of the areas of the hypocycloid and epicycloid.

I am, Sir,
Your obedient Servant,

WILLIAM WALTON.

Cambridge, July 31, 1854.

PROOF OF THE PARALLELOGRAM OF FORCES.

By ARTHUR COHEN.

THE only difficulty in the analytical proof of the parallelogram of forces consists in determining the direction of the resultant of two forces acting at right angles to one another.

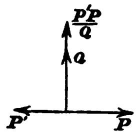
This problem may, in the following manner, be reduced to the solution of the functional equation

$$\phi(x) + \phi(y) = \phi(x+y),$$

which is easily solved by ordinary algebra.

[redacted]

Let P and Q be two forces acting at right angles to one another, R their resultant inclined to P at an angle θ. The direction of R and its ratio to P clearly depend only on the ratio $\frac{P}{Q}$. We may therefore put



$$\theta = \phi(a) \text{ where } \tan(a) = \frac{Q}{P}.$$

In direction of Q and in direction opposite to P' apply respectively two forces $\frac{P'P}{Q}$ and P', and let R' be their resultant; then, since $P : Q :: \frac{P'P}{Q} : P'$, it is clear that R' and R are at right angles to one another. Let the resultant of R' and R be inclined to R at an angle θ'. Then

$$\theta' = \phi(a') \text{ where } \tan(a') = \frac{R'}{R} = \frac{P'}{Q};$$

and since R' is also the resultant of the forces $P - P'$, $Q + \frac{P'P}{Q}$, therefore

$$\theta + \theta' = \phi(\beta) \text{ where } \tan(\beta) = \frac{Q + \frac{P'P}{Q}}{P - P'} = \frac{\frac{Q}{P} + \frac{P'}{Q}}{1 - \frac{P'}{Q} \cdot \frac{Q}{P}} = \tan(a + a');$$

therefore $\qquad \beta = a + a'$;

therefore $\qquad \phi(a) + \phi(a') = \phi(a + a')$:

the solution of which last equation gives $\phi(a) = Ca$, C being a constant, that must equal unity, inasmuch as when

$$P = Q \tan(\theta) = 1 \text{ and } \tan(a) = 1;$$

therefore $\theta = a$, or the direction of the resultant coincides with that of the diagonal of the rectangle constructed on the two component forces.

THEOREMS ON THE QUADRATURE OF SURFACES AND THE RECTIFICATION OF CURVES.

By Rev. Robert Carmichael, A.M.,

Fellow of Trinity College, Dublin, and Examiner in Mathematics for the year [illegible] in the Queen's University in Ireland.

1. It is well known that there are many plane curves whose equations are more easily expressed in polar than in rectangular coordinates, and for whose rectification we employ the formula

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\{r^2 + (D_\theta r)^2\}}\, d\theta.$$

Of this class are, the Spiral of Archimedes,

$$r = a\theta;$$

the Lituus,

$$r^2 = \left(\frac{a}{\theta}\right)^2;$$

the Lemniscate,

$$r^2 = a^2 \cos 2\theta;$$

the Logarithmic Spiral,

$$r = ce^{\frac{\theta}{a}};$$

and the Cardioid,

$$r = a(1 - \cos\theta).$$

2. I am not aware that any mathematician has attempted to trace the surfaces analogous to these; but, for the *quadrature* of such surfaces, when discovered, it is absolutely necessary that we should have a general expression *in polar coordinates* for the element of any surface. Such an expression is not found in the ordinary works upon the Differential and Integral Calculus. In the elaborate treatise upon this subject by M. L'Abbé Moigno (*Paris*, 1844, tom. II. p. 235), the expression is investigated, by the usual analytical method, transformation of coordinates, from the well-known expression in rectangular coordinates,

$$d\sigma = \sqrt{(1 + p^2 + q^2)}\, dx\, dy,$$

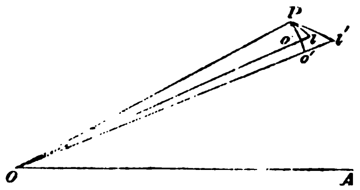
and is given in the following shape,

$$d\sigma = \sqrt{\{r^2 \sin^2\theta + \sin^2\theta (D_\theta r)^2 + (D_\phi r)^2\}}\; r\, d\theta\, d\phi.$$

A short geometrical deduction of this expression, whose merit I have great pleasure in sharing with my friend,

Alexander Jack, Esq., A.B., may not be unacceptable to the student.

Let P be any point on the surface. Through the axis OA and OP describe a plane, and round the axis describe, with the same line, a cone. The surface may then be supposed to be divided into its elements by planes and cones consecutive to these respectively, (the planes all



passing through the axis and the cones round it,) half of one such element being represented by $\iota P\iota'$. Then, remembering that the planes cut the cones orthogonally, we have

$$d\sigma = P\iota . P\iota' . \sin \iota P\iota' = P\iota . P\iota' . \surd(1 - \cos^2 \iota P\iota'),$$

whence

$$d\sigma = P\iota . P\iota' . \surd(1 - \sin^2 \iota Po . \sin^2 \iota' Po') = \surd(P\iota^2 . P\iota'^2 - o\iota^2 . o'\iota'^2),$$

o and o' being the points where the sphere described round the origin with radius OP intersects the consecutive radii vectores to the points ι, ι'; or

$$d\sigma = \surd[\{r^2 \sin^2\theta\, d\phi^2 + (D_\phi r)^2 d\phi^2\} . \{r^2 d\theta^2 + (D_\theta r)^2 d\theta^2\} - (D_\theta r)^2 d\theta^2 . (D_\phi r)^2 d\phi^2],$$

or finally,

$$d\sigma = \surd\{r^2 \sin^2\theta + \sin^2\theta\, (D_\theta r)^2 + (D_\phi r)^2\}\, r\, d\theta\, d\phi.$$

3. From this expression we may readily derive that for the perpendicular from the origin upon the tangent plane, in *polar* coordinates. In *rectangular* coordinates it is known to be

$$P = \frac{z - px - qy}{\surd(1 + p^2 + q^2)};$$

but the transformation of this to polar coordinates would be troublesome and tedious. We may easily derive the required expression from the volume of the elementary cone, for

$$Pd\sigma = r^3 \sin\theta\, d\theta\, d\phi,$$

and therefore

$$P = \frac{r^2 \sin\theta}{\surd\{r^2 \sin^2\theta + \sin^2\theta (D_\theta r)^2 + (D_\phi r)^2\}}.$$

4. As an example of the application of the formula for the quadrature of surfaces, let us suppose that it is re-

quired to investigate the quadrature, between given limits, of the surface

$$r = me^{-\phi} \cos\theta.$$

Then

$$D_\theta r = -me^{-\phi} \sin\theta, \quad D_\phi r = -m e^{-\phi} \cos\theta;$$

therefore

$$d\sigma = \sqrt{(m^2 e^{-2\phi} \cos^2\theta \sin^2\theta + m^2 e^{-2\phi} \sin^4\theta + m^2 e^{-2\phi} \cos^2\theta)}\, r\, d\theta\, d\phi,$$

or

$$d\sigma = m^2 e^{-2\phi} \cos\theta\, d\theta\, d\phi;$$

whence

$$\Sigma = m^2 \int e^{-2\phi} (\sin\theta_2 - \sin\theta_1)\, d\phi.$$

Let us suppose the limits to be given by the intersections, with the given surface, of the cones

$$\theta_2 = a\phi, \quad \theta_1 = b\phi,$$

and

$$\Sigma = m^2 \int_{\phi_1}^{\phi_2} e^{-2\phi} (\sin a\phi - \sin b\phi)\, d\phi,$$

an integral which is susceptible of easy reduction, since we know that

$$\int e^{-m\phi} \sin a\phi\, d\phi = -e^{-m\phi} \frac{m \sin a\phi + a \cos a\phi}{m^2 + a^2}.$$

5. As a second example, let it be proposed to investigate the quadrature, within given limits, of the surface

$$r = m \cos\phi \sin\theta.$$

Here

$$D_\theta r = m \cos\phi \cos\theta, \quad D_\phi r = -m \sin\phi \sin\theta,$$

and

$$d\sigma = m^2 \cos\phi \sin^2\theta\, d\theta\, d\phi;$$

whence

$$\Sigma = m^2 \int_{\theta_1}^{\theta_2} (\sin\phi_2 - \sin\phi_1) \sin^2\theta\, d\theta;$$

and, if the limits be given as before, there is no difficulty in determining the quadrature completely.

6. In the masterly treatise upon the Calculus of Variations by the Rev. Professor Jellett (*Dublin*, 1850, p. 262), it is shown that the surface which, within given limits, renders the double integral

$$\iint \sqrt{(p^2 + q^2)}\, dx\, dy,$$

[redacted]

or, γ being the angle made by the radius vector with the axis of z,

$$\iint \sin\gamma . d\sigma,$$

a minimum, is given by the partial differential equation

$$q^2 r - 2pqs + p^2 t = 0,$$

whose integral is known to be

$$y = xF_1(z) + F_2(z),$$

representing the gauche surface generated by a line which, gliding upon two fixed directrices, remains constantly parallel to a given plane; as indeed might be anticipated from a consideration of the question in its second form.

In the same manner it might be shewn that the surface which, within given limits, renders the double integral

$$\iint \sqrt{\{(D_\theta r)^2 + (D_\phi r)^2\}}\, d\theta\, d\phi$$

a minimum, is given by the equation

$$\phi = \theta F_1(r) + F_2(r).$$

If it be proposed to investigate the property of this surface corresponding to the character of the generation of the analogous surface in rectangular coordinates, as the latter character is exhibited by the supposition $z = const.$, so the former property may be investigated by the supposition $r = const.$ Let then the surface be supposed to intersect a sphere described round the origin, and let the nature of the curve of intersection be examined. If we resolve any element into its rectangular components, one such component is $rd\theta$ and the other $r \sin\theta\, d\phi$. Let i be the inclination of the element to the meridional plane described through its extremity and the fixed axis, and it is evident that

$$\tan i = \frac{r \sin\theta\, d\phi}{rd\theta} = F_1(c) \sin\theta,$$

c being the radius of the sphere; or *the tangent of the angle of inclination of the curve to the meridional plane is proportional to the sine of the angle made by the radius vector with the axis.*

7. It may be well here to indicate certain desiderata, the knowledge of which might lead to the discovery of some interesting properties of surfaces.

The *measure of curvature* at any point of a surface is expressed in rectangular coordinates by the formula

$$\frac{1}{R_1R_2} = \frac{rt - s^2}{(1 + p^2 + q^2)^2}:$$

we have no corresponding expression in polar coordinates. Such might be discovered by the investigation of the analogue of the known formula for plane curves

$$\rho = r\frac{dr}{dp}.$$

Again, the *sum of the curvatures* at any point of a surface is expressed by the formula, in rectangular coordinates,

$$\frac{1}{R_1} + \frac{1}{R_2} = -\frac{(1 + q^2)r - 2pqs + (1 + p^2)t}{(1 + p^2 + q^2)^{\frac{3}{2}}}:$$

we have no corresponding expression in polar coordinates. Other desiderata will readily suggest themselves.

8. With regard to the rectification of curves, it may be useful to make a few observations upon a subject which has recently attracted much attention among French mathematicians. In the Notes by M. Liouville to his valuable edition of the *Application de l'Analyse à la Géometrie* of the illustrious Monge, will be found (p. 558) the following remarks.

" M. Serret a fait usage de certaines variables qu' il avait déjà employées au tome XIII. du *Journal de Mathématiques*, pour resoudre le problème suivant: x, y, z, s, étant quatre fonctions d'une variable indépendente θ assujetties a verifier l'équation

$$dx^2 + dy^2 + dz^2 = ds^2,$$

exprimer sans forme finie et sans aucun signe d'intégration, les valeurs générales de ces fonctions. La solution de ce problème conduit, par exemple, a trouver des courbes à double courbure qui soient à la fois algébriques et rectifiables algébriquement, ou dont l'arc dépende d'une transcendante donnée. Le problème analogue pour les courbes planes dépend de l'équation plus simple

$$dx^2 + dy^2 = ds^2,$$

[redacted]

et se resont, comme on sait, par les formules

$$x = \psi'(\theta)\sin\theta + \psi''(\theta)\cos\theta,$$

$$y = \psi'(\theta)\cos\theta - \psi''(\theta)\sin\theta,$$

$$s = \psi(\theta) + \psi''(\theta),$$

ou la fonction ψ est arbitraire. Les formules de M. Serret pour l'équation

$$dx^2 + dy^2 + dz^2 = ds^2,$$

sont beaucoup plus compliquées, et, partant, beaucoup moins utiles."

It appears to me that the integration of these equations may be effected directly, and with great simplicity, by employing the Calculus of Quaternions.

Thus, in the notation of this Calculus, the first equation

$$dx^2 + dy^2 = ds^2$$

is equivalent to

$$-(i\,dx + j\,dy)^2 = -(d\rho)^2,$$

or

$$i\,dx + j\,dy = d\rho;$$

whence

$$ix + jy = \rho + \alpha,$$

α being an arbitrary vector; or, between given limits,

$$i(x_2 - x_1) + j(y_2 - y_1) = \rho_2 - \rho_1,$$

an identity, as it ought to be.

Similarly, the second equation

$$dx^2 + dy^2 + dz^2 = ds^2$$

is equivalent to

$$-(i\,dx + j\,dy + k\,dz)^2 = -(d\rho)^2$$

or

$$i\,dx + j\,dy + k\,dz = d\rho;$$

whence

$$ix + jy + kz = \rho + \alpha,$$

α being an arbitrary vector, or, between given limits,

$$i(x_2 - x_1) + j(y_2 - y_1) + k(z_2 - z_1) = \rho_2 - \rho_1,$$

an identity, as it ought to be.

5, Trinity College, Dublin, Nov. 1864.

ON THE INTEGRATION OF LINEAR AND PARTIAL DIFFERENTIAL EQUATIONS.

By Arthur H. Curtis, A.M.
Trinity College, Dublin.

In a paper published in the No. of the *Cambridge and Dublin Mathematical Journal* for November 1851, Mr. Carmichael has given a method for integrating partial differential equations of a certain class. This method rests on the properties of a symbol of operation, which Mr. Carmichael has termed the Index Symbol. I propose in the first part of this paper to shew, that for this symbol we may substitute a more general one, which may be employed in the integration of a wider class of equations. In the second part I propose to consider an equation which has been integrated by Dr. Hargreave in the *Philosophical Transactions* for 1848, and to deduce its solution in a form more readily interpretable.

I. Mr. Carmichael employs the following symbolic equations:—

$$\left.\begin{aligned}
&\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+\&\text{c.}\ldots+\alpha\right)^{-1}0=u_{-\alpha}\\
&\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+\&\text{c.}\ldots+\alpha\right)^{-m}0=u_{-\alpha}(\log x+\log y+\log z+\&\text{c.})^{m-1}\\
&\qquad\qquad+v_{-\alpha}(\log x+\log y+\log z+\&\text{c.})^{m-2}\\
&\qquad\qquad+\&\text{c.}\ldots\ldots\ldots\ldots+w_{-\alpha}\\
&\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+\&\text{c.}\ldots+\alpha\right)^{-1}f_{-\alpha}=\frac{1}{n}(\log x+\log y+\log z+\&\text{c.})f_{-\alpha}
\end{aligned}\right\}\ldots\ldots(1),$$

where $u_{-\alpha}$, $v_{-\alpha}$, &c. are *arbitrary* homogeneous functions of the degree $-\alpha$ of the variables x, y, z, &c.; $f_{-\alpha}$ a *given* function of the same character; and n the number of variables.

Now, if we consider that

$$\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&\text{c.}\ldots+\alpha\right)$$
$$=\left(x^{\frac{1}{a}}\frac{d}{dx^{\frac{1}{a}}}+y^{\frac{1}{b}}\frac{d}{dy^{\frac{1}{b}}}+z^{\frac{1}{c}}\frac{d}{dz^{\frac{1}{c}}}+\&\text{c.}\ldots+\alpha\right),$$

it is evident that by changing x, y, z, &c. in these equations

[redacted]

into $x^{\frac{1}{a}}$, $y^{\frac{1}{b}}$, $z^{\frac{1}{c}}$, &c. respectively, we obtain the following more general ones:—

$$\left.\begin{aligned}
&\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c.\ldots+\alpha\right)^{-1}0=u_{-\alpha}\\
&\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c.\ldots+\alpha\right)^{-m}0=u_{-\alpha}(\log x^{\frac{1}{a}}+\log y^{\frac{1}{b}}+\log z^{\frac{1}{c}}+\&c.)^{m-1}\\
&\qquad\qquad+v_{-\alpha}(\log x^{\frac{1}{a}}+\log y^{\frac{1}{b}}+\log z^{\frac{1}{c}}+\&c.)^{m-2}\\
&\qquad\qquad+\&c.\ldots\ldots\ldots\ldots+w_{-\alpha}\\
&\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c.\ldots+\alpha\right)^{-1}f_{-\alpha}\\
&\qquad\qquad=\frac{1}{n}(\log x^{\frac{1}{a}}+\log y^{\frac{1}{b}}+\log z^{\frac{1}{c}}+\&c.\ldots)f_{-\alpha}
\end{aligned}\right\}\ldots\ldots(2),$$

where $u_{-\alpha}$, $v_{-\alpha}$, &c. are arbitrary homogeneous functions of the degree $-\alpha$ of $x^{\frac{1}{a}}$, $y^{\frac{1}{b}}$, $z^{\frac{1}{c}}$, &c.; $f_{-\alpha}$ a given function of the same nature; and n the number of variables.*

The class, of equations immediately integrable by these symbolic formulæ is represented by the equation

$$\left\{\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c.\ldots+\alpha\right)\left(a'x\frac{d}{dx}+b'y\frac{d}{dy}+c'z\frac{d}{dz}+\&c.\ldots+\alpha'\right)\ldots\right\}u=V\ldots(3),$$

where V is a function of x, y, z, &c., which may be supposed to be of the form $\Sigma Ax^m y^n z^r\ldots\ldots$

The inversion of the operator involved in (3) gives

$$u=\frac{1}{\left(ax\frac{d}{dx}+by\frac{d}{dy}+\&c.\ldots+\alpha\right)\left(a'x\frac{d}{dx}+b'y\frac{d}{dy}+\&c.\ldots+\alpha'\right)\ldots}\Sigma Ax^m y^n z^r\ldots$$

$$+\frac{L}{\left(ax\frac{d}{dx}+by\frac{d}{dy}+\ldots+\alpha\right)}0+\frac{M}{\left(a'x\frac{d}{dx}+b'y\frac{d}{dy}+\ldots+\alpha'\right)}0+\&c.$$

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* The process by which I first inverted the symbol

$$\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c.\ldots..+\alpha\right),$$

was more artificial than that here employed, but the result once obtained, its form suggested to me the proof given above.

where L, M, N, &c. are determinate constants, each of which may be changed into unity, as

$$\frac{L}{\left(ax\frac{d}{dx}+by\frac{d}{dy}+\&c.\ldots a\right)}0=\frac{1}{\left(ax\frac{d}{dx}+by\frac{d}{dy}\ldots\right)}L0$$

$$=\frac{1}{\left(ax\frac{d}{dx}+by\frac{d}{dy}\ldots\right)}0.$$

Now $$\left(x\frac{d}{dx}\right)^s Ax^m y^n z^r\ldots=(m)^s Ax^m y^n z^r\ldots,$$

$$\left(y\frac{d}{dy}\right)^{s'} Ax^m y^n z^r\ldots=(n)^{s'} Ax^m y^n z^r\ldots,$$

or generally

$$\phi\left(x\frac{d}{dx},\ y\frac{d}{dy},\ z\frac{d}{dz},\ \&c.\right)Ax^m y^n z^r\ldots=\phi(m,n,r,\&c.)\,Ax^m y^n z^r\ldots;$$

and therefore

$$u=\Sigma\frac{Ax^m y^n z^r\ldots}{(am+bn+cr+\&c.\ldots+a)\,(a'm+b'n+c'r+\&c.\ldots+a')\ldots}$$
$$+f_{-a}\left(x^{\frac{1}{a}},\ y^{\frac{1}{b}}\ z^{\frac{1}{c}},\ \&c.\right)+f_{-a'}\left(x^{\frac{1}{a'}},\ y^{\frac{1}{b'}},\ z^{\frac{1}{c'}},\ \&c.\right)+\&c.,$$

f_{-a}, $f_{-a'}$, &c., now denoting *arbitrary* functions of the degree $-a$, $-a'$, &c., homogeneous in the variables which they contain.

I shall now illustrate this method by a few particular examples.

Ex. 1. $x\dfrac{dz}{dx}-y\dfrac{dz}{dy}=\dfrac{x^2}{y}$.

This equation is equivalent to

$$\left(x\frac{d}{dx}-y\frac{d}{dy}\right)z=\frac{x^2}{y};$$

therefore $$z=\frac{1}{x\frac{d}{dx}-y\frac{d}{dy}}\frac{x^2}{y}+f_0(x,y^{-1})$$

$$=\frac{1}{2-(-1)}\frac{x^2}{y}+f(x\div y^{-1})$$

$$=\frac{x^2}{3y}+f(x.y).$$

Gregory's *Examples*, p. 361, *2nd edit.*

[redacted]

Ex. 2. $$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} - y^2\frac{d^2z}{dy^2} - y\frac{dz}{dy} = 0.$$

This equation may be put into the form

$$\left\{\left(x\frac{d}{dx}\right)^2 - \left(y\frac{d}{dy}\right)^2\right\} z = 0,$$

or $$\left\{\left(x\frac{d}{dx} + y\frac{d}{dy}\right)\left(x\frac{d}{dx} - y\frac{d}{dy}\right)\right\} z = 0;$$

therefore

$$z = \left(x\frac{d}{dx} + y\frac{d}{dy}\right)^{-1} 0 + \left(x\frac{d}{dx} - y\frac{d}{dy}\right)^{-1} 0 = f_0(x, y) + F_0(x, y^{-1})$$

$$= f\left(\frac{x}{y}\right) + F(x.y).$$

Gregory's *Examples*, p. 367.

Ex. 3. $$\left(x^2\frac{d^2}{dx^2} - y^2\frac{d^2}{dy^2}\right) z = xy.$$

This equation may be written

$$\left\{x\frac{d}{dx}\left(x\frac{d}{dx} - 1\right) - y\frac{d}{dy}\left(y\frac{d}{dy} - 1\right)\right\} z = xy,$$

or $$\left(x\frac{d}{dx} - y\frac{d}{dy}\right)\left(x\frac{d}{dx} + y\frac{d}{dy} - 1\right) z = xy;$$

therefore

$$z = \frac{1}{\left(x\frac{d}{dx} - y\frac{d}{dy}\right)\left(x\frac{d}{dx} + y\frac{d}{dy} - 1\right)} xy + \left(x\frac{d}{dx} - y\frac{d}{dy}\right)^{-1} 0$$

$$+ \left(x\frac{d}{dx} + y\frac{d}{dy} - 1\right)^{-1} 0$$

$$= \frac{1}{\left(x\frac{d}{dx} - y\frac{d}{dy}\right)} xy + f(x.y) + F(x, y)$$

But, as xy is a homogeneous function of the degree zero in x and y^{-1}, we have, by the third of equations (2),

$$z = \tfrac{1}{2}(\log x + \log y^{-1})\, xy + f(xy) + xF\left(\frac{y}{x}\right),$$

and therefore

If $a = -1$, we get as the solution of

$$x^2r - 2xys + y^2t + px + qy - m^2z = 0,$$

$$z = x^{-m}\phi(x.y) + x^m\psi(x.y).$$

If $a = 1$, we obtain the solution of

$$x^2r + 2xys + y^2t + px + qy - m^2z = 0,$$

$$z = x^{-m}\phi\left(\frac{y}{x}\right) + x^m\psi\left(\frac{y}{x}\right).$$

If $m = 0$, the equation becomes, when reduced to its symbolic form,

$$\left(x\frac{d}{dx} + ay\frac{d}{dy}\right)^2 z = 0;$$

therefore

$$z = \left(x\frac{d}{dx} + ay\frac{d}{dy}\right)^{-2} 0 = f_0(x, y^{\frac{1}{a}}) + (\log x + \log y^{\frac{1}{a}})\, F_0(x, y^{\frac{1}{a}})$$

$$= f\left(\frac{x}{y^{\frac{1}{a}}}\right) + \log(x.y^{\frac{1}{a}})\, F\left(\frac{x}{y^{\frac{1}{a}}}\right)$$

$$= \phi\left(\frac{x^a}{y}\right) + \log(x^a.y)\, \psi\left(\frac{x^a}{y}\right).$$

Ex. 6. $x^2r - y^2t - 2ap + 2bq + (a.a+1 - b.b+1)z = 0$.

This equation is equivalent to

$$\left\{\left(x\frac{d}{dx}\right)^2 - (2a+1)x\frac{d}{dx} - \left(y\frac{d}{dy}\right)^2 \right.$$

$$\left. + (2b+1)y\frac{d}{dy} + a.a+1 - b.b+1\right\} z = 0,$$

or $\left\{x\frac{d}{dx} + y\frac{d}{dy} - (a+b+1)\right\}\left\{x\frac{d}{dx} - y\frac{d}{dy} - (a-b)\right\} z = 0;$

therefore

$$z = \left\{x\frac{d}{dx} + y\frac{d}{dy} - (a+b+1)\right\}^{-1} 0 + \left\{x\frac{d}{dx} - y\frac{d}{dy} - (a-b)\right\}^{-1} 0$$

$$= f_{a+b+1}(x, y) + F_{a-b}(x, y^{-1})$$

$$= x^{a+b+1}f\left(\frac{x}{y}\right) + x^{a-b}F(x.y).$$

[redacted]

Ex. 7. $$xy\frac{d^2z}{dx\,dy} + ax\frac{dz}{dx} + by\frac{dz}{dy} + abz = V.$$

This equation may be written in the form

$$\left(x\frac{d}{dx} + b\right)\left(y\frac{d}{dy} + a\right)z = V;$$

therefore

$$z = \frac{1}{\left(x\frac{d}{dx} + b\right)\left(y\frac{d}{dy} + a\right)}V + \left(x\frac{d}{dx} + b\right)^{-1}0 + \left(y\frac{d}{dy} + a\right)^{-1}0$$

$$= \frac{1}{\left(x\frac{d}{dx} + b\right)\left(y\frac{d}{dy} + a\right)}V + x^{-b}f(y) + y^{-a}F(x).$$

Gregory's *Examples*, p. 366.

When the form of V has been assigned, the term $\frac{1}{\left(x\frac{d}{dx} + b\right)\left(y\frac{d}{dy} + a\right)}V$ can be interpreted. In fact, if $V = \Sigma Ax^m y^n$, it is equivalent to $\Sigma\frac{Ax^m y^n}{(m+b)(n+a)}$.

By this method can be integrated all equations of the form

$$\left\{\left(x\frac{d}{dx} + b\frac{d}{dy} + \&c. + a\right)\left(x\frac{d}{dx} + b'\frac{d}{dy} + \&c. + a'\right)\ldots\right\}u = V,$$

where the operator affecting u, so far as it depends on any one of the variables, z for example, may contain either $\frac{d}{dz}$ or $z\frac{d}{dz}$, but not both. For if we put $y = \log y_1$, $\frac{d}{dy} = y_1\frac{d}{dy_1}$, and the equation is reduced to the form

$$\left\{\left(x\frac{d}{dx} + by_1\frac{d}{dy_1} + \&c. + a\right)\left(x\frac{d}{dx} + b'y_1\frac{d}{dy_1} + \&c. + a'\right)\ldots\right\}u = V,$$

which is comprehended under the class of equations already considered.

This being solved, we must change y_1 into e^y in the result.

As an example of this class, suppose

$$\left(x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + ax\frac{d^2}{dx\,dy} + b\frac{d^2}{dy^2}\right)z = xy,$$

or $$\left\{\left(x\frac{d}{dx}\right)^2 + ax\frac{d}{dx}\frac{d}{dy} + b\left(\frac{d}{dy}\right)^2\right\}z = xy,$$

or $$\left(x\frac{d}{dx} - \alpha\frac{d}{dy}\right)\left(x\frac{d}{dx} - \beta\frac{d}{dy}\right)z = xy,$$

α, β being the roots of the equation

$$u^2 + au + b = 0;$$

therefore

$$z = \frac{1}{\left(x\dfrac{d}{dx} - \alpha\dfrac{d}{dy}\right)\left(x\dfrac{d}{dx} - \beta\dfrac{d}{dy}\right)}xy$$

$$+ \left(x\frac{d}{dx} - \alpha\frac{d}{dy}\right)^{-1}0 + \left(x\frac{d}{dx} - \beta\frac{d}{dy}\right)^{-1}0,$$

but $$\left(x\frac{d}{dx} - \alpha\frac{d}{dy}\right)^{-1}0 = \left(x\frac{d}{dx} - \alpha y_1\frac{d}{dy_1}\right)^{-1}0 = f(x.y_1^{\frac{1}{\alpha}})$$

$$= f(x.e^{\frac{y}{\alpha}}) = f(e^{\log x}.e^{\frac{y}{\alpha}}) = f(e^{\frac{y}{\alpha}+\log x})$$

$$= \phi\left(\frac{y}{\alpha} + \log x\right), \text{ or, as } \phi \text{ is arbitrary,}$$

$$= \phi(y + \alpha\log x);$$

therefore

$$z = \frac{1}{\left(1 - \alpha\dfrac{d}{dy}\right)\left(1 - \beta\dfrac{d}{dy}\right)}xy + \phi(y + \alpha\log x) + \psi(y + \beta\log x),$$

but $$\frac{1}{\left(1 - \alpha\dfrac{d}{dy}\right)\left(1 - \beta\dfrac{d}{dy}\right)}xy = \frac{x}{\alpha\beta}\left(\frac{d}{dy} - \frac{1}{\alpha}\right)^{-1}\left(\frac{d}{dy} - \frac{1}{\beta}\right)^{-1}y$$

$$= \frac{x}{b}e^{\frac{y}{\alpha}}\int_y e^{-\frac{y}{\alpha}}e^{\frac{y}{\beta}}\int_y e^{-\frac{y}{\beta}}y = x(y - a),$$

and the solution is

$$z = x(y - a) + \phi(y + \alpha\log x) + \psi(y + \beta\log x).$$

II. Many of the most important differential equations of the second order and first degree as yet integrated, indeed all those given by Gregory in his *Examples* which are not reducible either to equations with constant coefficients, or to the class considered in the first part of this paper, are reducible to the following form:

$$\left(\frac{d^2}{dx^2} + 2Q\frac{d}{dx} + Q^2 + Q' \pm c^2 - \frac{m.m+1}{x^2}\right)u = 0 \ldots(a),$$

Q being a function of x and $Q' = \dfrac{d}{dx}Q$.

It is therefore a matter of some importance to obtain the solution of this equation in a form easily capable of interpretation in all cases, more particularly so, as in the instances which Gregory has considered the integration has been effected by various means, and frequently by series, a method which is always tedious, and involves the difficulty of reducing the result when obtained to a finite form. Dr. Hargreave has solved the equation

$$\left(\frac{d^2}{dx^2} + 2Q\frac{d}{dx} + Q^2 + Q' - c^2 - \frac{m.m-1}{x^2}\right)u = 0,*$$

which may be identified with (a) by changing m into $m+1$. His solution then involving $\left(\frac{d^2}{dx^2} - c^2\right)^m$ or $\left(e^{cx}\frac{d}{dx}e^{-2cx}\frac{d}{dx}e^{cx}\right)^m$, in order to interpret it we must perform $2m$ differentiations. But a solution involving only half that number, and therefore more immediately interpretable, may be obtained as follows.

The equation

$$\left(\frac{d^2}{dx^2} + 2Q\frac{d}{dx} + Q^2 + Q' \pm c^2 - \frac{m.m+1}{x^2}\right)u = 0$$

may be put into the form

$$\left\{\left(\frac{d}{dx} + Q\right)^2 \pm c^2 - \frac{m.m+1}{x^2}\right\}u = 0,$$

or $$\left\{\left(e^{-Q_1}\frac{d}{dx}e^{Q_1}\right)^2 \pm c^2 - \frac{m.m+1}{x^2}\right\}u = 0, \text{ where } Q_1 = \int Q\,dx,$$

or $$\left\{e^{-Q_1}\left(\frac{d}{dx}\right)^2 e^{Q_1} \pm c^2 - \frac{m.m+1}{x^2}\right\}u = 0;$$

therefore $$\left\{\frac{d^2}{dx^2} \pm c^2 - \frac{m.m+1}{x^2}\right\}e^{Q_1}u = 0.$$

Let $e^{Q_1}u = z$, and therefore $u = e^{-Q_1}z$,

where z is given by the equation

$$\left(\frac{d^2}{dx^2} \pm c^2 - \frac{m.m+1}{x^2}\right)z = 0.$$

This equation, by a very ingenious process, has been solved by Professor Boole in the following elegant form,

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* *Philosophical Transactions* for 1848, Part I.

$$z = \frac{1}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m \frac{1}{x^{2m-1}}\left(\frac{d^2}{dx^2}\pm c^2\right)^{-1}0,$$

which is equivalent to

$$z = \frac{1}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m \frac{1}{x^{2m-1}}\{A\cos(cx+\alpha)\},$$

or
$$z = \frac{1}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m \frac{1}{x^{2m-1}}(Ae^{cx}+Be^{-cx}),$$

according as the upper or lower sign before c^2 has been taken throughout.*

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* Professor Boole's paper being difficult of access to many readers of this Journal, I may be excused for giving here his method of obtaining the solution which I have assumed above. The equation

$$\left(\frac{d^2}{dx^2}\pm c^2 - \frac{m.m+1}{x^2}\right)u = 0,$$

by assuming $x = e^{\theta}$, can be reduced to the form

$$\{(D+m)(D-m-1)\pm c^2e^{2\theta}\}u = 0,$$

where $D = \frac{d}{d\theta}$. Let $u = e^{-m\theta}v$, and we get

$$\{D(D-2m-1)\pm c^2e^{2\theta}\}v = 0.$$

Assume $v = \phi(D)w$, and this equation becomes

$$\{D(D-2m-1)\phi(D)\pm c^2\phi(D-2)e^{2\theta}\}w = 0,$$

and therefore

$$\{D(D-2m-1)\frac{\phi(D)}{\phi(D-2)}\pm c^2e^{2\theta}\}w = \{\phi(D-2)\}^{-1}0.$$

This can be assimilated to the equation

$$\{D(D-1)\pm c^2e^{2\theta}\}w = 0,$$

by assuming $\{\phi(D-2)\}^{-1}0 = 0$, and determining $\phi(D)$ so as to satisfy the equation

$$D(D-2m-1)\frac{\phi(D)}{\phi(D-2)} = D.D-1, \quad\text{or}\quad \frac{\phi(D)}{\phi(D-2)} = \frac{D-1}{D-2m-1},$$

which may be done by assuming

$$\phi(D) = \frac{D-1.D-3.D-5\ldots}{D-2m-1.D-2m-3\ldots} = (D-1.D-3.D-5\ldots D-2m+1)$$

$$= e^{\theta}De^{-\theta}e^{3\theta}De^{-3\theta}\ldots De^{(-2m+1)\theta} = e^{-\theta}(e^{2\theta}D)^m e^{(-2m+1)\theta}$$

$$= \frac{1}{x}\left(x^3\frac{d}{dx}\right)^m\frac{1}{x^{2m-1}}, \quad\text{since } D = x\frac{d}{dx},$$

therefore $$u = x^{-m}v = x^{-m}\phi(D)w = \frac{1}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m\frac{1}{x^{2m-1}}w.$$

But the equation for w is equivalent to

$$\left\{x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)\pm c^2x^2\right\}w = 0,$$

The corresponding solutions of (a) are

$$u=\frac{e^{-Q_1}}{x^{m-1}}\left(x^2\frac{d}{dx}\right)^m\frac{A\cos(cx+\alpha)}{x^{2m-1}},$$

and
$$u=\frac{e^{-Q_1}}{x^{m-1}}\left(x^2\frac{d}{dx}\right)^m\frac{Ae^{cx}+Be^{-cx}}{x^{2m-1}}.$$

Suppose, for example, $Q=\dfrac{n}{x}$, and we get the solution of the equation

$$\left(D^2+\frac{2n}{x}D+\frac{n.n-1-m.m+1}{x^2}\pm c^2\right)u=0,\quad\text{where } D=\frac{d}{dx},$$

in the form

$$u=\frac{1}{x^{m+n-1}}\left(x^2\frac{d}{dx}\right)^m\frac{A\cos(cx+\alpha)}{x^{2m-1}},$$

or
$$u=\frac{1}{x^{m+n-1}}\left(x^2\frac{d}{dx}\right)^m\frac{Ae^{cx}+Be^{-cx}}{x^{2m-1}}.$$

If $m=1$, $n=1$, we obtain as the solution of

$$\left(D^2+\frac{2}{x}D+c^2-\frac{2}{x^2}\right)u=0,$$

$$u=\frac{d}{dx}\left\{\frac{A\cos(cx+\alpha)}{x}\right\}=-\frac{A}{x^2}\cos(cx+\alpha)-\frac{Ac}{x}\sin(cx+\alpha),$$

Gregory's *Examples*, p. 313;

while the solution of

$$\left(D^2+\frac{2}{x}D-c^2-\frac{2}{x^2}\right)u=0,$$

is $$u=\frac{d}{dx}\left\{\frac{Ae^{cx}+Be^{-cx}}{x}\right\}=A\left(\frac{c}{x}-\frac{1}{x^2}\right)e^{cx}-B\left(\frac{c}{x}+\frac{1}{x^2}\right)e^{-cx}.$$

The suppositions $n=1$, $m=0$, give the equation

$$\left(D^2+\frac{2}{x}D\pm c^2\right)u=0,$$

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or $\left(x^4\dfrac{d^2}{dx^2}\pm c^2x^4\right)w=0$, or $\left(\dfrac{d^2}{dx^2}\pm c^2\right)w=0$;

therefore
$$w=\left(\frac{d^2}{dx^2}\pm c^2\right)^{-1}0;$$

and therefore
$$u=\frac{1}{x^{m+1}}\left(x^2\frac{d}{dx}\right)^m\frac{1}{x^{2m-1}}\left(\frac{d^2}{dx^2}\pm c^2\right)^{-1}0.$$

and its solution is

$$u = \frac{1}{x}\{A\cos(cx+\alpha)\},$$

or

$$u = \frac{1}{x}\{Ae^{cx} + Be^{-cx}\},$$

according as we take the upper or lower sign before c^2 in the proposed equation.

Gregory's *Examples*, p. 312.

Let $Q = c = \frac{1}{2}q$, and take the negative sign before c^2; the equation is then

$$\left(D^2 + qD - \frac{m.m+1}{x^2}\right)u = 0,$$

and its solution

$$u = \frac{e^{-\frac{1}{2}qx}}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m \frac{Ae^{\frac{1}{2}qx} + Be^{-\frac{1}{2}qx}}{x^{2m-1}}.$$

If, for instance, $m = 1$, we obtain the equation

$$\left(D^2 + qD - \frac{2}{x^2}\right)u = 0,$$

and its solution

$$u = e^{-\frac{1}{2}qx}x\frac{d}{dx}\frac{Ae^{\frac{1}{2}qx} + Be^{-\frac{1}{2}qx}}{x}$$

$$= C_1\left(1 - \frac{2}{qx}\right) + C_2\left(1 + \frac{2}{qx}\right)e^{-qx}.$$

Gregory's *Examples*, p. 349.

Let $Q = \frac{-2}{x}$, $m = 2$, and take the positive sign before c^2, we thus get the equation

$$\left(D^2 - \frac{4}{x}D + c^2\right)u = 0,$$

and its solution

$$u = \frac{1}{x}\left(x^3\frac{d}{dx}\right)^2 \frac{A\cos(cx+\alpha)}{x^3}$$

$$= x^2\frac{d}{dx}x^3\frac{d}{dx}\frac{A\cos(cx+\alpha)}{x^3}$$

$$= A\{(3 - c^2x^2)\cos(cx+\alpha) + 3cx\sin(cx+\alpha)\}.$$

Gregory's *Examples*, p. 350.

If in equation (α) we put $Q = \frac{n}{x}$, and transform, as Dr. Hargreave has done in the paper already alluded to, by changing the independent variable so as to cause the second term to vanish, we get the equation

$$\left\{\frac{d^2}{dz^2} \pm \left(\frac{c}{2n-1}\right)^2 z^{-\frac{4n}{2n-1}} + (n.n-1-m.m+1)\, z^{-2}\right\} u = 0, \text{ where } x = z^{-\frac{1}{2n-1}},$$

which for $m = n - 1$ reduces itself to

$$\left\{\frac{d^2}{dz^2} \pm \left(\frac{c}{2n-1}\right)^2 z^{-\frac{4n}{2n-1}}\right\} u = 0 \;\ldots\ldots\ldots\ldots (\beta);$$

the solution of which therefore is

$$u = x^{-2n}\left(x^3\frac{d}{dx}\right)^{n-1} \frac{1}{x^{2n-3}} \left(\frac{d^2}{dx^2} \pm c^2\right)^{-1} 0,$$

which is equivalent to

$$\left.\begin{aligned} u &= x^{-2n}\left(x^3\frac{d}{dx}\right)^{n-1} \frac{A\cos(cx+\alpha)}{x^{2n-3}} \\ \text{or}\quad u &= x^{-2n}\left(x^3\frac{d}{dx}\right)^{n-1} \frac{Ae^{cx} + Be^{-cx}}{x^{2n-3}} \end{aligned}\right\}, \text{ where } x = z^{-\frac{1}{2n-1}},$$

according as the upper or lower sign before c^2 is taken in (β).

If in equation (α) we put $Q = \frac{-n}{x}$, and change the independent variable so as to cause the second term to vanish, or change, in the transformed equation above, n into $-n$, we obtain the equation

$$\left\{\frac{d^2}{dz^2} \pm \left(\frac{c}{2n+1}\right)^2 z^{-\frac{4n}{2n+1}} + (n.n+1-m.m+1) z^{-2}\right\} u = 0, \text{ where } x = z^{\frac{1}{2n+1}}.$$

If we assume $m = n$, this equation reduces itself to

$$\left\{\frac{d^2}{dz^2} \pm \left(\frac{c}{2n+1}\right)^2 z^{-\frac{4n}{2n+1}}\right\} u = 0 \;\ldots\ldots\ldots\ldots (\gamma),$$

the solution of which is

$$u = x^{-1}\left(x^3\frac{d}{dx}\right)^{n} \frac{1}{x^{2n-1}} \left(\frac{d^2}{dx^2} \pm c^2\right)^{-1} 0,$$

which is equivalent to

$$\left.\begin{aligned} u &= x^{-1}\left(x^3\frac{d}{dx}\right)^n \frac{A\cos(cx+\alpha)}{x^{2n-1}} \\ \text{or}\qquad u &= x^{-1}\left(x^3\frac{d}{dx}\right)^n \frac{Ae^{cx}+Be^{-cx}}{x^{2n-1}} \end{aligned}\right\}, \text{ where } x = z^{\frac{1}{2n+1}}.$$

according as the upper or lower sign is taken in the second term of (γ).

If in (β) and (γ) we substitute $e^{\int_z v}$ for u, they will be reduced to Riccati's forms; but the solutions are not interpretable unless n be an integer, which is Riccati's restriction.

Substituting unity for (n) in (β), we get the equation

$$\left(\frac{d^2}{dz^2} - \frac{c^2}{z^4}\right)u = 0,$$

and its solution

$$u = x^{-3}\frac{(Ae^{cx}+Be^{-cx})}{x^{-1}}, \text{ where } x = \frac{1}{z},$$

or $$u = z\left(Ae^{\frac{c}{z}} + Be^{-\frac{c}{z}}\right);$$

while the solution of $\left(\frac{d^2}{dz^2} + \frac{c^2}{z^4}\right)u = 0$ is $u = Az\cos\left(\frac{c}{z}+\alpha\right)$.

Gregory's *Examples*, p. 345.

In (γ) let $n = 1$, and taking the negative sign before c^2 it becomes

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{3}\right)^2 z^{-\frac{4}{3}}\right\}u = 0;$$

the solution of which is

$$\begin{aligned} u &= x^{-1}\left(x^3\frac{d}{dx}\right)\left(\frac{Ae^{cx}+Be^{-cx}}{x}\right), \text{ where } x = z^{\frac{1}{3}}, \\ &= A(cx-1)e^{cx} - B(cx+1)e^{-cx} \\ &= A(cz^{\frac{1}{3}}-1)e^{cz^{\frac{1}{3}}} - B(cz^{\frac{1}{3}}+1)e^{-cz^{\frac{1}{3}}}; \end{aligned}$$

or supposing $c = 3k$, the solution of $\left(\frac{d^2}{dz^2} - k^2 z^{-\frac{4}{3}}\right)u = 0$ is

$$u = C_1\left(z^{\frac{1}{3}} - \frac{1}{3k}\right)e^{3kz^{\frac{1}{3}}} + C_2\left(z^{\frac{1}{3}} + \frac{1}{3k}\right)e^{-3kz^{\frac{1}{3}}}.$$

Gregory's *Examples*, p. 345.

In the same manner the solution of $\left(\dfrac{d^2}{dz^2} + k^2 z^{-\frac{4}{3}}\right) u = 0$ is

$$u = C\left\{z^{\frac{1}{3}} \sin(3kz^{\frac{1}{3}} + \alpha) + \frac{1}{3k}\cos(3kz^{\frac{1}{3}} + \alpha)\right\}.$$

In (γ) let $n = 2$; and taking the negative sign before c^2 we get the equation

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{5}\right)^2 z^{-\frac{8}{5}}\right\} u = 0,$$

the solution being

$$u = x^{-1}\left(x^3 \frac{d}{dx}\right)^2 \frac{Ae^{cx} + Be^{-cx}}{x^3}, \quad \text{where } x = z^{\frac{1}{5}},$$

or $\qquad u = A(c^2x^2 - 3cx + 3)\, e^{cx} + B(c^2x^2 + 3cx + 3)\, e^{-cx},$

or $\qquad u = A(c^2z^{\frac{2}{5}} - 3cz^{\frac{1}{5}} + 3)\, e^{cz^{\frac{1}{5}}} + B(c^2z^{\frac{2}{5}} + 3cz^{\frac{1}{5}} + 3)\, e^{-cz^{\frac{1}{5}}};$

and therefore the solution of $\left(\dfrac{d^2}{dz^2} - k^2 z^{-\frac{8}{5}}\right) u = 0$ is

$$u = C_1\left(z^{\frac{2}{5}} - \frac{3}{5k} z^{\frac{1}{5}} + \frac{3}{(5k)^2}\right) e^{5kz^{\frac{1}{5}}} + C_2\left(z^{\frac{2}{5}} + \frac{3}{5k} z^{\frac{1}{5}} + \frac{3}{(5k)^2}\right) e^{-5kz^{\frac{1}{5}}}.$$

This equation is given by Gregory (p. 345), but his solution is incorrect, as is evident from inspection, since the particular solutions should only differ in the sign of k.

We have hitherto in equation (α) considered Q to be a function of x only, and c to be a constant, in which case (α) will represent a class of linear differential equations. But if we suppose Q a function of x and $\dfrac{d}{dy}$, and c a function of $\dfrac{d}{dy}$, (α) will then represent a class of partial differential equations, the most general form of which will be

$$\left[\frac{d^2}{dx^2} + 2f\left(x, \frac{d}{dy}\right)\frac{d}{dx} + \left\{f\left(x, \frac{d}{dy}\right)\right\}^2 + f'\left(x, \frac{d}{dy}\right) - \frac{m.m+1}{x^2} - \left\{F\left(\frac{d}{dy}\right)\right\}^2\right] u = 0 \ldots(\alpha'),$$

where $f'\left(x, \dfrac{d}{dy}\right)$ stands for $\dfrac{d}{dx} f\left(x, \dfrac{d}{dy}\right)$, the corresponding solution being

$$u = \frac{e^{-\int f\left(x,\, \frac{d}{dy}\right) dx}}{x^{m+1}} \left(x^3 \frac{d}{dx}\right)^m \frac{1}{x^{2m-1}} \left[\frac{d^2}{dx^2} - \left\{F\left(\frac{d}{dy}\right)\right\}^2\right]^{-1} 0.$$

If $F\left(\dfrac{d}{dy}\right) = c\,\dfrac{d}{dy}$, this solution becomes

$$u = \frac{e^{-\int f\left(x,\, \frac{d}{dy}\right) dx}}{x^{m+1}} \left(x^3 \frac{d}{dx}\right)^m \frac{1}{x^{2m-1}} \left\{e^{cx\frac{d}{dy}}\phi(y) + e^{-cx\frac{d}{dy}}\psi(y)\right\},$$

or $$u = \frac{e^{-\int f\left(x,\, \frac{d}{dy}\right) dx}}{x^{m+1}} \left(x^3 \frac{d}{dx}\right)^m \frac{1}{x^{2m-1}} \left\{\phi(y + cx) + \psi(y - cx)\right\}.$$

Suppose, for example,

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - a^2\frac{d^2}{dy^2}\right) u = 0.$$

In this case $f\left(x, \dfrac{d}{dy}\right) = \dfrac{1}{x}$, $m = 0$, and the solution is

$$u = \frac{1}{x}\left\{\phi(y + ax) + \psi(y - ax)\right\}.$$

Gregory's *Examples*, p. 367.

Let $$\frac{d^2}{dy^2}u = a^2\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{2}{x^2}\right)u,$$

or $$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{2}{x^2} - \frac{1}{a^2}\frac{d^2}{dy^2}\right)u = 0.$$

Here $f\left(x, \dfrac{d}{dy}\right) = \dfrac{1}{x}$, $m = 1$, $c = \dfrac{1}{a}$, and the solution consequently is

$$u = \frac{d}{dx}\frac{1}{x}\left\{\phi\left(y + \frac{x}{a}\right) + \psi\left(y - \frac{x}{a}\right)\right\},$$

or, as ϕ and ψ are arbitrary,

$$u = \frac{d}{dx}\frac{1}{x}\left\{\phi(x + ay) + \psi(x - ay)\right\}$$

$$= \frac{1}{x}\left\{\phi'(x + ay) + \psi'(x - ay)\right\} - \frac{1}{x^2}\left\{\phi(x + ay) + \psi(x - ay)\right\}.$$

Gregory's *Examples*, p. 368.

$$= a\phi'(y+ax) - a\psi$$

Let

$$\left(\frac{d^2}{dx^2} + \right.$$

Here $f\left(x, \frac{d}{dy}\right) = \frac{a}{2}\,\frac{d}{dy}$

$$u = e^{-\frac{a}{2}x\frac{d}{dy}}\, x\,\frac{d}{dx}\,\frac{1}{x}\left\{\phi\left(y + \frac{ax}{2}\right)\right.$$

$$= e^{-\frac{a}{2}x\frac{d}{dy}}\left[\frac{a}{2}\left\{\phi'\left(y + \frac{ax}{2}\right) - \psi'\left(\right.\right.\right.$$

$$= \frac{a}{2}\phi'(y) - \frac{1}{x}\phi(y) - \frac{a}{2}\psi'(y -$$

or, as ϕ and ψ are arbitrary,

$$u = a\phi'(y) - \frac{2}{x}\phi(y) + a\psi$$

Let

$$\left(\frac{d^2}{dx^2} - 2\frac{p}{x}\frac{d}{dx} - \right.$$

Here $f\left(x, \frac{d}{dy}\right) = -\frac{p}{x}$ and $m =$

the solution of which therefore is

$$u = x^3 \frac{d}{dx}\, x^3 \frac{d}{dx}\, \frac{\phi(y+ax) + \psi(y-ax)}{x^3}$$

$$= x^3 \frac{d}{dx}\left[\frac{-3}{x}\{\phi(y+ax)+\psi(y-ax)\} + a\{\phi'(y+ax) - \psi'(y-ax)\}\right]$$

$$= 3\{\phi(y+ax)+\psi(y-ax)\} - 3ax\{\phi'(y+ax)-\psi'(y-ax)\}$$
$$+ a^2x^2\{\phi''(y+ax)+\psi''(y-ax)\}.$$

Gregory's *Examples*, p. 369.

If in (β) and (γ) we change c into $c\dfrac{d}{dy}$, we obtain the equations

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{2n-1}\right)^2 z^{-\frac{4n}{2n-1}} \frac{d^2}{dy^2}\right\} u = 0 \;\ldots\ldots\ldots (\beta'),$$

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{2n+1}\right)^2 z^{-\frac{4n}{2n+1}} \frac{d^2}{dy^2}\right\} u = 0 \;\ldots\ldots\ldots (\gamma');$$

the solutions of which therefore are respectively

$$u = x^{-2n}\left(x^3\frac{d}{dx}\right)^{n-1} \frac{\phi(y+cx)+\psi(y-cx)}{x^{2n-3}}, \text{ where } x = z^{-\frac{1}{2n-1}},$$

$$u = x^{-1}\left(x^3\frac{d}{dx}\right)^{n} \frac{\phi(y+cx)+\psi(y-cx)}{x^{2n-1}}, \quad \text{where } x = z^{\frac{1}{2n+1}}.$$

In (β') let $n = 1$; it will then become

$$\left(\frac{d^2}{dz^2} - \frac{c^2}{z^4}\frac{d^2}{dy^2}\right) u = 0,$$

the solution of which therefore is

$$u = x^{-1}\{\phi(y+cx) + \psi(y-cx)\}, \text{ where } x = z^{-1},$$

or
$$u = z\left\{\phi\left(y+\frac{c}{z}\right) + \psi\left(y - \frac{c}{z}\right)\right\}.$$

Gregory's *Examples*, p. 368.

In (γ') let $n = 1$, and we obtain the equation

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{3}\right)^2 z^{-\frac{4}{3}} \frac{d^2}{dy^2}\right\} u = 0,$$

the solution of which consequently is

$$u = x^3 \frac{d}{dx} \frac{\phi(y+cx) + \psi(y-cx)}{x}, \text{ where } x = z^{\frac{1}{3}}$$

$$= -\{\phi(y+cx) + \psi(y-cx)\} + cx\{\phi'(y+cx) - \psi'(y-cx)\}$$

$$= -\{\phi(y+cz^{\frac{1}{3}}) + \psi(y-cz^{\frac{1}{3}})\} + cz^{\frac{1}{3}}\{\phi'(y+cz^{\frac{1}{3}}) - \psi'(y-cz^{\frac{1}{3}})\}.$$

The solution of $\left(\frac{d^2}{dz^2} - a^2 z^{-\frac{4}{3}} \frac{d^2}{dy^2}\right) u = 0$ consequently is

$$u = -\{\phi(y+3az^{\frac{1}{3}}) + \psi(y-3az^{\frac{1}{3}})\} + 3az^{\frac{1}{3}}\{\phi'(y+3az^{\frac{1}{3}}) - \psi'(y-3az^{\frac{1}{3}})\};$$

or, as ϕ and ψ are arbitrary,

$$u = z^{\frac{1}{3}}\{\phi'(y+3az^{\frac{1}{3}}) + \psi'(y-3az^{\frac{1}{3}})\} - \frac{1}{3a}\{\phi(y+3az^{\frac{1}{3}}) - \psi(y-3az^{\frac{1}{3}})\}.$$

Gregory's *Examples*, p. 368.

6, *Trinity College, Dublin, Nov.* 24, 1864.

THE END.

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